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Article

SPECIAL PROPERTIES OF PLANE SOLENOIDAL FIELDS

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Abstract: There are given algebraic and integral identities for a pair or a triple of plane solenoidal fields. As applications, we obtain sufficient potentiality conditions for a plane vector field. The integral identities are also important for exact a priori estimates.

Keywords: solenoidal field; potential field; Helmholmtz-Weyl decomposition

Key Contribution: There are given some constructions of a plane potential field with an application of one or two solenoidal fields.

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0. Introduction

The well-known classical Helmholtz result connected with decomposition of a vector field by a sum of solenoidal and potential components admits an generalization. This is known as Helmholtz–Weyl decomposition (see, for example,[1]). A more exactly, Lebesgue space $L_2(R^n)$ of vector fields $u = (u_1, \dots, u_n)$ has decomposition

$$L_2(R^n) = H_2(R^n) \oplus G_2(R^n)$$

where $H_2(R^n)$ is a closure of all smooth solenoidal fields and $G_2(R^n)$ is a closure of all smooth potential fields with respect to the norm of space $L_2(R^n)$. In [2], [3] and [4] for different dimensions it is noted that $H_2(R^n)$ can be consider as a closure of all smooth finite solenoidal fields. We mark that to replace space R^n onto a domain $\Omega \neq R^n$ it is impossible (see,[5]). By the way, some conditions for these replacements can be found in [6].

Helmholtz–Weyl decomposition implies integral identity:

$$\int_{R^n} u \cdot g dx = 0 \tag{1}$$

for any fields $u \in H_2(R^n)$ and $g \in G_2(R^n)$, which gives a key for studying of the Navier–Stokes equations. In principle, different results connected with integral identities for solenoidal fields in space were obtained in [7]. For another dimensions there are extended in [8], [9].

But applying the curle of a solenoidal field and new integral identites (see [10]) there was managed to emphasize a distinction in properties of plane and spatial solenoidal fields. For the first time this idea there was considered in [11]. Various examples of flows in space confirm a poorness of plane–parallel fluids in the comparison with spatial fluids.

The main goal of this work is to give new integral identities for plane solenoidal fields. These identities can be considered as an origin of new a priori estimates (see, for example,[10]), as an origin of new conservation laws which can be connected with an initial data in the Cauchy problem. As a corollary we have sufficient conditions for the field to be a potential field on a plane. These integral identities we can

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consider as elements of latent symmetry. Hydrodynamics is very rich by such elements (see, for example, the important review [12]). Therefore, the comparison of different symmetry applications is very useful in this way.

0.1. Notations

Let $u : R^2 \to R^2,\ u = (u_1, u_2)$, be an arbitrary vector field. Symbols

$$u_{k,\,i} = \frac{\partial u_k}{\partial x_i},\ u_{k,\,ij} = \frac{\partial^2 u_k}{\partial x_i \partial x_j}$$

so forth mean a partial differentiation or differentiation in distributions. They are also noted by another symbol

$$D^\alpha u,$$

where $\alpha = (\alpha_1, \alpha_2)$ is a multi–index of a partial derivative. Naturally Δ is the Laplace operator. Below if I don't mark separately the repeated lower indices i, j, k, m mean summation there, where they change. For example,

$$u_{i,\,jm} u_{k,\,im} u_{k,\,j} = \sum_{i,j,m,k=1}^{2} u_{i,\,jm} u_{k,\,im} u_{k,\,j},$$

$$u_i u_{k,\,i} \triangle u_k = \sum_{i,k=1}^{2} u_i u_{k,\,i} \triangle u_k$$

and etc. Further, rotor coordinates are noted by

$$c_{ki}(u) = u_{k,i} - u_{i,k}$$

and there are interpreted as elements of a skew-symmetric matrix C of the second order.

As usually, a finite field vanishes out of a some disk.

1. Main results

For simplicity, we will confine ourselves to the formulation of the main results for the entire plane and smooth fields. The main results are described by Theorem 1 and Theorem 2. Their proof relies on Lemma 1 and Lemma 2. These simple lemmas may be interesting as separate statements.

