The spontaneous pricing order in the noncooperative game of monopolistic manufacturer and many retailers

Tieyan Si

Harbin Institute of Technology, Harbin 150080, China

Abstract

We consider the collective pricing orders in a minimum supply chain that is composed of a monopolistic manufacturer and many retailers that belong to the same chain store firm. The retailers have the freedom to raise or lower the local price. The chain store firm sets up the commercial rules for local retail stores to maximize its total payoff. The monopolistic manufacturer firm controls the total quantity supplied for the market to achieve maximum benefits. We applied the two dimensional Ising model in statistical physics to map the collective distribution of microscopic strategy of local retailers into the macroscopic total payoff of the chain store firm. The local stores choose to raise the price or lower the price based their own mind when the supply in market surpasses the demand. When the supply in market is far less than the demand, the stores synchronously raise prices, even though a local store only have the incomplete information of their nearest neighboring supermarket. We find the critical equation for the balance point between the action of supplier and the action of chain store management based on game theory and statistical physics. The critical equation can identify the Nash equilibrium point of the non-cooperative game between the manufacturer and the chain-store seller, and reveal different levels of collective operations. This statistical physics method also holds for more complicate supply chains and economic systems.

Keywords Spontaneous order, Price formation, Non-cooperative game theory, Statistical physics, Supply chain, two dimensional Ising model.

1   Introduction

Hayek believed that spontaneous order emerges in economics without a global planer if individual operator achieves the maximum degree of freedom [1]. Hayek's notion of spontaneous order is formalized within the framework of an Arrow-Debreu economy theory [2]. The operators in economics are not completely free instead they strongly interact with one another to spontaneously form firms with special function in the network of human society. Even though economic activities are highly localized, the failure of one firm could induce

a chain of failures in global market in a financial crisis, leading to a vast destruction of economic order. The collective behavior of economic activity shows similar phenomena as biological system and many particle physics. The natural rule and mathematical principal for the generation and evolution of collective cooperation are still unknown due to the high sophistication of society [3]. In order to understand the formation of spontaneous economic order and predict the occurrence of economic crisis in a quantitative way, we focus on the spontaneous pricing order of many local retailers in market.

In the classical market mechanism, the market price of a good grows higher with respect to an increasing demand and a decreasing supply, which attracts more firms to produce the same good with the same quantity at a lower price. When the total supply surpasses the market demand, the price keeps dropping until it reaches a balance point between supply and demand [4]. In practical economic activities, both supply and demand are highly fluctuating and uncertain [5]. The price is an unknown function of supply and demand, which results in an inevitable obstacle in mathematical modeling of market mechanism. Based on different assumptions about the functional dependence of price on demand, game theory provides an effective framework for the operations in many different aspects of economics [6], such as the mean-field game model of electricity market [9], noncooperative game theory of two competitive firms that sell the same product [7], noncooperative bargaining games in network service [8], noncooperative competition in supply chain [10], Nash-Cournot model with nonsmooth demand functions in electric power market [12] and dynamic bargaining game in supply chain [11]. The complex economic system goes far beyond the framework of game theory. Other mathematical theories are developed to study price formation, such as the general price and assortment competition model [13], the Boltzmann-type pricing model [14], stochastic latent moment model of electricity price [15], dynamic pricing of perishable assets [16] and network service [17]. Statistical physics was also proved an effective theory for understanding the financial markets, trading and communications networks [18]. The one dimensional Ising model in statistical physics theory was introduced into economics theory to study the price formation in financial markets, in which the up-spin represents a trader placing buy orders, a down-spin represents a trader placing sell orders [19].

In this manuscript, we generalized the two dimensional Ising model to build a game theory for the price formation in a minimum supply chain from the point of view of statistical physics. The minimum supply chain is composed of many identical local retailers and a monopolistic manufacturer. The nearest neighboring stores play either a noncooperative or cooperative game. These local stores are collectively controlled by the strategy of the management of chain store firm to play a non-cooperative game with the monopolistic manufacturer. This oversimplified supply chain model would demonstrate collective phase transition like that in statistical physics [23]. The sudden transition from one stable collective strategy to another one results in singularity of total payoff function. Other collective economic observable also show singularity and obey scaling laws at critical point. Renormalization group theory [25] offered a natural explanation on the transitions between collective strategy patterns by non-cooperative behavior between two competing powers. We developed the exact correspondence between non-cooperative game theory and renormalization group theory for phase transition, and established the multi-phase coexistence equation, which
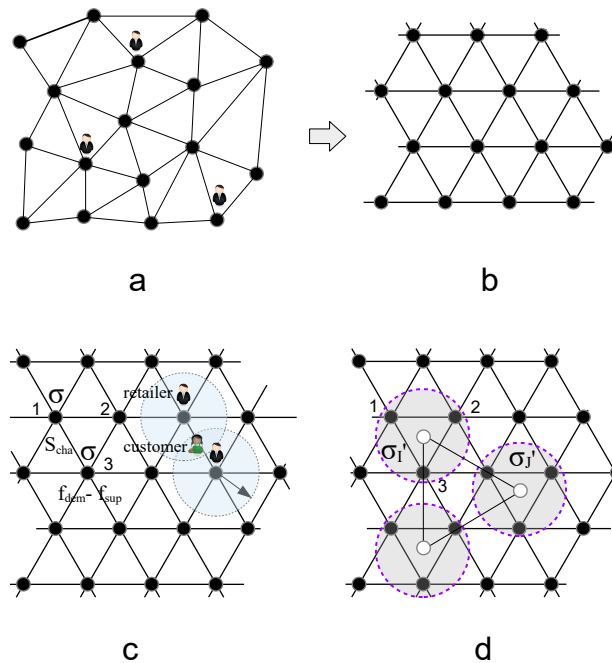
Figure 1: (a) The random distribution of retailers in a local region construct a network of triangles. (b) The random network of retailers are transformed topologically into a regular triangular lattice. (c) The distribution of retailers in a triangular lattice with customers located on the bond that connects two retailers. (d) Three nearest neighboring retailers form a minimum triangular coalition, which is represented by a block spin.

has promising applications in predicting sudden changes in supply chain and financial market.

2 Noncooperative game model of the monopolistic manufacturer and chain store

2.1 The two dimensional Ising model for the competition between chain store retailers and monopolistic manufacturer

The supply chain is a network that links the raw material supplier, manufacturer, distributor, retailer and customer. The whole network is too complex to build a mathematical physics model that is capable of leading to accurate predictions. In order to reach a quantitative understanding on the competition relationship within a supply chain, we combine the raw material supplier and manufacturer into one monopolistic manufacturer firm, and combine the distributor and retailers into one chain store firm. The chain store firm governs many local retailers that randomly distributed in a regional market as showed in Fig.

1 (a), in which the local stores located at the vertex of the network. We further assume one local retailer only competes with his nearest neighboring retailers, therefore the influence of the distance between the nearest neighboring retailers in the random network is smeared out. Then the random network is regularized into a homogeneous triangular lattice in Fig. 1 (b) for the convenience of mathematical modeling. We assume that each local store has two strategies: raising one unit of price or lowering one unit of price, which is correspondingly expressed by Ising spin $\sigma_i = +1$ or $\sigma_i = -1$ in Fig. 1 (c). The local store is the microscopic game player. It chooses its strategy according to the strategy played by its nearest neighbors and the macroscopic quantity of supply versus demand. The payoff of the retailer located at the ith vertex is quantified by the payoff function

$$P_i := P_0 - H_i = P_0 + S_{cha}(\sigma_0 + \sigma_i)(\sigma_0 + \sigma_j) + (f_{dem} - f_{sup})(\sigma_0 + \sigma_i), \quad (1)$$

where P_0 is a reference payoff value to ensure the minimum income of local store. S_{cha} is the commercial rule for the products of two nearest neighboring stores, which is defined by the management layer of the chain store firm. A positive S_{cha} represents the collective strategy that sell complementary products so that they cooperate with each other to maximize the total payoff. The collective strategy with a negative S_{cha} requires that two nearest neighboring stores sell the same product with the same quality, as a result, they compete with each other for constant customers by lowering price. The strategy of $S_{cha} = -1$ puts two stores in the Prisoner's dilemma of the noncooperative game. σ_0 is the reference price basis. f_{dem} indicate the supply quantity in market and f_{dem} represent the demand quantity of market. When the demand surpasses the supply, $f_{dem} - f_{sup} > 0$, a store raises its price. While on the contrary case, the store lowers its price. H_i defines the Hamiltonian function in statistical physics. Two nearest neighboring stores play a noncooperative game with the following payoff matrix,

		σ_2	σ_2
		$+1$	-1
σ_1	$+1$	$[P_1(\sigma_1 = +1),\ P_2(\sigma_2 = +1)]$	$[P_1(\sigma_1 = +1),\ P_2(\sigma_2 = -1)]$
σ_1	-1	$[P_1(\sigma_1 = -1),\ P_2(\sigma_2 = +1)]$	$[P_1(\sigma_1 = -1),\ P_2(\sigma_2 = -1)]$

