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Binomial Cubic Fibonacci Sums

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Abstract

We evaluate some binomial cubic Fibonacci sums.

1 Introduction

As noted by Nagy et al. [4], there is a paucity of binomial cubic Fibonacci and Lucas identities in existing literature.

Let F_j and L_j be the j^{th} Fibonacci number and Lucas number, respectively.

Our main goal is to evaluate the following sums,

$$\sum_{k=0}^{n}\binom{n}{k}F_{k+s}^3,\quad \sum_{k=0}^{n}(-1)^k\binom{n}{k}F_{k+s}^3,\quad \sum_{k=0}^{n}2^k\binom{n}{k}F_{k+s}^3,$$

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}2^{n-k}F_{k+s}^3,\quad \sum_{k=0}^{n}(-1)^k\binom{n}{k}3^kF_{k+s}^3,\quad \sum_{k=0}^{n}\binom{n}{k}3^{n-k}F_{k+s}^3,$$
$$\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}F_{2k+s}^3,\quad \sum_{k=1}^{\lceil n/2\rceil}\binom{n}{2k-1}F_{2k+s}^3,$$

and the corresponding series involving Lucas numbers, for any non-negative integer n and any integer s.

The Fibonacci numbers, F_j, and the Lucas numbers, L_j, are defined, for $j \in \mathbb{Z}$, through the recurrence relations

$$F_j = F_{j-1} + F_{j-2}, (j \ge 2),\quad F_0 = 0,\ F_1 = 1;$$

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and

$$L_j = L_{j-1} + L_{j-2}, (j \geq 2), \quad L_0 = 2,\, L_1 = 1;$$

with

$$F_{-j} = (-1)^{j-1} F_j, \quad L_{-j} = (-1)^j L_j.$$

Throughout this paper, we denote the golden ratio, $(1+\sqrt{5})/2$, by α and write $\beta = (1-\sqrt{5})/2 = -1/\alpha$, so that $\alpha\beta = -1$ and $\alpha + \beta = 1$.

Explicit formulas (Binet formulas) for the Fibonacci and Lucas numbers are

$$F_j = \frac{\alpha^j - \beta^j}{\alpha - \beta}, \quad L_j = \alpha^j + \beta^j, \quad j \in \mathbb{Z}. \tag{1}$$

Koshy [3] and Vajda [6] have written excellent books dealing with Fibonacci and Lucas numbers.

2 Required identities

Lemma 1. *For real or complex z, let a given well-behaved function $h(z)$ have, in its domain, the representation $h(z) = \sum_{k=c_1}^{c_2} g(k) z^{f(k)}$ where $f(k)$ and $g(k)$ are given real sequences and $c_1, c_2 \in [-\infty, \infty]$. Let j be an integer. Then,*

$$\begin{aligned} &5\sqrt{5} \sum_{k=c_1}^{c_2} g(k) z^{f(k)} F_{jf(k)}^3 \\ &\qquad = h(\alpha^{3j}) - h(\beta^{3j}) - 3\left(h\left((-1)^j \alpha^j z\right) - h\left((-1)^j \beta^j z\right)\right), \end{aligned} \tag{F}$$

$$\begin{aligned} &\sum_{k=c_1}^{c_2} g(k) z^{f(k)} L_{jf(k)}^3 \\ &\qquad = h(\alpha^{3j}) + h(\beta^{3j}) + 3\left(h\left((-1)^j \alpha^j z\right) + h\left((-1)^j \beta^j z\right)\right). \end{aligned} \tag{L}$$

Proof. Set $m = 3$ in Adegoke [1, identities (F) and (L)]. □

Lemma 2. *Let a, b, c and d be rational numbers and λ an irrational number. Then,*

$$a + \lambda b = c + \lambda d \iff a = c, \quad b = d\,.$$

Lemma 3. *For p and q integers,*

$$1 + (-1)^p \alpha^{2q} = \begin{cases} (-1)^p \alpha^q F_q \sqrt{5}, & \text{if } p \text{ and } q \text{ have different parity;} \\ (-1)^p \alpha^q L_q, & \text{if } p \text{ and } q \text{ have the same parity.} \end{cases} \tag{2}$$

$$1 - (-1)^p \alpha^{2q} = \begin{cases} (-1)^{p-1} \alpha^q L_q, & \text{if } p \text{ and } q \text{ have different parity;} \\ (-1)^{p-1} \alpha^q F_q \sqrt{5}, & \text{if } p \text{ and } q \text{ have the same parity.} \end{cases} \tag{3}$$