Lemma 1. *For every three smooth solenoidal plane fields $u,\ v,\ w$ the following algebraic identities are true:*

$$w_{i,\,j} u_{k,\,i} v_{k,\,j} + u_{i,\,j} w_{k,\,i} v_{k,\,j} \equiv 0, \tag{2}$$

$$w_{i,\,jm} u_{k,\,im} v_{k,\,j} + u_{i,\,jm} w_{k,\,im} v_{k,\,j} \equiv 0, \tag{3}$$

$$u_{i,\,jm} u_{k,\,im} w_{k,\,j} \equiv 0, \tag{4}$$

$$D^\alpha u_{i,\,jm} D^\alpha u_{k,\,im} D^\beta w_{k,\,j} \equiv 0, \tag{5}$$

$$D^\alpha w_{i,\,jm} D^\beta u_{k,\,im} D^\gamma v_{k,\,j} + D^\beta u_{i,\,jm} D^\alpha u_{k,\,im} D^\gamma v_{k,\,j} \equiv 0. \tag{6}$$

In particular,

$$\Delta u_{i,\,j} u_{k,\,i} \Delta u_{k,\,j} + u_{i,\,j} \Delta u_{k,\,i} \Delta u_{k,\,j} \equiv 0, \tag{7}$$

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$$u_{i,\,jl}u_{k,\,im}u_{k,\,jml} \equiv 0. \tag{8}$$

Proof of Lemma 1. Grouping of terms in (2) implies to equality

$$w_{i,\,j}u_{k,\,i}v_{k,\,j} + u_{i,\,j}w_{k,\,i}v_{k,\,j} =$$

$$= 2(u_{1,\,1}v_{1,\,1}w_{1,\,1} + u_{2,\,2}v_{2,\,2}w_{2,\,2}) + (u_{1,\,2}w_{2,\,1} + u_{2,\,1}w_{1,\,2})divv +$$

$$(v_{1,\,2}w_{1,\,2} + v_{2,\,1}w_{2,\,1})divu + (u_{1,\,2}v_{1,\,2} + u_{2,\,1}v_{2,\,1})divw.$$

Hence, we have the first equality of lemma because $u_{1,\,1} = -u_{2,\,2}$ and etc.

Fix m. Replacements u onto $u_{,m}$, w onto $w_{,m}$ in (2) imply (3) after summation with respect to m. Taking in (3) $w = u$ and replacing v by w we have (4).

Replacements u onto $D^\alpha u$, w onto $D^\beta w$ in (4) give (5). Formula (6) follows from (2) since fields $w_{,\,m}$, $u_{,\,m}$, $D^\alpha w$, $D^\beta u$, $D^\gamma v$ are solenoidal. Identity (7) is corollary of (2). Equality (8) is verified by the same way. It is sufficient to group

$$u_{i,\,jl}u_{k,\,im}u_{k,\,jml} = (u_{1,\,1l}u_{1,\,1m}u_{1,\,1ml} + u_{2,\,2l}u_{2,\,1m}u_{2,\,2ml}) + (u_{2,\,1l}u_{1,\,2m}u_{1,\,1ml} + u_{1,\,2l}u_{2,\,1m}u_{2,\,2ml}) +$$

$$(u_{1,\,2l}u_{1,\,1m}u_{1,\,2ml} + u_{2,\,2l}u_{1,\,2m}u_{1,\,2ml}) + (u_{1,\,1l}u_{2,\,1m}u_{2,\,1ml} + u_{2,\,1l}u_{2,\,2m}u_{2,\,1ml}).$$

Here, every sum in brackets vanishes.

Lemma is proved. □

Lemma 2. *For every pair of smooth finite solenoidal plane fields u, w the following integral identities are true:*

$$\int_{R^2} w_{i,\,jm}u_{k,\,im}u_{k,\,j}dx = 0 \tag{9}$$

$$\int_{R^2} u_{i,\,jm}w_{k,\,im}u_{k,\,j}dx = 0, \tag{10}$$

$$\int_{R^2} \Delta u_{i,\,j}u_{k,\,im}u_{k,\,jm}dx = 0, \tag{11}$$

$$\int_{R^2} D^\beta w_{i,\,jm}D^\alpha u_{k,\,im}D^\alpha u_{k,\,j}dx = 0, \tag{12}$$