(2)

For an exemplar case of parameter settings: $(P_0 = 0, S_{cha} = 1, \sigma_0 = 3, f_{dem} - f_{sup} = 1)$, the explicit payoff matrix reads

		σ_2	σ_2
		$+1$	-1
σ_1	$+1$	$[20,\ 20]$	$[10,\ 2]$
σ_1	-1	$[2,\ 10]$	$[6,\ 6]$

(3)

This example shows the payoff reaches a maximal value when two stores cooperate with each other. Since all local stores belong to the same chain store firm, the management layer of the chain store aims at maximizing the total payoff of

all local stores, which is the sum of local payoff Eq. (1)

$$
\begin{aligned}
P_{tot} \quad &:= \quad \sum_i P_i = \sum_i P_0 - \sum_i H_i = P_{tot,0} - H_{tot}. \\
&= \quad P_{tot,0} + S_{cha}\sum_{\langle ij \rangle}(\sigma_0 + \sigma_i)(\sigma_0 + \sigma_j) + (f_{dem} - f_{sup})\sum_i(\sigma_0 + \sigma_i). \quad (4)
\end{aligned}
$$

Maximizing the total payoff P_{tot} is equivalent to minimizing the energy function generated by the action of Hamiltonian H_{tot} on collective strategy state. In the case of constant customer, the chain store competes with the monopolistic manufacture for total payoff. The payoff gained by the chain store is the loss of the monopolistic manufacture. Therefore the total payoff is the output of a noncooperative game played by the strategy of chain store S_{cha} and the supply quantity $(f_{dem} - f_{sup})$ that is controlled by the monopolistic manufacture, here we assume that the demand quantity f_{dem} is constant for simplicity. Therefore besides the microscopic games between local stores, there is a macroscopic game played between the chain store firm and the manufacture. For an extreme case of balanced market supply and demand $(f_{dem} - f_{sup}) = 0$, the total payoff is utterly determined by the strategy of the chain store firm. The manufacture only takes the minimum payoff. On the contrary extreme case of an absent strategy of chain store firm $S_{cha} = 0$, the manufacture takes full control of the market. In a general case, the interactive decision-making process between the chain store firm and manufacture firm drives the game into a Nash equilibrium point. The corresponding payoff function of the chain store firm and the monopolistic manufacturer are

$$
\begin{aligned}
P_{cha}(S_{cha}) \quad &:= \quad P_{tot,0}/2 + S_{cha}\sum_{\langle ij \rangle}(\sigma_0 + \sigma_i)(\sigma_0 + \sigma_j), \\
P_{sup}(f_{man}) \quad &:= \quad P_{tot,0}/2 + f_{man}\sum_i(\sigma_0 + \sigma_i). \\
f_{man} \quad &= \quad f_{dem} - f_{sup}. \quad (5)
\end{aligned}
$$

The total payoff is the sum of the two payoff functions in Eq. (5),

$$
P_{tot} := P_{cha} + P_{sup}. \quad (6)
$$

$S_{cha} > 0$ and $S_{cha} < 0$ represent the cooperative strategy and non-cooperative strategy respectively defined by the management of the chain store. $f_{dem} - f_{sup} > 0$ and $f_{dem} - f_{sup} < 0$ represent the short supply and excess supply strategy respectively, which is controlled by the supply quantity of the monopolistic manufacture. The payoff matrix of this macro noncooperative game reads

		S_{cha}	f_{man}
		> 0	< 0
S_{cha}	> 0	$[P_{cha}(>0),\ P_{sup}(>0)]$	$[P_{cha}(>0),\ P_{sup}(<0)]$
f_{man}	< 0	$[P_{cha}(<0),\ P_{sup}(>0)]$	$[P_{cha}(<0),\ P_{sup}(<0)]$

$$(7)$$

For a general strategy vector, (S_{cha}, f_{man}) in which $S_{cha} \in \mathbf{S}_{cha}$, $f_{man} \in \mathbf{S}_{man}$ with $\mathbf{S}_{cha}$ and $\mathbf{S}_{man}$ the strategy space of the chain store firm and manufacture firm respectively. A decision rule for a strategy S_{cha} of the chain store is an operator $\hat{f}_A : f_{man} \Rightarrow S_{cha}$. When the chain store knows that manufacture

plays a strategy f_{man}, it plays the strategy $S_{cha} \in \hat{f}_A f_{man}$. The decision rule for the manufacture is also a map $\hat{f}_B$ from $\mathbf{S}_{cha}$ to $\mathbf{S}_{man}$. When a pair of strategies (S_{cha}, f_{man}) satisfies

$$\bar{S}_{cha} \in \hat{f}_A \bar{f}_{man}, \quad \bar{f}_{man} \in \hat{f}_B \bar{S}_{cha}, \tag{8}$$

they form a pair of consistent pair of strategies, which is exactly the fixed-point of mapping rules of this game, $(\hat{f}_A, \hat{f}_B)$. The matrix representation of this game rules reads

$$\begin{pmatrix} \bar{S}_{cha} \\ \bar{f}_{man} \end{pmatrix} = \begin{pmatrix} 0 & \hat{f}_A \\ \hat{f}_B & 0 \end{pmatrix} \begin{pmatrix} \bar{S}_{cha} \\ \bar{f}_{man} \end{pmatrix}. \tag{9}$$

Here we denote the decision rule matrix as

$$\hat{U} = \begin{pmatrix} 0 & \hat{f}_A \\ \hat{f}_B & 0 \end{pmatrix} \tag{10}$$

For an initial strategy vector, $\vec{S} = (S_{cha}, f_{man})^T$, repeated action of the decision matrix after n rounds of action would finally drive the game to a fixed point,

$$\vec{S} = \hat{U}^n \vec{S}, \tag{11}$$

The existence of fixed point was proved by optima and equilibria theory [20]. For a practical case in economic system, the big challenge is to find out the decision rule matrix. The real economic system is usually too complex to admit such a simple operation matrix. However when two competing firms reach a critical state of a zero-sum game. A firm excites up all of its subbranches in global region in order to win until its rival firm bankrupts. Only one firm survives in the end. This dynamic game process is mathematically equivalent to the renormalization group transformation approach to the critical point of phase transition in physics [21]. Therefore the physics theory provides a natural way of deriving decision rule matrix for games.

2.2 The decision rule matrix derived from statistical physics theory

The total payoff function in game theory has an exact one-to-one correspondence with the Hamiltonian function in statistical physics. The physics theory of minimizing potential energy is mathematically equivalent to maximizing the total payoff in economics system. With redefined parameters, the total payoff Eq. (4) is expressed into the same form as Ising model without any approximation and assumption,

$$\begin{aligned} P_{tot} &= \gamma_1 \sum_{\langle ij \rangle}^{N} \sigma_i \sigma_j + \gamma_2 \sum_i^{N} \sigma_i + \gamma_3, \\ \gamma_3 &= P_{tot,0} + S_{cha} \sigma_0^2 + (f_{dem} - f_{sup}) \sigma_0, \\ \gamma_2 &= 2 S_{cha} + (f_{dem} - f_{sup}), \\ \gamma_1 &= S_{cha}. \end{aligned} \tag{12}$$

where $\langle ij \rangle$ indicates the nearest neighbors. The conventional Hamiltonian of Ising model is reexpressed by the total payoff Eq. (4),

$$H_{tot} = P_{tot,0} - P_{tot}, \tag{13}$$

6

Substituting the Ising-type payoff Eq. (12) into the Hamiltonian Eq. (13) leads to an effective Hamiltonian with a new parameter $\bar{\gamma}_3$,

$$
\begin{aligned}
H_{tot} &= -\gamma_1 \sum_{<ij>}^{N} \sigma_i \sigma_j - \gamma_2 \sum_{i}^{N} \sigma_i - \bar{\gamma}_3, \\
\bar{\gamma}_3 &= -S_{cha}\sigma_0^2 - (f_{dem} - f_{sup})\sigma_0,
\end{aligned}
\tag{14}
$$

where γ_1 and γ_2 are the same as that in Eq. (12). The solution of Ising model holds exactly for the total payoff Eq. (1). Each total payoff value corresponds to an energy value derived from the total Hamiltonian H_{tot}.