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Proof. We have

$$\begin{aligned}(-1)^{p+q}+(-1)^p\alpha^{2q}&=\alpha^{p+q}\beta^{p+q}+\alpha^{p+2q}\beta^p\\&=\alpha^{p+q}\beta^p(\alpha^q+\beta^q)\\&=(-1)^p\alpha^qL_q.\end{aligned}\tag{4}$$

Similarly,

$$(-1)^{p+q}-(-1)^p\alpha^{2q}=(-1)^{p-1}\alpha^qF_q\sqrt{5}.\tag{5}$$

□

Corresponding to (4) and (5) we have

$$(-1)^{p+q}+(-1)^p\beta^{2q}=(-1)^p\beta^qL_q\tag{6}$$

and

$$(-1)^{p+q}-(-1)^p\beta^{2q}=(-1)^p\beta^qF_q\sqrt{5}.\tag{7}$$

Identities (4), (5), (6) and (7) imply

$$(-1)^q+\alpha^{2q}=\alpha^qL_q,\tag{8}$$

$$(-1)^q-\alpha^{2q}=-\alpha^qF_q\sqrt{5},\tag{9}$$

$$(-1)^q+\beta^{2q}=\beta^qL_q,\tag{10}$$

$$(-1)^q-\beta^{2q}=\beta^qF_q\sqrt{5}.\tag{11}$$

Lemma 4 (Hoggatt et al [2])**.** *For p and q integers,*

$$L_{p+q}-L_p\alpha^q=-\beta^pF_q\sqrt{5},\tag{12}$$

$$L_{p+q}-L_p\beta^q=\alpha^pF_q\sqrt{5},\tag{13}$$

$$F_{p+q}-F_p\alpha^q=\beta^pF_q,\tag{14}$$

$$F_{p+q}-F_p\beta^q=\alpha^pF_q.\tag{15}$$

Lemma 5. *We have*

$$1-\alpha=\beta,\quad 1-\beta=\alpha,\quad 1+\alpha^3=2\alpha^2,\quad 1+\beta^3=2\beta^2,\tag{16}$$

$$1+\alpha=\alpha^2,\quad 1+\beta=\beta^2,\quad 1-\alpha^3=-2\alpha,\quad 1-\beta^3=-2\beta,\tag{17}$$

$$1-2\alpha=-\sqrt{5},\quad 1-2\beta=\sqrt{5},\quad 1+2\alpha^3=\alpha^3\sqrt{5},\quad 1+2\beta^3=-\beta^3\sqrt{5},\tag{18}$$

$$2+\alpha=\alpha\sqrt{5},\quad 2+\beta=-\beta\sqrt{5},\quad 2-\alpha^3=-\sqrt{5},\quad 2-\beta^3=\sqrt{5},\tag{19}$$

$$1+3\alpha=\alpha^2\sqrt{5},\quad 1+3\beta=-\beta^2\sqrt{5},\quad 1-3\alpha^3=-2\alpha^2\sqrt{5},\quad 1-3\beta^3=2\beta^2\sqrt{5},\tag{20}$$

$$3-\alpha=-\beta\sqrt{5},\quad 3-\beta=\alpha\sqrt{5},\quad 3+\alpha^3=2\alpha\sqrt{5},\quad 3+\beta^3=-2\beta\sqrt{5}.\tag{21}$$

Proof. Each identity is obtained by making appropriate substitutions for p and q in the identities given in Lemma 4. □

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3 Binomial cubic Fibonacci identities

