Article

A semi-deterministic random walk with resetting

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Abstract: We consider a discrete-time random walk (x_t) which at random times is reset to the starting position and performs a deterministic motion between them. We show that the quantity $\Pr\Big(x_{t+1} = n+1 | x_t = n\Big), n \to \infty$ determines if the system is averse, neutral or inclined towards resetting. It also classifies the stationary distribution. Double barrier probabilities, first passage times and the distribution of the escape time from intervals are determined.

Keywords: Random walk with resetting, Escape probabilities, Exit times

1. Preliminaries

In a previous paper [1] we introduced the *Sisyphus random walk* as an infinite Markov chain that moves on the space state $\mathbb{N} = \{0, 1, 2, \ldots, \infty\}$ and that at every step can either jump one unit rightward, or return to the initial state, from where it is restarted. The system was named after the king of Ephyra, Sisyphus, who was condemned to lift a heavy stone in an endless cycle.

Here we generalize the above idea and consider a random walk on the integers $(x_t)_{t\in\mathbb{N}}$ whose dynamics alternates deterministic linear motion with resets which drive the system to the starting point at the random times $(\mathbf{t}_n)_{n\geq 1}$. At every clock tick the position of the random walker is such that $|x_t|$ either increases one unit, or returns to the *ground state*, whereupon the evolution continues. Such resetting occurs through an independent mechanism superimposed to the original semi-deterministic evolution. Once (x_t) is reset to the origin at $\mathbf{t}_1$, it begins the evolution anew from scratch, which is deterministic between resets.

Using translational invariance we can suppose that $x_0 = 0$ with no loss of generality. Concretely, starting from $x_0 = 0$, three possibilities open for the future position x_1: the system may remain at $x_1 = 0$ provided a reset occurs at $t = 1$; otherwise, it goes one unit to the right with probability ρ or to the left with probability $\bar{\rho} = 1 - \rho$. In addition, if the system has wandered into the positives so at a certain time $t \geq 0$ is $x_t > 0$ (respectively $x_t < 0$) then at time $t + 1$, it may either be reset to the origin $x_{t+1} = 0$ with arbitrary probability or else increase (respectively, decrease) one unit to $x_{t+1} = x_t + 1$ (respectively $x_{t+1} = x_t - 1$).

Such apparent simplicity is misleading as this simple evolution law can exhibit a surprisingly complex and rich behavior. Indeed, at each site we allow arbitrary probabilities for the random walk to reset to the origin and, additionally, the possibility to move both in the positive and negative integers. The only restriction in this general dynamics is the requirement that $(x_t)_{t\in\mathbb{N}}$ be a Markov chain. The resulting system is a natural, non trivial generalization of that of [1], which is recovered when the reset probability is independent of the location and when $\rho = 1$.

In a different setting such system may be used as an idealized model of the random dynamics of a “mobile" in a trap, say, who is trying to climb stepwise a ladder or wall given that at every step there is a common probability of slipping to the bottom, resulting in the need to restart again. Here, the natural question would be the determination of the location probability and expected time to escape the trap.

A related mechanism–Sisyphus cooling– was proposed by Claude Cohen-Tannoudji in certain optical contexts to the effect that an atom may climb up a potential hill, till suddenly it is returned to some ground state where it can restart anew. The hallmark of such systems is the possibility to display "back-to-square-one" behavior, a feature common in real life systems. Indeed, the study of stochastic processes subject to random resets is a problem that has attracted a great interest in recent years after the seminal work of Manrubia and Zanette [2] and Evans and Majumadar [3]. Presently the dynamics of systems with resets is being subjected to intense study, see the recent review [4]. Other mechanisms for random walks that are suddenly refreshed to the starting position are considered in [5,6,35]. Brownian motion with resets is considered in [3,8] while in [9] the propagator of Brownian motion under time-dependent resetting is obtained (see also [18] for further elaboration). In [10] these ideas are applied to the case of a compound Poisson process with positive jumps and constant drift. Further elaboration appears in [11]. Reset mechanisms have been also thoroughly applied to search strategies in mathematical and physical contexts as well as to behavioral ecology, see [12–18]. Surprisingly, strategies that incorporate reset to pure search are advantageous in certain contexts in ecology and biophysics and molecular dynamics, [19–22]. A generañization of the the Kardar-Parisi-Zhang (KPZ) equation that describes fluctuating interfaces and polymers under resetting is covered in [23]. Dynamical systems with resets have also been used as proxies of the classical integrate-and-fire models of neuron dynamics, see [24,25]. In the context of Lévy flights with resetting see the interesting papers [26,27]. For other applications see also the recent papers [28–35]