Fig. (1) shows a homogeneous distribution of the local retailers that belong to the same chain store firm, every store locates at a vertex of a triangular lattice. The collective strategy distribution of these local retailers can be expressed by a sequence of $+1$ or -1 with the position index of the ± 1 runs over every lattice sites of the triangular lattice,

$$
|\psi\rangle = |+1-1-1\cdots-1_{ij}\cdots-1+1\rangle,
\tag{15}
$$

where ij is a summary of the position vector $\vec{r} = i\mathbf{e}_a + j\mathbf{e}_b$ with $\mathbf{e}_a$ and $\mathbf{e}_b$ the basis vector of the triangular lattice. When the nearest neighboring stores sell two different products, A and B. The chain store firm expects to promote the sale of product A, the management layer would command the store that sell product B to raise the price and command the store that sell product A to lower the price. In that case, the collective strategy distribution is an alternative distribution of $+1$ and -1, which is called anti-ferromagnetic phase in magnetism theory,

$$
|\psi\rangle_{af} = |+1-1+1-1\cdots+1_{ij}-1_{ij+1}\cdots+1-1\rangle.
\tag{16}
$$

The eigenemergy of this ferromagnetic state is

$$
E_{af} = {}_{af}\langle\psi|H_{tot}|\psi\rangle_{af} = +NS_{cha} + S_{cha}\sigma_0^2 + (f_{dem} - f_{sup})\sigma_0.
$$

The corresponding total payoff reads,

$$
P_{af} = P_{tot,0} - NS_{cha} - S_{cha}\sigma_0^2 - (f_{dem} - f_{sup})\sigma_0.
$$

When the nearest neighboring stores sell complementary products, A and B, all stores choose to raise the price to maximize its own profit. The collective strategy distribution of this homogeneous state is called ferromagnetic state in physics,

$$
|\psi\rangle_f = |+1+1+1+1\cdots+1_{ij}+1_{ij+1}\cdots+1+1\rangle.
\tag{17}
$$

The eigenemergy of this ferromagnetic state is

$$
\begin{aligned}
E_f &= {}_f\langle\psi|H_{tot}|\psi\rangle_f \\
&= -NS_{cha} - N2S_{cha} - N(f_{dem} - f_{sup}) + S_{cha}\sigma_0^2 + (f_{dem} - f_{sup})\sigma_0.
\end{aligned}
\tag{18}
$$

The corresponding total payoff reads,

$$
P_f = P_{tot,0} + NS_{cha} + N2S_{cha} + N(f_{dem} - f_{sup}) - S_{cha}\sigma_0^2 - (f_{dem} - f_{sup})\sigma_0. \tag{19}
$$

Comparing the total payoff Eq. (19) and Eq. (17) suggests that the total payoff of the antiferromagnetic strategy is lower than the total payoff of ferromagnetic strategy. The promotion strategy in the antiferromagnetic case caused a decline of profit.

A general collective strategy of the chain store is far more complicate than the antiferromagnetic or ferromagnetic strategy. The energy function with respect to a general collective strategy with stochastic spatial distribution of local strategies

$$|\psi\rangle = |+1-1+1+1\cdots-1_{ij}-1_{ij+1}\cdots-1+1\rangle, \qquad (20)$$

is given by $E = \langle\psi|H_{tot}|\psi\rangle$, which leads to the total payoff equation $P_{tot} = P_{tot,0} - E$. Every total payoff value has a certain probability to last for a period in market due to the activity levels of economics. In a hot economic system with high activity level, every store makes frequent transactions and high average profit. In a cold economic environment, both the transaction rate and average profit are kept at a low value.

Definition, *The temperature T in the minimal chain supply model is proportional to the activity level of economics, which is proportional to the transaction rate of local retailer, or the amount of business transactions within unit time period. The temperature also measures the amount of customers the store attracts within an unit time period.*

Proposition *In an economic system, if the total payoff is defined by $P_{tot} = P_{tot,0} - H_{tot}$, maximizing the total payoff P_{tot} is equivalent to minimizing the eigenenergy E_{tot} with respect to the Hamiltonian $H_{tot} = P_{tot,0} - P_{tot}$.*

When a retailer changes his strategy to maximize his payoff, he either loses or wins the game, caused a dynamic transfer of payoff between different retailers. When this transferring dynamics reaches an equilibrium state, the probability of finding a collective strategy pattern $|\psi\rangle$ with eigenenergy E_{tot} obeys Boltzmann distribution, $P_\psi = \exp[-\beta E_{tot}]$, where $\beta = 1/T$ is the inverse of temperature. The probability of a high energy state decreases exponentially with the eigenenergy. The equivalent probability distribution with respect to the total payoff P_{tot} reads $P_\psi = \exp[-\beta(P_{tot,0} - P_{tot})]$. Here $P_{tot,0}$ is a constant. The probability of a certain payoff increases exponentially with the total payoff,

$$P_\psi = \exp[-\beta P_{tot,0}]\exp[\beta P_{tot}]. \qquad (21)$$

The probability of the reference total payoff $P_{tot,0}$ decreases exponentially, the P_{tot} counts the amount of payoff increased or decreased curing the non-cooperative games. The sum of the probability function of all possible payoff values $P_{tot,n}$ defines the partition function in economics,

$$Z = e^{-\beta P_{tot,0}} \sum_n e^{\beta P_{tot,n}} = \sum_n e^{-\beta E_{tot,n}}. \qquad (22)$$

The relative probability for a ceratin payoff value $P_{tot,n}$ is defined by

$$P_n = e^{-\beta P_{tot,0}} \frac{e^{\beta P_{tot,n}}}{Z}. \qquad (23)$$

When the local retailers competes to reach the maximal payoff state, the energy functions reaches its minimum value. Therefore, the dynamic rules for a physical

system approaching the minimum energy state is the inverse decision rule matrix for the non-cooperative game in this minimum supply chain model.

Proposition *The Nash equilibrium point of a non-cooperative game between two rivalry firms is a critical point of phase transition between collective ordered states.*

A phase transition describes the sudden change of collective states in many particle system. In this minimum supply chain model, the collective phase indicates a regular collective pattern of local strategies. When the collective strategy distribution of many local retailers transforms from one pattern to another one, it fulfills the phase transition phenomena. The correlation length at phase transition point reaches infinity. The strategy of a local retailer has a strong influence on any other local retailer that is locates at an arbitrary distance away. If one retailer on one boundary of a finite triangular lattice changes his strategy, another retailer on the opposite boundary would change his strategy simultaneously.

Here we apply the renormalization group transformation theory in statistical mechanics on this minimum supply chain model to derive the decision rule matrix for the noncooperative game. The effective Hamiltonian Eq. (14) describes the total energy of many retailers located periodically on a triangular lattice. The retailers are independent sellers and choose to raise price or lower price at their own will before the whole lattice of retailers arriving at the critical point. As the noncooperative game between the chain store firm and manufacture is performed, the three nearest neighboring retailers on the minimum triangle united to form coalition (as showed in Fig. 1 (b)), they choose the same strategy according the majority rule. If more the two of the three retailers choose the same strategy, the third retailer must play the same strategy no matter he agrees or not. The collective strategy of the triangular coalition is denoted as the block spin in Fig. 1 (b). The block spin is $\sigma_I^{'} = +1$, if $(\sigma_1, \sigma_2, \sigma_3) = (+1, +1, -1); (+1, -1, +1); (-1, +1, +1); (+1, +1, +1)$. The block spin is $\sigma_I^{'} = -1$, if $(\sigma_1, \sigma_2, \sigma_3) = (+1, -1, -1); (-1, +1, -1); (-1, -1, +1); (-1, -1, -1)$. This coalition formation implemented the Kadanoff block process [22] in renormalization group transformation theory [25]. Since triangular coalition has the same strategy set as that of single retailer, the partition function of these triangular coalitions shares the same form as that of many retailers if the total number of retailers grows up to infinity,

$$Z \;=\; \sum_{|\sigma_i|} e^{-\beta H_{tot,i}(\gamma_1, \gamma_2, \gamma_3, \sigma_i)} = \sum_{|\sigma_I|} e^{-\beta H'_{tot,I}(\gamma'_1, \gamma'_2, \gamma'_3, \sigma_I)}. \qquad (24)$$

where $|\sigma_i|$ represents all possibilities of the collective strategy configuration of retailers. $|\sigma_I|$ represents the collective strategy configuration of triangular coalitions. This scale invariance of partition function is realized in a straightforward way of keeping the Hamiltonian of triangular coalition groups in the same form as that of original retailers. Since the triangular coalition groups are also located on a triangular lattice, three triangular coalition groups could unit to form a bigger coalition that enclosed nine original retailers. This grouping process continuous until the monopolistic coalition forms, which decimates the degree of freedom of thousands of retailers into fewer degree of freedom of one monopoly. As a result, the effective Hamiltonian Eq. (14) transform into the same formu-

lation with redefined parameters as following,

$$H'_{tot} = -\gamma'_1 \sum_{<IJ>}^{N'/2} \sigma'_I \sigma'_J - \gamma'_2 \sum_{I}^{N'} \sigma'_I - \bar{\gamma}'_3. \tag{25}$$

The new parameters in the Hamiltonian equation is a function of the original parameters of retailers, which have different formulations with respect to different regroup process and different approximations of computing partition function. For the partition function of triangular block spins on triangular lattice with Taylor expansion of exponential function up to the second order, the transformations between the new parameter and old parameters can be derived from statistical physics theory [23],

$$\gamma'_1 = 2\gamma_1 \left(\frac{e^{3\gamma_1} + e^{-\gamma_1}}{e^{3\gamma_1} + 3e^{-\gamma_1}}\right)^2, \quad \gamma'_2 = 3\gamma_2 \left(\frac{e^{3\gamma_1} + e^{-\gamma_1}}{e^{3\gamma_1} + 3e^{-\gamma_1}}\right). \tag{26}$$