$$\int_{R^2} D^\alpha u_{i,\,jm}D^\beta w_{k,\,im}D^\alpha u_{k,\,j}dx = 0, \tag{13}$$

$$\int_{R^2} u_{i,\,jl}u_{k,\,iml}u_{k,\,jm}dx = 0. \tag{14}$$

Proof of Lemma 2. For the second integral we exchange summation indices j and m. Then

$$2\int_{R^2} w_{i,\,jm}u_{k,\,im}u_{k,\,j}dx = \int_{R^2} w_{i,\,jm}u_{k,\,im}u_{k,\,j}dx + \int_{R^2} w_{i,\,jm}u_{k,\,ij}u_{k,\,m}dx =$$

$$= \int_{R^2} w_{i,\,jm}(u_{k,\,m}u_{k,\,j})_{,i}dx = -\int_{R^2} divw_{,\,jm}(u_{k,\,m}u_{k,\,j})dx = 0.$$

Equality (9) is proved.

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Replace in (9) w onto u. After that we replace u onto $u + tw$, where t is an arbitrary number. Then

$$\int_{R^2} (u_{i,\, jm} + tw_{i,\, jm})(u_{k,\, im} + tw_{k,\, im})(u_{k,\, j} + tw_{k,\, j})dx = 0$$

for any t. Therefore, the coefficient at the first power t vanishes that is

$$\int_{R^2} (w_{i,\, jm}u_{k,\, im}u_{k,\, j} + u_{i,\, jm}w_{k,\, im}u_{k,\, j} + u_{i,\, jm}u_{k,\, im}w_{k,\, j})dx = 0.$$

Therefore, from (9) and (3) it follows (10). Now in (9) we take $w = \Delta u$. Integrating by parts we obtain (11) from (2) where we choose $w = \Delta u,\ v = u$ and u in (2) is replaced by Δu.

If in (9) and (10) we replace w onto $D^\beta w$, u onto $D^\alpha u$, then we get (12) and (13). Formula (14) follows immediately from (11) and (8) since

$$\int_{R^2} \Delta u_{i,\, j}u_{k,\, im}u_{k,\, jm}dx = -\int_{R^2} (u_{i,\, jl}u_{k,\, iml}u_{k,\, jm} + u_{i,\, jl}u_{k,\, im}u_{k,\, jml})dx.$$

Lemma is proved. □

Remark 1. *The integral identity* (9) *is true for spatial solenoidal fields where the integral over plane must be replaced by the integral over whole space.*

Theorem 1. *Let $u,\ v$ be a pair of smooth solenoidal plane fields and one of them is finite. Then*

1) a vector field $g^1 = (g^1_1, g^1_2)$ where

$$g^1_k = u_{i,\, k}\triangle v_i + u_{i,\, kj}v_{i,\, j} + u_{i,\, j}v_{k,\, ij},\ k = 1, 2, \tag{15}$$

is potential;

2) a vector field $g^2 = (g^2_1, g^2_2)$ where

$$g^2_k = u_{k,\, i}\triangle v_i + v_{k,\, i}\triangle u_i + u_{k,\, ij}v_{i,\, j} + u_{i,\, j}v_{k,\, ij},\ k = 1, 2, \tag{16}$$

is potential;

3) a vector field $g^3 = (g^3_1, g^3_2)$ where

$$g^3_k = u_{i,\, k}\triangle v_i + u_{i,\, j}c_{ki,\, j}(v),\ k = 1, 2, \tag{17}$$

is potential;

4) a vector field $g^4 = (g^4_1, g^4_2)$ where

$$g^4_k = c_{ki}(u)\triangle v_i + c_{ki}(v)\triangle u_i,\ k = 1, 2, \tag{18}$$

is potential.

Proof of Theorem 1. Equality (2) we integrate over entire plane and apply integration formula by parts removing derivatives of the field w. Then

$$\int_{R^2} (w_i(u_{k,\, i}\triangle v_k + u_{k,\, ij}v_{k,\, j}) + w_k u_{i,\, j}v_{k,\, ij})dx = 0. \tag{19}$$

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Now, in the first sum we exchange summation indices $i,\ k$. Hence,

$$\int_{R^2} w_k(u_{i,\,k}\triangle v_i + u_{i,\,kj}v_{i,\,j} + u_{i,\,j}v_{k,\,ij})dx = 0$$

for every smooth solenoidal field w. In the fact, we have the identity similar (1). Therefore, from Helmholtz–Weyl decomposition it follows that the vector field $g^1 = (g_1^1, g_2^1)$ defining by (15) is potential.