Lemma 6. *For non-negative integer n, integers j, r and s and real or complex x and z,*

$$\begin{aligned}5\sqrt{5}\sum_{k=0}^{n}\binom{n}{k}x^{n-k}z^kF_{j(rk+s)}^3 &= \alpha^{3js}(x+\alpha^{3jr}z)^n-\beta^{3js}(x+\beta^{3jr}z)^n\\ &\quad -(-1)^{js}3\alpha^{js}(x+(-1)^{jr}\alpha^{jr}z)^n\\ &\quad +(-1)^{js}3\beta^{js}(x+(-1)^{jr}\beta^{jr}z)^n,\end{aligned}\tag{F1}$$

$$\begin{aligned}\sum_{k=0}^{n}\binom{n}{k}x^{n-k}z^kL_{j(rk+s)}^3 &= \alpha^{3js}(x+\alpha^{3jr}z)^n+\beta^{3js}(x+\beta^{3jr}z)^n\\ &\quad +(-1)^{js}3\alpha^{js}(x+(-1)^{jr}\alpha^{jr}z)^n\\ &\quad +(-1)^{js}3\beta^{js}(x+(-1)^{jr}\beta^{jr}z)^n.\end{aligned}\tag{L1}$$

Proof. Set $m=3$ in Adegoke [1, identities (BF$'$) and (BL$'$)]. □

Theorem 1. *For non-negative integer n and any integer s,*

$$\sum_{k=0}^{n}\binom{n}{k}F_{k+s}^3=\frac{1}{5}(2^nF_{2n+3s}+3F_{n-s}),\tag{22}$$

$$\sum_{k=0}^{n}\binom{n}{k}L_{k+s}^3=2^nL_{2n+3s}+3L_{n-s},\tag{23}$$

Proof. Set $x=1$, $z=1$, $j=1$, $r=1$ in (F1), utilizing identity (16), to obtain

$$5\sqrt{5}\sum_{k=0}^{n}\binom{n}{k}F_{k+s}^3=2^n(\alpha^{3s+2n}-\beta^{3s+2n})+3(\alpha^{n-s}-\beta^{n-s});$$

and hence identity (22). To prove identity (23), use these $(x,z,j,\ldots)$ values in (L1). □

The $s=0$ special case of (22) was obtained by Stanica [5].

Theorem 2. *For non-negative integer n and any integer s,*

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^kF_{k+s}^3=\frac{1}{5}((-1)^n2^nF_{n+3s}-(-1)^s3F_{2n+s}),\tag{24}$$

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}L_{k+s}^3=(-1)^n2^nL_{n+3s}+(-1)^s3L_{2n+s},\tag{25}$$

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Proof. To prove identity (24), set $x = 1$, $z = -1$, $j = 1$, $r = 1$ in (F1), noting the identities in (17), to get

$$5\sqrt{5}\sum_{k=0}^{n}(-1)^k\binom{n}{k}F_{k+s}^3 = (-1)^n 2^n(\alpha^{n+3s} - \beta^{n+3s}) - 3(-1)^s(\alpha^{2n+s} - \beta^{2n+s}),$$

from which the identity follows. The proof of (25) is similar. Use these values in (L1). □

Stanica [5] also found the $s = 0$ case of identity (24).

Theorem 3. *For non-negative integer n and any integer s,*

$$\sum_{k=0}^{n}\binom{n}{k}2^k F_{k+s}^3 = \begin{cases} 5^{n/2-1}(F_{3n+3s} - (-1)^s 3F_s), & n \text{ even;} \\ 5^{(n-3)/2}(L_{3n+3s} + (-1)^s 3L_s) & n \text{ odd,} \end{cases} \tag{26}$$

$$\sum_{k=0}^{n}\binom{n}{k}2^k L_{k+s}^3 = \begin{cases} 5^{n/2}(L_{3n+3s} + (-1)^s 3L_s), & n \text{ even;} \\ 5^{(n+1)/2}(F_{3n+3s} - (-1)^s 3F_s) & n \text{ odd.} \end{cases} \tag{27}$$

Proof. The proof of (26) proceeds with the choice $j = 1$, $r = 1$, $x = 1$, $z = 2$ in (F1), employing the set of identities (18), giving

$$\begin{aligned} 5\sqrt{5}\sum_{k=0}^{n}2^k\binom{n}{k}F_{k+s}^3 &= (\sqrt{5})^n(\alpha^{3n+3s} - (-1)^n\beta^{3n+3s}) \\ &\quad - 3(-1)^{n+s}(\sqrt{5})^n(\alpha^s - (-1)^n\beta^s), \end{aligned}$$

from which the identity follows in accordance with the parity of n. The proof of (27) is similar. Use these $(x, z, j, \dots)$ values in (L1). □