This special transformation relation connotate a semigroup transformation. This transformation matrix defines the game rule between the renormalized strategy of the chain store firm and the strategy of the monopolistic manufacture. When the chain store firm and manufacture modifies their strategy values, the strategy flows in the new parameter spaces are attracted or expelled out of some fixed points. A non-trivial fixed point $\vec{\gamma}^*$ is invariant under this renormalization group transformation,

$$\vec{\gamma}^* = \hat{U}\vec{\gamma}^*. \tag{27}$$

Only when the fixed point $\vec{\gamma}^* = (\gamma^*_1, \gamma^*_2)^T$ is a saddle point, it corresponds to a Nash equilibrium point in game theory. Both the theoretical and experimental research on the phase transition phenomena suggest that phase transition only occurs at a saddle point [23]. Near the saddle point, renormalization group transformation theory shares the mathematical structure with non-cooperative game theory, and admits a game theory explanation. Whenever the chain store firm and manufacture modifies their strategy values according to the renormalization group matrix $\hat{U}$, the strategy pair in the parameter space approaches to the fixed point. In the strategy space of a non-cooperative game, the strategy surface is a saddle surface with its saddle point as an unstable fixed point. When this game reaches a Nash equilibrium point, none of the two strategy parameters has further options to increase its payoff. The winner takes all profit at the saddle point and the loser wins nothing, this results in a sudden change of the total payoff, which indicates the occurrence of phase transition.

For the special renormalization group transformation Eq. (26), the saddle point is derived by equating the new parameters with the old parameters: ($\gamma_1 = \gamma'_1 = \gamma^*_1$, $\gamma_2 = \gamma'_2 = \gamma^*_2$). The solution of the saddle point is ($\gamma^*_1 = \frac{1}{4}\ln(1 + 2\sqrt{2})$, $\gamma^*_2 = 0$). In the minimum supply chain model with the payoff Eq. (12), this fixed point locates at the special strategy value of the chain store and the manufacture,

$$
\begin{aligned}
S_{cha} &= \frac{1}{4}\ln(1 + 2\sqrt{2}), \\
(f_{dem} - f_{sup}) &= -2S_{cha}.
\end{aligned}
\tag{28}
$$

Therefore the Nash equilibrium point of the non-cooperative game between the chain store and monopolistic manufacture is in a special case of short supply.

Not every fixed point is a saddle point. Certain fixed point are source point or sink point. The saddle point can be identified explicitly by the flow vector field in its vicinity. In order to show the flow vector field around the fixed point, we perform Taylor expansion around $\vec{\gamma}^* = (\gamma_1^*, \gamma_2^*)^T$ and make a truncation up to the first order, the vector field in the vicinity of fixed point evolves under the renormalization group flow map operator U_f,

$$\left[\begin{array}{c} \gamma_1' - \gamma_1^* \\ \gamma_2' - \gamma_2^* \end{array} \right] = \left[\begin{array}{cc} \frac{\partial \gamma_1'}{\partial \gamma_1} & \frac{\partial \gamma_1'}{\partial \gamma_2} \\ \frac{\partial \gamma_2'}{\partial \gamma_1} & \frac{\partial \gamma_2'}{\partial \gamma_2} \end{array} \right]^* \left[\begin{array}{c} \gamma_1 - \gamma_1^* \\ \gamma_2 - \gamma_2^* \end{array} \right]. \tag{29}$$

Here the strategy variation vectors is denoted as $\delta\gamma' = (\gamma_1' - \gamma_1^*, \gamma_2' - \gamma_2^*)^T$ and $\delta\gamma = (\gamma_1 - \gamma_1^*, \gamma_2 - \gamma_2^*)^T$. The transformation matrix reads

$$U_f = \left[\begin{array}{cc} \frac{\partial \gamma_1'}{\partial \gamma_1} & \frac{\partial \gamma_1'}{\partial \gamma_2} \\ \frac{\partial \gamma_2'}{\partial \gamma_1} & \frac{\partial \gamma_2'}{\partial \gamma_2} \end{array} \right]^*, \tag{30}$$

When higher order of Taylor expansion is taken into account, the renormalization group flow operator has more complex forms. In the most general case, the renormalization group transformation reads

$$\delta\gamma' = \hat{U}_f \delta\gamma. \tag{31}$$

The vector flow field around a saddle point always has an in-flow along the curve of one parameter and an out-flow along the curve of the other parameter. According to the phase transition theory of statistical mechanics, a collective observable obeys power law at the critical point of phase transition. The index of these power law can be derived by the eigenvalue of the flow operator $\hat{U}_f$. The collective observable in this monimum supply chain model can be well-defined in a similar way of statistical mechanics.

3   The sudden change of macroscopic observable in economics as a signal of phase transition

In a highly active market, all retailers of a chain store firm trade frequently with many customers. Here the activity of the market is quantified by a parameter called temperature T. In a highly active market, the local retailer choose his strategy at his own will to either raise price or lower the price. When we make a strategy map of all retailers in a local region to label the strategy of each retailer, usually it shows a disorder distribution of $+1$ and -1. However, when the activity level of the market drops to a low value, local retailers began to form small local coalitions to maximize their profit. When the activity level of the market drops below a critical point, all retailers suddenly choose the same strategy synchronously. This critical point is a phase transition point and Nash equilibrium point. The collective observable always show some sudden change at the critical point.

The free payoff is a macroscopic observable in this minimum supply chain model. In the collective payoff function Eq. (4), $P_{tot} = P_{tot,0} - H_{tot}$, the reference total payoff $P_{tot,0}$ is a fixed value which does not join in the currency

flow of trade market. Here we define a free payoff function F_{pay} to quantify the variation of payoff in the game,

$$F_{pay} = -T \, ln \, Z; \quad Z = e^{-\beta P_{tot,0}} \sum_n e^{\beta P_{tot,n}}. \tag{32}$$

Here the total payoff $P_{tot,n}$ the value with respect to the nth strategy distribution of local retailers in a local region.

The average price of a product in many local retailers is another macroscopic observable in this minimum supply chain model. The price of a product is different from one retailer to another one, since a retailer makes an independent decision at his own will. Suppose there are N retailers in total, the average price is defined as

$$P_{pri} = \frac{1}{N} \sum_{i=1}^{N} (\sigma_0 + \sigma_i) = \sigma_0 + \frac{1}{N} \sum_{i=1}^{N} \sigma_i. \tag{33}$$

This average price can be derived from partition function by

$$P_{pri} = T \frac{\partial \ln Z}{\partial \gamma_2}. \tag{34}$$

If the price fluctuation is $\sigma_i = \pm 1$, the second terms on the right hand side of Eq. (33) is called magnetization in the two dimensional Ising model of spins [23]. According to the exact solution of two dimensional Ising [24], if the retailers in this minimum supply chain model are located on a two dimensional square lattice, the average price P_{pri} shows a sudden change at a critical temperature,

$$
\begin{aligned}
P_{pri} &= \sigma_0 + \frac{(1 + e^{-4\beta S_{cha}})^{1/4}(1 - 6e^{-4\beta S_{cha}} + e^{-8\beta S_{cha}})^{1/8}}{\sqrt{1 - e^{-4\beta S_{cha}}}}, \quad T < T_c. \\
P_{pri} &= \sigma_0. \qquad T > T_c.
\end{aligned}
\tag{35}
$$

where $\beta = 1/T$ is the inverse of temperature. S_{cha} is the collective strategy value of the chain store firm. The critical temperature is $T_c = 2S_{cha}/\ln(1 + \sqrt{2})$.