Equality (2) we integrate again and apply integration formula by parts removing derivatives of the field v. Then we have the following equality:

$$\int_{R^2} v_k(u_{k,\,i}\triangle w_i + u_{k,\,ij}w_{i,\,j} + u_{i,\,j}w_{k,\,ij} + w_{k,\,i}\triangle u_i)dx = 0.$$

Exchanging fields v and w we get

$$\int_{R^2} w_k(u_{k,\,i}\triangle v_i + u_{k,\,ij}v_{i,\,j} + u_{i,\,j}v_{k,\,ij} + v_{k,\,i}\triangle u_i)dx = 0 \tag{20}$$

for every smooth solenoidal field w. Hence, from Helmholtz–Weyl decomposition vector field $g^2 = (g_1^2, g_2^2)$ defining by (16) is potential.

The field from (17) is also potential. It follows from equality

$$g_k^1 = g_k^3 + (u_{i,\,j}v_{i,\,j})_{,k},\ k = 1, 2,$$

and the fact that the field g^1 is potential.

Now, we suppose that a solenoidal field w is smooth and finite. Then (see Theorem 2 from [10]) we have

$$\int_{R^2} w_i c_{ki}(u)\triangle u_k dx = 0, \tag{21}$$

Let us replace in (21) the field u onto the field $u + tv$, where v is an arbitrary smooth solenoidal field and t is any number.Then

$$\int_{R^2} w_i c_{ki}(u + tv)(\triangle(u_k + tv_k)dx \equiv 0$$

for every t. Therefore, a coefficient at the first power t must equal to zero. That is

$$\int_{R^2} w_i(c_{ki}(u)\triangle v_k + c_{ki}(v)\triangle u_k)dx = 0$$

for any smooth solenoidal field w. Then from Helmholtz–Weyl decomposition the vector field $g^4 = (g_1^4, g_2^4)$ defining by formula (18) is potential. Theorem is proved. □

Theorem 2. *Let u be a smooth solenoidal plane field. Then*

1) a vector field $g^5 = (g_1^5, g_2^5)$ where

$$g_k^5 = u_{k,\,ij}\triangle u_{i,\,j} + u_{i,\,jm}u_{k,\,ijm},\ k = 1, 2,$$

is potential;

2) a vector field $g^6 = (g_1^6, g_2^6)$, where

$$g_k^6 = D^\beta(D^\alpha u_{k,\,im}\triangle D^\alpha u_{i,\,j} + D^\alpha u_{i,\,jm}D^\alpha u_{k,\,ijm}),\ k = 1, 2,$$

is potential;

3) a vector field $g^7 = (g_1^7, g_2^7)$*, where*

$$g_k^7 = D^\beta(D^\alpha u_{k,\,i} D^\gamma(\triangle v_i) + D^\alpha u_{k,\,ij} D^\gamma v_{i,\,j} + D^\alpha u_{i,\,j} D^\gamma v_{k,\,ij} + D^\alpha v_{k,\,i} D^\gamma(\triangle u_i)),\ k = 1, 2,$$

is potential.

Proof of Theorem 2. Let w be a smooth solenoidal finite field. In equality (10) it follows to fulfill reiterated integration by parts removing derivatives of the vector field w. Then repeating arguments from proof of Theorem 1 we have necessary statement. The second part we obtain from (13). Potentiality of the field g^7 it follows immediately from (20) if we replace w onto $D^\beta w$, u onto $D^\alpha u$, v onto $D^\gamma v$. Theorem is proved. □

Remark 2. *Integral identities with two solenoidal fields may be very useful for conservation laws because we can take into account an initial velocity in the Cauchy problem for Navier–Stokes equations.*

Remark 3. *Choosing multi–indices* α *and* β *and applying item 3) from Theorem 2 we can construct diverse potential fields summing over* α *and* β*.*

2. Conclusions

There are shown new algebraic and integral identities for plane solenoidal vector fields. Applying them there are offered constructions of a potential field with applications of two solenoidal plane fields.

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