Theorem 4. *For non-negative integer n and any integer s,*

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}2^{n-k}F_{k+s}^3 = \begin{cases} 5^{n/2-1}((-1)^{s-1}3F_{n+s} + F_{3s}), & n \text{ even;} \\ 5^{(n-3)/2}((-1)^{s-1}3L_{n+s} - L_{3s}), & n \text{ odd;} \end{cases} \tag{28}$$

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}2^{n-k}L_{k+s}^3 = \begin{cases} 5^{n/2}((-1)^s 3L_{n+s} + L_{3s}), & n \text{ even;} \\ 5^{(n+1)/2}((-1)^s 3F_{n+s} - F_{3s}), & n \text{ odd.} \end{cases} \tag{29}$$

Proof. The coice $x = 2$, $z = -1$, $j = 1$, $z = 1$ in (F1), noting the set of identities (19) gives

$$\begin{aligned} 5\sqrt{5}\sum_{k=0}^{n}(-1)^k\binom{n}{k}2^{n-k}F_{k+s}^3 &= (\sqrt{5})^n(-1)^n(\alpha^{3s} - (-1)^n\beta^{3s}) \\ &\quad - (\sqrt{5})^n(-1)^s 3(\alpha^{n+s} - (-1)^n\beta^{n+s}); \end{aligned}$$

from which we get (28). The proof of (29) is similar. □

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Theorem 5. *For non-negative integer n and any integer s,*

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}3^kF_{k+s}^3=\begin{cases}5^{n/2-1}(2^nF_{2n+3s}-(-1)^s3F_{2n+s}), & n\ \text{even};\\ -5^{(n-3)/2}(2^nL_{2n+3s}+(-1)^s3L_{2n+s}), & n\ \text{odd};\end{cases}\tag{30}$$

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}3^kL_{k+s}^3=\begin{cases}5^{n/2}(2^nL_{2n+3s}+(-1)^s3L_{2n+s}), & n\ \text{even};\\ -5^{(n+1)/2}(2^nF_{2n+3s}-(-1)^s3F_{2n+s}), & n\ \text{odd}.\end{cases}\tag{31}$$

Proof. Choose $x=1$, $z=-3$, $j=1$, $r=1$ in (F1). This gives, with the use of the identities in (20),

$$\begin{aligned}5\sqrt5\sum_{k=0}^{n}(-1)^k\binom{n}{k}3^kF_{k+s}^3&=(\sqrt5)^n(-1)^n2^n(\alpha^{2n+3s}-(-1)^n\beta^{2n+3s})\\&\quad-(\sqrt5)^n(-1)^s3(\alpha^{2n+s}-(-1)^n\beta^{2n+s}).\end{aligned}$$

Identity (30) now follows. The proof of (31) is similar. □

Theorem 6. *For non-negative integer n and any integer s,*

$$\sum_{k=0}^{n}\binom{n}{k}3^{n-k}F_{k+s}^3=\begin{cases}5^{n/2-1}(2^nF_{n+3s}+3F_{n-s}), & n\ \text{even};\\ 5^{(n-3)/2}(2^nL_{n+3s}+3L_{n-s}), & n\ \text{odd};\end{cases}\tag{32}$$

$$\sum_{k=0}^{n}\binom{n}{k}3^{n-k}L_{k+s}^3=\begin{cases}5^{n/2}(2^nL_{n+3s}+3L_{n-s}), & n\ \text{even};\\ 5^{(n+1)/2}(2^nF_{n+3s}+3F_{n-s}), & n\ \text{odd}.\end{cases}\tag{33}$$

Proof. Set $x=3$, $z=1$, $j=1=r$ in (F1) and use the set of identities in (21) to obtain

$$\begin{aligned}5\sqrt5\sum_{k=0}^{n}\binom{n}{k}3^{n-k}F_{k+s}^3&=(\sqrt5)^n2^n(\alpha^{n+3s}-(-1)^n\beta^{n+3s})\\&\quad+(\sqrt5)^n3(\alpha^{n-s}-(-1)^n\beta^{n-s});\end{aligned}$$

from which (32) follows. The proof of (33) is similar. Use the same $(x,z,\ldots)$ values in (L1). □