In analogy of the magnetic susceptibility in magnetic system, we define a price susceptibility χ_{pri} to measure the price response to the combined strategy of chain store firm and monopolistic manufacturer $\gamma_2 = 2S_{cha} + (f_{dem} - f_{sup})$,

$$\chi_{pri} = \frac{1}{N} \frac{\partial P_{pri}}{\partial \gamma_2} = \frac{T}{N} \frac{\partial^2 \ln Z}{\partial \gamma_2^2}. \tag{36}$$

According to fluctuation-dissipation theorem, this price susceptibility is proportional to the difference between the average of price square and the square of average price,

$$
\begin{aligned}
\chi_{pri} &= \frac{1}{T}[\langle \bar{P}_{pri}^2 \rangle - \langle \bar{P}_{pri} \rangle^2] \\
&= \frac{1}{NT}[\langle \sigma_i \sigma_j \rangle - \langle \sigma_i \rangle \langle \sigma_j \rangle],
\end{aligned}
\tag{37}
$$

where $\langle \bar{P}_{pri} \rangle = \langle \sum_{i=1}^{N} \sigma_i \rangle$ and $\langle \bar{P}_{pri}^2 \rangle = \langle (\sum_{i=1}^{N} \sigma_i)^2 \rangle$. The price susceptibility is proportional to the correlation function between two retailer locates at the ith

site and the jth site, $C_{ij} = \langle \sigma_i \sigma_j \rangle - \langle \sigma_i \rangle \langle \sigma_j \rangle$. Below the critical temperature T_c, the correlation length approaches to infinity, which lead to a divergent price susceptibility, $\chi_{pri} = \infty$. The price susceptibility function obeys a scaling law in the vicinity of critical point,

$$\chi_{pri} \propto \left| \frac{T - \frac{2S_{cha}}{\ln(1+\sqrt{2})}}{\frac{2S_{cha}}{\ln(1+\sqrt{2})}} \right|^{-\gamma}, \quad \gamma = \frac{7}{4}. \tag{38}$$

Other similar macroscopic observable in statistical mechanics can also be introduced into this minimum supply chain model. Here we define the specific heat C_p of free payoff as

$$C_p = -T \frac{\partial^2 F_{pay}}{dT^2}. \tag{39}$$

The second order derivative of free payoff with respect to the strategy parameter γ_2 and temperature also defines a macroscopic observable

$$\alpha = \frac{\partial^2 F}{\partial T \partial \gamma_2}, \tag{40}$$

which is similar to the compressibility in thermodynamics. All of these macroscopic observable obey a scaling law near critical point, for example,

$$P_{pri} \propto - \left| \frac{T - \frac{2S_{cha}}{\ln(1+\sqrt{2})}}{\frac{2S_{cha}}{\ln(1+\sqrt{2})}} \right|^{-1/8}, \quad C_P \propto \left| \frac{T - \frac{2S_{cha}}{\ln(1+\sqrt{2})}}{\frac{2S_{cha}}{\ln(1+\sqrt{2})}} \right|^{0.0127}. \tag{41}$$

The specific heat C_p of free payoff shows a logarithmic divergence. These scaling law is derived based on the exact solutions of two dimensional Ising model [24]. The indexes of these scaling laws obey certain universal scaling relations. We developed the phase coexistence to derive the scaling relations in the next section.

Another useful macroscopic variable for this minimum supply chain model is pressure. In analogy of pressure in thermodynamic physics, we introduce an internal state function **x** into the payoff function to characterized the average distance between two nearest neighboring retailers. When the density of retailers in a local region grows, the distance between neighboring retailers decreases. If the total number of customer is constant, these retailers encounter stronger competition and pressure to keep the same level of profit. Their we define a macroscopic factor call ed pressure,

Definition *The pressure on local retailers P is inversely proportional to the distance between two nearest neighboring retailers and the number of customers within an unit area.*

Increasing pressure P reduces the zone of influenced customers. The retailers are pushed into narrower region to compete for finite customers. On the other side, when the activity level of market increases, the zone of influence customers is expanded to wider region to gain more profit. Therefore the temperature and pressure are two competing factors in economics. When the pressure controller P dominates the market, the market falls into an ice phase. When the activity of market dominates the trade, the retailers are not confined in a fixed site, instead they search new locations to expand their business.

3.1 The characterization of the discontinuity of macroscopic observable

In thermodynamics physic, Ehrenfest use the discontinuity of free energy for two phases to characterize the order of phase transition [26]. Here we use the discontinuity of free payoff function to define the phase transition in this minimum supply chain model. For the zeroth order phase transition, the free payoff function of two phases F_{pay}^A and F_{pay}^B is not continuous, $F_{pay}^A \neq F_{pay}^B$. For the first order phase transition, the free payoff function of two phases is continuous $F_{pay}^A = F_{pay}^B$, with a discontinuous first order derivative,

$$\frac{dF_{pay}^A}{dT} \neq \frac{dF_{pay}^B}{dT}, \quad \frac{dF_{pay}^A}{dP} \neq \frac{dF_{pay}^B}{dP}. \tag{42}$$

The second order phase transition is characterized by the discontinuity of macroscopic observable that is defined by the second order derivative of free payoff function, such as the specific heat C_p Eq. (39) and the analogue compressibility α Eq. (40). Higher order of phase transition is defined by the discontinuity of the higher order of the derivative of the free payoff function. Ehrenfest's definition provided a straightforward way to classify different order of phase transitions. However the inequality equations are not sufficient enough to reveal the detail geometry of free payoff function. Here we developed a Lie group theory definition to classify the phase transition.

The payoff matrix of prisoner dilemma shows the payoff of two rivalry players always reach the same value at Nash equilibrium. If one player gets more profit than the other players, the game will continuous until they gain equal profit. At the equilibrium point, exchange the role of the two players will not change the outcome payoffs. This invariance indicates a local symmetry in the vicinity of saddle point. We apply the Riemanian geometry theory to the free payoff manifold, and introduce two basis of the tangent vector space in the vicinity of saddle point,

$$\mathbf{e}_1 = \frac{\partial}{\partial \gamma_1}, \quad \mathbf{e}_2 = \frac{\partial}{\partial \gamma_2}. \tag{43}$$

The metric tensor is given in a bilinear form by the inner product of two basis vectors, $g_{ij} = < e_i, e_j >$. Since the two strategy parameters compete at critical point to construct a non-cooperative game, the two strategies plays opposite action at the critical point. The generator of $SO(2)$ group naturally embed the antisymmetric character of the two competitors, $\hat{L}_{SO(2)}(\gamma_1, \gamma_2) = -\hat{L}_{SO(2)}(\gamma_2, \gamma_1)$. Thus we expand the tangential vector field on the free energy surface by the generator of $SO(2)$ group,

$$\hat{L}_{SO(2)} = \gamma_2 \frac{\partial}{\partial \gamma_1} - \gamma_1 \frac{\partial}{\partial \gamma_2}. \tag{44}$$

Then the exponential operator becomes the group element of the $SO(2)$ symmetry group according to the Lie-group exponential map, $Lie : \hat{L}_{SO(2)} \Rightarrow Exp(\hat{L}_{SO(2)})$, which is a smooth map of the Lie algebra to the Lie group. As $p \to \infty$, it reaches the full $SO(2)$ group element $U(\theta)$. Thus a more complete definition for the pth

14

order phase transition is proposed as following,

$$
\begin{aligned}
U^{(p)} &= \sum_0^p \frac{1}{p!}(\hat{L}_{SO(2)}\theta)^p, \\
U^{(p-1)}F_{pay}^A &= U^{(p-1)}F_{pay}^B, \\
(U^{(p)} - U^{(p-1)})F_{pay}^A &\neq (U^{(p)} - U^{(p-1)})F_{pay}^B.
\end{aligned}
\tag{45}
$$

The discontinuity of free payoff function is fully encoded in this definition, which spontaneously encoded the competition relation between the two parameters. This definition equation unifies Ehrenfest's definition for different orders of phase transition. For example, when $n = 0$, $U^{(0)} = I$, it yields $F_{pay}^A = F_{pay}^B$. For $n = 1$, $U^{(1)} = I + \theta\hat{L}_{SO(2)}$, Eq. (45) reads,

$$
\begin{aligned}
(I + \theta\hat{L}_{SO(2)})F_{pay}^A &= (I + \theta\hat{L}_{SO(2)}F_{pay}^B, \\
\Rightarrow \quad F_{pay}^A = F_{pay}^B, \quad &\hat{L}_{SO(2)}F_{pay}^A \neq \hat{L}_{SO(2)}F_{pay}^B. \\
\gamma_2\frac{\partial F_{pay}^A}{\partial \gamma_1} - \gamma_1\frac{\partial F_{pay}^A}{\partial \gamma_2} &\neq \gamma_2\frac{\partial F_{pay}^B}{\partial \gamma_1} - \gamma_1\frac{\partial F_{pay}^B}{\partial \gamma_2}.
\end{aligned}
\tag{46}
$$

When $p \to (p-1) \to \infty$, phase transition can not be detected by the discontinuous of free payoff function and its derivative up to any order.

3.2 The phase coexistence equation for identifying Nash equilibrium point

The non-cooperative game between the chain store firm and monopolistic manufacture can be described by the differential game theory [27]. The strategy space of chain store firm and manufacture are listed as a consequence of strategy pairs,

$$
S_1 = \{\gamma_1(t_1), \gamma_1(t_2), ..., \gamma_1(t_n)\}, \quad S_2 = \{\gamma_2(t_1), \gamma_2(t_2), ..., \gamma_2(t_n)\}.
\tag{47}
$$

When the two players play the strategy pairs above alternatively, they finally reach a fixed point (γ_1^*, γ_2^*), at which the free payoff encounters a saddle point,

$$
F_{pay}[\mathbf{x}, \gamma_1^*, \gamma_2] \leq F_{pay}[\mathbf{x}, \gamma_1^*, \gamma_2^*] \leq F_{pay}[\mathbf{x}, \gamma_1, \gamma_2^*].
\tag{48}
$$

Where $\mathbf{x}$ is the internal state function that describes internal degree of freedom of the strategy. The fixed point of the strategy pair is the optimal play for the two players as well as the optimal outcome at $\mathbf{x}$. There is a theorem states that if the value function of a differential game exists it is unique [27]. Therefore the Nash equilibrium point in this differential game is an unique saddle point, at which neither of two players can improve their guaranteed results.

The zeroth order phase transition is demonstrated by the sudden change of collective payoff. If the collective payoff is continuous but the growing speed of collective payoff shows a sudden change, this indicates the first order transition. So does the second order, the third order phase transition, and so on. Whenever a phase transition occurs, certain competing factors in market must have reached a Nash equilibrium. In order to find out the hidden competing powers that govern the marketing behavior, it is a key challenge to identify the Nash equilibrium point.

Since Nash equilibrium point is a saddle point, which is the maximal point of free payoff along the coordination curve of γ_1 and the minimal point along the coordination curve of γ_2. The saddle point Eq. (48) is equivalent to the equation of the first order derivative of the collective payoff

$$\frac{dF_{pay}}{d\gamma_1}\frac{dF_{pay}}{d\gamma_2} < 0. \tag{49}$$

The two competitors must share the same growing peed of payoff in order to reach an equilibrium, therefore the first order phase coexistence equation at the critical point reads

$$\frac{dF_{pay}}{d\gamma_1} + \frac{dF_{pay}}{d\gamma_2} = 0. \tag{50}$$

This phase coexistence equation only exist at the Nash equilibrium point. Because Nash equilibrium point is saddle point, the tangent vector field around the saddle point carries topological charge. In topology theory. the sum of all topological charges around the singular points is a topological invariant.

To find out the phase coexistence equation for higher order derivative of free payoff function, we first build a general tangent vector by the pth order derivative of the free payoff surface with respect to the two strategy values,

$$\phi^1 = \partial_T^{p-1}F_{pay}, \quad \phi^2 = \partial_P^{p-1}F_{pay}, \tag{51}$$

then map the tangent vector to an unit vector field $\vec{n}$,

$$n^a = \phi^a/||\phi||, \ \ n^a n^a = 1, \tag{52}$$

with $||\phi|| = \sqrt{\phi^1\phi^1 + \phi^2\phi^2}$. Substituting this unit vector field into Duan's topological current expression of Gaussian curvature on two dimensional manifold [28],

$$\Omega = \sum_{i,j,a,b=1}^{2} \epsilon^{ij}\epsilon_{ab}\frac{\partial n^a}{\partial\gamma_i}\frac{\partial n^b}{\partial\gamma_j}. \tag{53}$$

leads to a topological invariant of the free payoff manifold under the integration of the Gaussian curvature $C_1 = \int_M \Omega$, where C_1 is the first Chern number. (γ_1, γ_2) is explained as the strategy of the chain store firm and manufacturer. The first Chern number is also called Euler characteristic number on a compact Riemannian manifold. Duan's $\phi-$mapping topological current proved [28]

$$\Omega = \delta^2(\vec{\phi})D(\frac{\phi}{\gamma}) = \delta^2(\vec{\phi})\{\phi^1, \phi^2\}, \tag{54}$$

where $D(\phi/\gamma) = \frac{1}{2}\sum_{i,j,a,b=1}^{2}\epsilon^{jk}\epsilon_{ab}\partial_j\phi^a\partial_k\phi^b$ is the Jacobian vector, which is equivalent to the Poisson bracket of ϕ^1 and ϕ^2, $\{\phi^1, \phi^2\} = \sum_i(\frac{\partial\phi^1}{\partial\gamma_i}\frac{\partial\phi^2}{\partial\gamma_j} - \frac{\partial\phi^1}{\partial\gamma_i}\frac{\partial\phi^2}{\partial\gamma_j})$. The zero points of $\vec{\phi}$ are the singular points of the unit vector field $\vec{n}$, which maps the tangent vector $\vec{\phi}$ into an unit sphere. We denote the m isolated solutions of the equations $\vec{\phi} = 0$ as $\vec{\gamma}_k = (\gamma_k^1, \gamma_k^2), \ (k = 1, 2, \ldots m)$. The non-zero parts of Guassian curvature locates exactly at these singular points [28], $\Omega = \sum_{k=1}^{m} W_k\delta(\gamma - \gamma_k^1)\delta(\gamma - \gamma_k^2)$, where W_k is the winding number around the kth singular point. The sum of these local winding numbers is the first Chern

number $Ch = \int \Omega d^2\gamma = \sum_{k=1}^{l} W_k$. The stable singular points are located at the vanishing point of tangent vector field $\phi = 0$ under regular condition $D(\phi/\gamma) \neq 0$, which comes from the implicit function theorem [30]. When the regular condition is violated at certain point, i.e., $D(\phi/\gamma) = 0$, a definite solution of equation $\phi = 0$ is not available. This point corresponds to the unstable singular point, at which two topological current branch to different directions [28][29].

The free payoff reaches the maximal point along the coordination curve of γ_1, but reaches the minimal point in the direction of γ_2. The derivative of free payoff function with respect to γ_1 must carry an opposite sign as that of γ_2. This leads to the coexistence equation for different phases at the saddle point of Nash equilibrium. We apply Duan's topological theorem to describe the vector flow field around this saddle point of the free payoff function, which is explicitly constructed by renormalization group transformation theory. The mathematical constrain equation for the occurrence of branching dynamics $D(\phi/\gamma) = 0$ is essentially an universal phase coexistence equation. We take the two-phase coexistence equation

$$D(\phi/\gamma) = \{\phi^1, \phi^2\} = 0 \qquad (55)$$

as an example verify its effectiveness for thermodynamic physics theory. The base manifold is the variation of free energy $\delta F = F^A - F^B$, the two game players are temperature $\gamma_1 = T$ and pressure $\gamma_2 = P$. The universal phase coexist equation $\{\phi^1, \phi^2\} = 0$ unified the coexistence equations for different orders of phase transitions.

For the first order phase transition, we chose the vector order parameter as the variation of free energy, $\phi = \delta F$. Then we further define the Jacobian vector of a scalar function $\phi = \delta F$ as

$$D(\frac{\phi}{q}) = (\frac{\partial F^B}{\partial T} - \frac{\partial F^A}{\partial T}) + (\frac{\partial F^B}{\partial P} - \frac{\partial F^A}{\partial P}) = 0. \qquad (56)$$

Combining the thermodynamic relation $\frac{\partial F}{\partial T} = -S$ and $\frac{\partial F}{\partial P} = V$ with the generalized Jacobian vector $D(\frac{\phi}{q}) = 0$, we found

$$\frac{dP}{dT} = \frac{(S^B - S^A)}{(V^B - V^A)}. \qquad (57)$$

This is the Clapeyron equation in thermodynamic theory. This equation agrees exactly with the saddle point Eq. (50). For the second order phase transition, the vectorial order parameter is $\phi^1 = \partial_T \delta F$ and $\phi^2 = \partial_P \delta F$. Substituting the vector order parameter into the Jacobian vector equation

$$\{\phi^1, \phi^2\} = D(\phi/q) = \frac{\partial \phi^1}{\partial T}\frac{\partial \phi^2}{\partial P} - \frac{\partial \phi^1}{\partial P}\frac{\partial \phi^2}{\partial T} = 0, \qquad (58)$$

and using the relations

$$\partial_T \partial_T \delta F = \frac{C_p^A - C_p^B}{T}, \quad \partial_P \partial_P \delta F = V(\kappa_T^A - \kappa_T^B),$$

$$\partial_P \partial_T \delta F = V(\alpha^B - \alpha^A) \qquad (59)$$

with the specific heat C_p, α, κ defined as

$$C_p = T(\frac{dS}{dT})_P = -T\frac{\partial^2 F}{dT^2},$$
$$\alpha = \frac{1}{V}(\frac{\partial V}{\partial T})_P = \frac{1}{V}\frac{\partial^2 F}{\partial T \partial P},$$
$$\kappa = -\frac{1}{V}(\frac{\partial V}{\partial P})_P = -\frac{1}{V}\frac{\partial^2 F}{\partial P^2}, \tag{60}$$

we finally derived

$$D(\phi/q) = \frac{V}{T}(C_p^B - C_p^A)(\kappa_T^B - \kappa_T^A) - (V\alpha^B - V\alpha^A)^2. \tag{61}$$

The phase coexistence equation Eq. (61) exact coincides with the Ehrenfest equations in thermodynamics physics,

$$\frac{dP}{dT} = \frac{\alpha^B - \alpha^A}{\kappa^B - \kappa^A}, \quad \frac{dP}{dT} = \frac{C_p^B - C_p^A}{TV(\alpha^B - \alpha^A)}. \tag{62}$$