Lemma 7. *For non-negative integer n, integers j, r and s and real or complex z,*

$$\begin{aligned}&5\sqrt5\sum_{k=0}^{\lfloor n/2\rfloor}2\binom{n}{2k}z^{2k}F_{j(2rk+s)}^3\\&=\alpha^{3js}(1+\alpha^{3jr}z)^n+\alpha^{3js}(1-\alpha^{3jr}z)^n-\beta^{3js}(1+\beta^{3jr}z)^n-\beta^{3js}(1-\beta^{3jr}z)^n\\&\quad-(-1)^{js}\alpha^{js}3(1+(-1)^{jr}\alpha^{jr}z)^n-(-1)^{js}\alpha^{js}3(1-(-1)^{jr}\alpha^{jr}z)^n\\&\qquad+(-1)^{js}\beta^{js}3(1+(-1)^{jr}\beta^{jr}z)^n+(-1)^{js}\beta^{js}3(1-(-1)^{jr}\beta^{jr}z)^n,\end{aligned}\tag{F2}$$

$$
\begin{aligned}
&2\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}z^{2k}L^3_{j(2rk+s)}\\
&\quad=\alpha^{3js}(1+\alpha^{3jr}z)^n+\alpha^{3js}(1-\alpha^{3jr}z)^n+\beta^{3js}(1+\beta^{3jr}z)^n+\beta^{3js}(1-\beta^{3jr}z)^n\\
&\qquad+(-1)^{js}\alpha^{js}3(1+(-1)^{jr}\alpha^{jr}z)^n+(-1)^{js}\alpha^{js}3(1-(-1)^{jr}\alpha^{jr}z)^n\\
&\qquad\quad+(-1)^{js}\beta^{js}3(1+(-1)^{jr}\beta^{jr}z)^n+(-1)^{js}\beta^{js}3(1-(-1)^{jr}\beta^{jr}z)^n,
\end{aligned}\tag{L2}
$$

$$
\begin{aligned}
&5\sqrt5\sum_{k=1}^{\lceil n/2\rceil}2\binom{n}{2k-1}z^{2k-1}F^3_{j(2rk+s)}\\
&\quad=\alpha^{3j(r+s)}(1+\alpha^{3jr}z)^n-\alpha^{3j(r+s)}(1-\alpha^{3jr}z)^n-\beta^{3j(r+s)}(1+\beta^{3jr}z)^n+\beta^{3j(r+s)}(1-\beta^{3jr}z)^n\\
&\qquad-(-1)^{j(r+s)}\alpha^{j(r+s)}3(1+(-1)^{jr}\alpha^{jr}z)^n+(-1)^{j(r+s)}\alpha^{j(r+s)}3(1-(-1)^{jr}\alpha^{jr}z)^n\\
&\qquad\quad+(-1)^{j(r+s)}\beta^{j(r+s)}3(1+(-1)^{jr}\beta^{jr}z)^n-(-1)^{j(r+s)}\beta^{j(r+s)}3(1-(-1)^{jr}\beta^{jr}z)^n,
\end{aligned}\tag{F3}
$$

$$
\begin{aligned}
&2\sum_{k=1}^{\lceil n/2\rceil}\binom{n}{2k-1}z^{2k-1}L^3_{j(2rk+s)}\\
&\quad=\alpha^{3j(r+s)}(1+\alpha^{3jr}z)^n-\alpha^{3j(r+s)}(1-\alpha^{3jr}z)^n+\beta^{3j(r+s)}(1+\beta^{3jr}z)^n-\beta^{3j(r+s)}(1-\beta^{3jr}z)^n\\
&\qquad+(-1)^{j(r+s)}\alpha^{j(r+s)}3(1+(-1)^{jr}\alpha^{jr}z)^n-(-1)^{j(r+s)}\alpha^{j(r+s)}3(1-(-1)^{jr}\alpha^{jr}z)^n\\
&\qquad\quad+(-1)^{j(r+s)}\beta^{j(r+s)}3(1+(-1)^{jr}\beta^{jr}z)^n-(-1)^{j(r+s)}\beta^{j(r+s)}3(1-(-1)^{jr}\beta^{jr}z)^n.
\end{aligned}\tag{L3}
$$