The phase coexistence equation $D(\phi/q) = 0$ also hold for higher-order phase transition. For example, we consider a free energy function of temperature T and magnetic field B, then the Clausius-Clapeyron equation becomes $dB/dT = -\Delta S/\Delta M$. If the entropy and the magnetization are continuous across the phase boundary, the transition must be of higher order. For the pth order phase transition, the vector field is chosen as the $(p-1)$th derivative of F_{pay}, $\phi^1 = \partial_T^{p-1} F_{pay}$, $\phi^2 = \partial_B^{p-1} F_{pay}$. Substituting (ϕ^1, ϕ^2) into Eq. (58), we arrive

$$D(\phi/q) = \frac{\partial^p F_{pay}}{\partial T^p}\frac{\partial^p F_{pay}}{\partial B^p} - \frac{\partial \partial^{p-1} F_{pay}}{\partial B \partial T^{p-1}}\frac{\partial \partial^{p-1} F_{pay}}{\partial T \partial B^{p-1}} = 0. \tag{63}$$

Considering the heat capacity $\frac{\partial^2 F}{\partial T^2} = -\frac{C_B}{T}$ and the susceptibility $\frac{\partial^2 F}{\partial B^2} = \chi$, the phase coexistence equation $D(\phi/q) = 0$ reads

$$\left[\frac{dB}{dT}\right]^p = (-1)^p \frac{\Delta \partial^{p-2} C/\partial T^{p-2}}{T_c \Delta \partial^{p-2}\chi/\partial B^{p-2}}. \tag{64}$$

There are many layers of tangential space above the base manifold of free energy. An arbitrary tangential vector on the pth layer of tangential space, which is defined by the pth order derivative of free energy, can split it into the two independent direction, $\vec{\phi} = \phi^1 e^1 + \phi^2 e^2$. Then the coexistence curve equation is

$$\{\phi^1, \phi^2\} = \frac{\partial \phi^1}{\partial \gamma_1}\frac{\partial \phi^2}{\partial \gamma_2} - \frac{\partial \phi^1}{\partial \gamma_2}\frac{\partial \phi^2}{\partial \gamma_1} = 0. \tag{65}$$

The phase coexistence Eq. (64) is only special case of a series of coexistence equations for the pth order of phase transition, the complete coexistence curve equations are given by

$$\frac{\partial^i F_{pay}}{\partial T^i}\frac{\partial^{2p-i} F_{pay}}{\partial B^{2p-i}} - \frac{\partial^j \partial^{p-j} F_{pay}}{\partial B^j \partial T^{p-j}}\frac{\partial^k \partial^{p-k} F_{pay}}{\partial T^k \partial B^{p-k}} = 0,$$
$$(i, j, k = 1, 2, ..., p). \tag{66}$$

18

This universal coexistence equation revealed the non-trivial saddle point in higher order tangent space of free energy.

A further verification of this phase coexistence equation is realized by the scaling laws of macroscopic observable near critical point. Macroscopic observable obeys scaling laws with fractional index near critical point. These fractional index suggest a fractal space at the singular critical point. Usually the divergent physical quantity are defined by the second order derivative of the free energy. Such as the susceptibility $\chi = -\partial^2 F/\partial H^2$, H is magnetic field. The two phase coexistence equation $\{\phi^1, \phi^2\} = 0$ produce all of the scaling relations, such as the Fisher relation $\nu d = 2 - \alpha$, Widom relation $\hat{\gamma} = \beta(\delta - 1)$, Rushbrooke relation $\alpha + 2\beta + \hat{\gamma} = 2$, and so on. We take the Rushbrooke relation as example to verify the coexistence equation. The Gibbs free energy is $F = U - TS$, its differentiation is $dF = -SdT + VdP - MdH$. Experimental and numerical calculation found that three thermodynamic quantities obey the following scaling laws in the vicinity of critical point,

$$
\begin{aligned}
M &= -(\frac{\partial F}{\partial H}) \sim |T|^{\beta}, \ \ \chi = -\frac{1}{N}(\frac{\partial^2 F}{\partial H^2}) \sim |T|^{-\gamma}. \\
C_P &= -T(\frac{\partial^2 F}{\partial T^2})_{P,H} \sim |T|^{-\alpha}.
\end{aligned}
\tag{67}
$$

We choose the tangential vector field

$$
\phi^1 = (\frac{\partial F}{\partial H}), \ \ \ \ \phi^2 = (\frac{\partial F}{\partial T}).
\tag{68}
$$

and substitute the vector field into the phase coexistence equation $\{\phi^1, \phi^2\} = 0$, it yields

$$
\frac{\partial^2 F}{\partial T^2}\frac{\partial^2 F}{\partial H^2} - \frac{\partial^2 F}{\partial T \partial H}\frac{\partial^2 F}{\partial H \partial T} = 0.
\tag{69}
$$

Now we substitute the thermodynamic quantities Eq. (67) into the phase coexistence equation, it leads to

$$
|T|^{-\alpha-\gamma} = \beta^2 |T|^{2\beta-2}.
\tag{70}
$$

Comparing the index of temperature at the left hand side and the right hand side of Eq. (70), it shows the Rushbrooke relation $\alpha + 2\beta + \hat{\gamma} = 2$. Other scaling relation can be verified following similar procedure. These relations were firstly found by numerical simulation and experiments but was still not well-explained by mathematical theory so far. Here we provide rigorous explanation based on game theory and renormalization group theory. The scaling law of macroscopic observable near phase transition point in a complex system show wide existence beyond thermodynamics physics. has solid numerical and experimental foundation. Here it must be pointed out that the commutable relation $\partial_T \partial_H = \partial_H \partial_T$ have been used in the calculation. This suggests that the partial differential corresponding to different variables are commutable in the vicinity of critical point. Further more, one may choose different tangent vector field for the coexistence equation and obtain other scaling relations in the vicinity of critical point. For example, for the tangent vector in the pth layer of tangent space above a macroscopic observable surface O_{out},

$$
\phi^i = \partial_{\gamma^1}^{p-1-i_1} \partial_{\gamma^2}^{p-1-i_2} ... \partial_{\gamma^m}^{p-1-i_m} \delta O_{out}(\vec{\gamma}).
\tag{71}
$$

the phase coexistence equation $D(\phi/\gamma) = 0$ is the same Poisson bracket as before,

$$\{\phi^1, \phi^2\} = 0. \tag{72}$$

It is an unification of the special coexistence equations of different orders of phase transition.

3.3   The phase coexistence equation for multi-player games

In multi-player games with more than two players, its free payoff surface is expanded by more strategy parameters space. The phase coexistence equation has a straightforward extension from two phase coexistence to multi-phase coexistence equation. For a multiplayer game with m players, each of which plays a strategy γ_i, then we choose the a vector in the m dimensional tangent vector space,

$$\phi^i = \partial_{\gamma_1}^{p-1-a_{i1}} \partial_{\gamma_2}^{p-1-a_{i2}} ... \partial_{\gamma_m}^{p-1-a_{im}} F_{pay}(\vec{\gamma}),$$

where $\sum_i a_i = 0$. Following Duan's topological current theory[28], this high dimensional tangent vector can also be mapped into a high dimensional unit vector field $\vec{n}$ with $n^a = \phi^a/||\phi||$, $n^a n^a = 1$, where $||\phi|| = \sqrt{\phi^a \phi^a}$ ($a = 1, 2, ..., m$). Then the high dimensional topological current is expressed as

$$\Omega = \sum_{i,j,...,k;a,b,...,c=1}^{m} \epsilon^{ij...k} \epsilon_{ab...c} \frac{\partial n^a}{\partial \gamma_i} \frac{\partial n^b}{\partial \gamma_j} ... \frac{\partial n^c}{\partial \gamma_k}, \tag{73}$$

where $(a, b, ..., c = 1, 2, ..., m)$, $(i, j, ..., k = 1, 2, ..., m)$, $\epsilon_{ab...c}$ is antisymmetric tensor. On even dimensional manifold, this topological current is exactly equivalent to the Riemanian curvature tensor which directly leads to the Gaussian curvature in two dimensions. Applying Laplacian Green function relation, it can be proved that $\Omega = \delta(\vec{\phi}) D(\frac{\phi}{\gamma})$, where the Jacobian $D(\frac{\phi}{\gamma})$ is defined as

$$D(\frac{\phi}{\gamma}) = \sum_{i,j,...,k;a,b,...,c=1}^{m} \epsilon^{ij...k} \epsilon_{ab...c} \frac{\partial \phi^a}{\partial \gamma_i} \frac{\partial \phi^b}{\partial \gamma_j} ... \frac{\partial \phi^c}{\partial \gamma_k}.$$

The multi-phase coexistence equation is $D(\frac{\phi}{\gamma}) = 0$. In $m = 2n$ dimensional manifold, Duan's topological current theory proved that the topological charge of this current is the Chern number $Ch = \int \Omega d^2 q = \sum_{k=1}^{l} W_k$, which is the sum of the winding number around the fixed points of multi-player game.

For example, if the game has three players, the free payoff is a function of three parameters, each of them represent the strategy played by the three players. The three phases intersects with one another at the coexistence points which sit at the solutions of

$$\{\phi^i, \phi^j, \phi^k\} = 0, \tag{74}$$

where $\{\phi^i, \phi^j, \phi^k\}$ is the generalized Poisson bracket. For a n-player game, we need to introduce a n-dimensional renormalization group transformation on the free payoff function. The transformation operator expand the tangent vector space around the identity on the manifold. We denote a vector operator as $\vec{L}$, a

group element is given by $U = e^{i\theta\vec{n}\cdot\vec{L}}$. The basic tangent vector field for phase transition is

$$\vec{\phi} = (\phi_{\gamma_1}, \phi_{\gamma_2}, ..., \phi_{\gamma_n}) = \frac{\delta^p[U(\theta)\delta\langle 0|\hat{O}|0\rangle]}{\delta\theta^p}|_{\theta=0}.$$
(75)

Then the multi-phase coexistence equation $D(\frac{\phi}{\gamma}) = 0$ is equivalent to the generalized Poisson bracket in mechanics [31],

$$\{\phi^1, \phi^2, ..., \phi^n\} = \frac{\partial(\phi^1, \phi^2, ..., \phi^n)}{\partial(\gamma_1, \gamma_2, ..., \gamma_n)},$$
(76)

which has an explicit expression of tensor equation

$$\{\phi^1, \phi^2, ..., \phi^n\} = \sum_{i,j,...,k}^{n} \epsilon^{ij...k} \frac{\partial\phi^1}{\partial\gamma_i}\frac{\partial\phi^2}{\partial\gamma_j}...\frac{\partial\phi^c}{\partial\gamma_n} = 0.$$
(77)

The multi-phase coexistence equation (77) governs the boundaries between many phases, which intersect at a branch point. When the minimum supply chain model is extended to cover the more aspects of the production process, including the raw material supplier, product manufacturer, logistic firm, retailers of the chain store and customers. The reference price σ_0 is expanded to include the cost price of raw material σ_{raw}, the cost price of manufacture σ_{man} and the cost price of logistic firm σ_{log}, i.e, $\sigma_0 = \sigma_{raw} + \sigma_{man} + \sigma_{log}$. Here the price parameters are also fluctuating variables that are controlled by the corresponding firms of raw material, manufacture and logistic firm. The supply function is compose of three independent sources of raw material, manufacture and logistic firms, $f_{sup} = f_{raw} + f_{man} + f_{log}$. In this case, the strategy parameters in the phase coexistence are explicitly assigned as following. $\gamma_1 = S_{cha}$, $\gamma_2 = f_{raw}$, $\gamma_3 = f_{man}$, $\gamma_4 = f_{log}$, $\gamma_5 = f_{dem}$. The total payoff matrix is a tensor equation of these strategies.

4 Conclusion

The statistical physics model suggests that a stable pricing order of many retailers emerges without a global planer, even if the retailer only has information about its nearest neighboring competitors. When the supply in the market is more enough to fulfill the demand of every retailer, most retailers competes with their neighbors by raising or lowering price. In a global scope, the price strategy of these retailers are randomly distributed in the region. When the supply reduces to certain critical point, the disordered phase suddenly transforms into an ordered phase in which all retailers choose the same strategy. The correlation between arbitrary two retailers exponentially decays in the disorder phase. As the strategies of chain store and the monopolistic manufacturer are performed to reach the critical point, the correlation between two retailers increases up to infinity. A Nash equilibrium is arrived at the critical point, at which a small change results in a sudden change of collective payoff. The critical point is always an unstable saddle point in phase transition theory of statistical physics. We mapped the renormalization group transformation theory exactly into the

non-cooperative game theory and shows that renormalization group transformation theory describes exactly the local coalition formation process. At the critical point, all local retailers together form one strong coalition to compete with the monopolistic manufacturer. This strong coalition is an emergent coalition because a local retailer does not compete with other retailers except his neighbors. The existence probability of a certain strategy distribution obeys a Boltzmann distribution function that varies with respect to its corresponding total payoff. As the activity level of economics decrease, all possible strategy distributions become possible, a local price fluctuation of one retailer rapidly propagates to another retailer far away. This long ranger correlation offered a mathematical quantification for the notion of Hayek's spontaneous order in economics. In order to predict the critical strategy for an ordered phase to break down, we derived an unified equation of phase coexistence, which is further verified by the thermodynamic physics theory. This minimal supply chain model could extend to a more complete model by including the strategy of raw material supplier, logistic firm and bargaining customers, and so on. It sheds a new light on the spontaneous orders in economics.

5　Compliance with Ethical Standards

The author declares that there is no conflict of interest.

References

[1] F. A, Hayek, *Individualism and economic order*. The University of Chicago Press, (1948).

[2] Y. Tao, *Spontaneous economic order*, Journal of Evolutionary Economics **26**, 467 (2016).

[3] E. Pennisi, *How Did Cooperative Behavior Evolve*, Science **309**, 93 (2005).

[4] R. Pindyck and D. Rubinfeld, *Microeconomics*, Pearson Prentice Hall, (2013).

[5] J. Peck, J. Rampal, *Optimal Monopoly Mechanisms with Demand Uncertainty*, Mathematics of Operations Research, https://doi.org/10.1287/moor.2020.1120, (2021).

[6] M. Shubik, *Game Theory and Operations Research: Some Musings 50 Years Later*, Operations Research, **50**, 192 (2002).

[7] P. K. Kopalle and R. A. Shumsky, *Game Theory Models of Pricing*, "The oxford Handbook of Pricing Management", Edited by Ö. Özer and R. Phillips, (2012).

[8] T. Nguyen, *Coalitional Bargaining in Networks*, Operations Research **63**, 501, (2015).

[9] D. A. Gomes, J. Saude, *A Mean-Field Game Approach to Price formation*, Dynamic Games and Application,**11**, 29 (2021).

[10] R. P. Parker, R. Kapuściński, *Managing a Noncooperative Supply Chain with Limited Capacity*, Operations Research, **59**, 866 (2011).

[11] Q. Feng, G. Lai, L. X. Lu, *Dynamic Bargaining in a Supply Chain with Asymmetric Demand Information*, Management Science **61**, 301 (2015).

[12] B. F. Hobbs, J. S. Pang, *Nash-Cournot Equilibria in Electric Power Markets with Piecewise Linear Demand Functions and Joint Constraints*, Operations Research **55**, 113 (2007).

[13] A. Federgruen, M. Hu, *Technical NoteGlobal Robust Stability in a General Price and Assortment Competition Model*, Operations Research **69**, 164 (2021).

[14] M. Burger, L. Caffarelli, P. A. Markowich, M. T. Wolfram, *On a Boltzmann-type price formation model*, Proc. R. Soc. A,**469**, 20130126 (2013).

[15] A. Gianfreda, D. Bunnc, *A Stochastic Latent Moment Model for Electricity Price Formation*, Operation Research, **66**, 1189 (2018).

[16] G. Gallego, Ming Hu, *Dynamic Pricing of Perishable Assets Under Competition*, Management Science, **60**, 1241 (2014).

[17] Y. Masuda, S. Whang, *Dynamic Pricing for Network Service: Equilibrium and Stability*, Management Science, **45**, 857 (1999).

[18] M. Schulz, *Statistical Physics and Economics: Concepts, Tools, and Applications*, Springer, (2003).

[19] K. Sznajd-Weron, R. Weron, *A Simple Model of Price Formation*, Int. J. Mod. Phys. C,**13** 115 (2002).

[20] Jean-Pierre Aubin, *Optima and Equilibria—An Introduction to Nonlinear Analysis* Springer-verlag, (1991).

[21] T. Si, *Game Theory and Topological Phase Transition*, arXiv:cond-mat/0601014v4 (2008).

[22] L. P. Kadanoff, *Notes on Migdal's Recursion Formulas*, Ann. Phys. (N.Y.) **100**, 359 (1976).

[23] H. E. Stanley, *Introduction to phase transition and Critical Phenomena*, Oxford, 2nd ed. (1983).

[24] C. N. Yang, *The Spontaneous Magnetization of a Two-Dimensional Ising Model*, Phys. Rev. **85**, 808 (1952).

[25] K. G. Wilson, *The renormalization group and critical phenomena*, Rev. Mod. Phys. **55**, 583 (1983).

[26] P. Ehrenfest, Proc. R. Acad. Sci. Amsterdam, **36**, 153 (1933).

[27] J. Lewin, *Differential Game—Theory and Methods for Solving Game Problems with Singular Surface*, Springer-Verlag, (1994).

[28] Y. S. Duan, *The structure of the topological current*, SLAC-PUB-3301, (1984).

[29] Y. S. Duan and H. Zhang, *Line defects of a two-component vector order parameter*, Phys. Rev. E, **60**, 2568, (1999).

[30] E. Goursat, *A Course in Mathematical Analysis*, vol. I, translated by E. R. Hedrick, Dover, New York, (1904).

[31] Y. Nambu, *Generalized Hamiltonian Dynamics*, Phys. Rev. D **7**, 2405 (1973).

[32] S. Nash and S. Sen, *Topology and Geometry for physicists*, Academic. London, (1983).

[33] J. A. Schouten, *Tensor Analysis for Physicists*, Clarendon Oxford, (1951).