Proof. In the identities

$$h_1(z)=2\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}z^{2rk+s}=z^s(1+z^r)^n+z^s(1-z^r)^n,$$

$$h_2(z)=2\sum_{k=1}^{\lceil n/2\rceil}\binom{n}{2k-1}z^{2rk+s}=z^{r+s}(1+z^r)^n-z^{r+s}(1-z^r)^n,$$

identify

$$g(k)=2\binom{n}{2k},\quad f(k)=2rk+s,\quad c_1=0,\quad c_2=\lfloor n/2\rfloor,\quad h(z)=z^s(1+z^r)^n+z^s(1-z^r)^n,$$

and use these in (F) and (L) to obtain (F2) and (L2).

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Similarly, use of

$$g(k)=2\binom{n}{2k-1},\quad f(k)=2rk+s,\quad c_1=1,\quad c_2=\lceil n/2\rceil\,,\quad h(z)=z^s(1+z^r)^n-z^s(1-z^r)^n,$$

in (F) and (L) gives (F3) and (L3). □

Theorem 7. *For non-negative integer n and any integer s,*

$$\begin{aligned}10\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}F_{2k+s}^3&=2^n(F_{2n+3s}+(-1)^nF_{n+3s})\\&\quad-3(-1)^s(F_{2n+s}-(-1)^sF_{n-s}),\end{aligned}\tag{34}$$

$$\begin{aligned}2\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}L_{2k+s}^3&=2^n(L_{2n+3s}+(-1)^nL_{n+3s})\\&\quad+3(-1)^s(L_{2n+s}+(-1)^sL_{n-s}).\end{aligned}\tag{35}$$

Proof. The choice of $z=1=j=r$ in (F2) gives

$$\begin{aligned}10\sqrt5\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}F_{2k+s}^3&=2^n(\alpha^{2n+3s}-\beta^{2n+3s})+(-1)^n2^n(\alpha^{n+3s}-\beta^{n+3s})\\&\quad+3(-1)^s(\beta^s\alpha^n-\alpha^s\beta^n)-3(-1)^s(\alpha^{2n+s}-\beta^{2n+s});\end{aligned}$$

from which identity (34) follows. The proof of (35) is similar; use $z=1=j=r$ in (L2). □

Corollary 8. *For non-negative integer n and any integer s,*

$$10\sum_{k=0}^{n}\binom{2n}{2k}F_{2k+s}^3=\begin{cases}4^nL_nF_{3n+3s}-(-1)^s3F_{n+s}L_{3n}, & n\ \text{even};\\ 4^nF_nL_{3n+3s}-(-1)^s3L_{n+s}F_{3n}, & n\ \text{odd};\end{cases}\tag{36}$$

$$2\sum_{k=0}^{n}\binom{2n}{2k}L_{2k+s}^3=\begin{cases}4^nL_nL_{3n+3s}+(-1)^s3L_{n+s}L_{3n}, & n\ \text{even};\\ 5(4^nF_nF_{3n+3s}+(-1)^s3F_{n+s}F_{3n}), & n\ \text{odd}.\end{cases}\tag{37}$$

Proof. Write $2n$ for n in each of the identities (34) and (35). Simplification is achieved by the use of the following well-known Fibonacci identities which are valid for any two integers u and v having the same parity:

$$F_u+(-1)^{(u-v)/2}F_v=L_{(u-v)/2}F_{(u+v)/2},\tag{38}$$

$$F_u-(-1)^{(u-v)/2}F_v=F_{(u-v)/2}L_{(u+v)/2},\tag{39}$$

$$L_u+(-1)^{(u-v)/2}L_v=L_{(u-v)/2}L_{(u+v)/2},\tag{40}$$

$$L_u-(-1)^{(u-v)/2}L_v=5F_{(u-v)/2}F_{(u+v)/2}.\tag{41}$$

□

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Corollary 9. *For non-negative integer n,*

$$10\sum_{k=0}^{n}\binom{2n-1}{2k}F_{2k}^3=\begin{cases}(2^{2n-1}-3)F_{2n-1}L_{n-1}L_n, & n\ \textit{even};\\ (2^{2n-1}-3)5F_{2n-1}F_{n-1}F_n, & n\ \textit{odd};\end{cases}\tag{42}$$

$$2\sum_{k=0}^{n}\binom{2n}{2k}L_{2k}^3=\begin{cases}(4^n+3)L_nL_{3n}, & n\ \textit{even};\\ (4^n+3)5F_nF_{3n}, & n\ \textit{odd}.\end{cases}\tag{43}$$

Proof. To prove (42), write $2n-1$ for n in (34) and set $s=0$. To prove (43), set $s=0$ in identity (37). □

Theorem 10. *For non-negative integer n and any integer s,*

$$\begin{aligned}10\sum_{k=1}^{\lceil n/2\rceil}\binom{n}{2k-1}F_{2k+s}^3&=2^n(F_{2n+3s+3}-(-1)^nF_{n+3s+3})\\&\quad-(-1)^s3(F_{2n+s+1}-(-1)^sF_{n-s-1}),\end{aligned}\tag{44}$$

$$\begin{aligned}2\sum_{k=1}^{\lceil n/2\rceil}\binom{n}{2k-1}L_{2k+s}^3&=2^n(L_{2n+3s+3}-(-1)^nL_{n+3s+3})\\&\quad+(-1)^s3(L_{2n+s+1}+(-1)^sL_{n-s-1}).\end{aligned}\tag{45}$$

Proof. Set $z=1=j=r$ in identity (F3) to obtain

$$\begin{aligned}&10\sqrt{5}\sum_{k=1}^{\lceil n/2\rceil}\binom{n}{2k-1}F_{2k+s}^3\\&=2^n(\alpha^{2n+3s+3}-\beta^{2n+3s+3})-(-1)^n2^n(\alpha^{n+3s+3}-\beta^{n+3s+3})\\&\quad+(-1)^{s+1}3(\alpha^{2n+s+1}-\beta^{2n+s+1})+(-1)^{s+1}3(\alpha^n\beta^{s+1}-\alpha^{s+1}\beta^n);\end{aligned}$$

from which identity (44) follows. The proof of (45) is similar. □

Corollary 11. *For non-negative integer n and any integer s,*

$$10\sum_{k=1}^{n}\binom{2n}{2k-1}F_{2k+s}^3=\begin{cases}4^nF_nL_{3n+3s+3}-(-1)^s3L_{n+s+1}F_{3n}, & n\ \textit{even};\\ 4^nL_nF_{3n+3s+3}-(-1)^s3F_{n+s+1}L_{3n}, & n\ \textit{odd};\end{cases}\tag{46}$$

$$2\sum_{k=1}^{n}\binom{2n}{2k-1}L_{2k+s}^3=\begin{cases}5(4^nF_nF_{3n+3s+3}+(-1)^s3F_{n+s+1}F_{3n}), & n\ \textit{even};\\ 4^nL_nL_{3n+3s+3}+(-1)^s3L_{n+s+1}L_{3n}, & n\ \textit{odd}.\end{cases}\tag{47}$$

Proof. Write $2n$ for n in each of the identities (44) and (45) and make use of identities (38) – (41). □

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Corollary 12. *For non-negative integer n,*

$$10\sum_{k=1}^{n}\binom{2n-1}{2k-1}F_{2k-1}^{3}=\begin{cases}(2^{2n-1}+3)5F_{2n-1}F_{n-1}F_{n}, & n \text{ even};\\ (2^{2n-1}+3)F_{2n-1}L_{n-1}L_{n}, & n \text{ odd},\end{cases} \tag{48}$$

$$2\sum_{k=1}^{n}\binom{2n}{2k-1}L_{2k-1}^{3}=\begin{cases}(4^{n}-3)5F_{n}F_{3n}, & n \text{ even};\\ (4^{n}-3)L_{n}L_{3n}, & n \text{ odd}.\end{cases} \tag{49}$$

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