As commented the main aim of this paper is to study the main features of the semi-deterministic random walk with resets $(x_t), t = 0, 1, \ldots \infty$. The evolution rules for such random walk are described in section 2. We then study the propensity towards resetting of the system. According to this important property we divide systems as reset averse, neutral or reset-inclined, and characterize them in terms of the transition probabilities and behavior of $\Pr\left(x_{t+1} = n+1 | x_t = n\right), n \to \infty$. In section 3 we study the stationary distribution that the system approaches for large time. Section 4 considers first-passage problems and, in particular, two-sided exit probabilities; concretely, given levels $a, b \in \mathbb{N}$, we study the probabiity that x reaches $a > 0$ before having reached $-b$cand distributions of the escape time. First passage times (FPT) play also a key role in statistical decision models, or to devise optimal strategies for seeking information, the rate at which a Brownian particle under the influence of a metastable potential, escapes from a potential well is also a critical subject in the study of polymers. The so called Kramers problem [36], is a classical subject in statistical physics.

Under the simplest election $\rho = 1$ and $q_n := \Pr\left(x_{t+1} = n+1 | x_t = n\right) = q_1$ constant we have that the distribution of the FPT to level $k \geq 1$ is that of the *number of trials required in an unfair coin-toss to obtain k consecutive successes*, a classical problem in probability. Even with $k = 2$ the distribution of such problem is not trivial.

2. The Model

Here we define the model at hand. Let $x_0 := 0$ be the initis al position. The evolution rules for the random walk $(x_t), t = 0, 1, \ldots \infty$ are as follows. We suppose that, if for any $t \geq 0$ is $x_t = 0$ then the random walk satisfies

$$\Pr\left(x_{t+1} = n | x_t = 0\right) = \bar{q}_1 \delta_{n0} + q_1 \rho \delta_{n1} + q_1 \bar{\rho} \delta_{n,-1}, j \in \mathbb{Z} \tag{1}$$

where we denote $q_1 = \Pr(x_{t+1} \neq 0 | x_t = 0) \in (0,1)$ the probability that, starting form zero, the system moves away from the origin at the next instant and $\rho := \Pr(x_{t+1} = 1 | x_t = 0, x_{t+1} \neq 0) \in [0,1]$ the probability that *if the system abandons the origin*at time

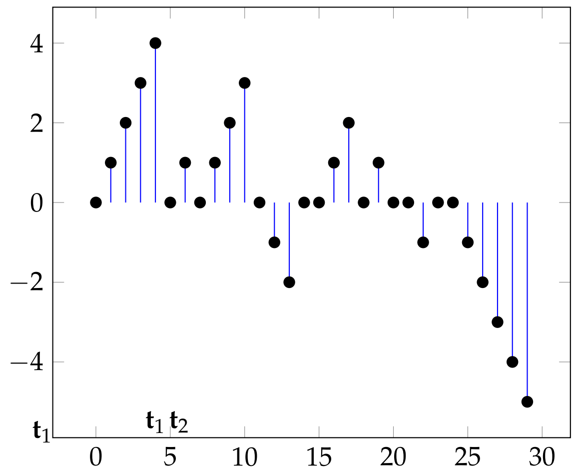

Figure 1. A typical showing sample paths of the process where $\mathbf{t}_1 = 5, \mathbf{t}_2 = 7, \ldots$ and $x_1 = x_6 = 1$

t it goes to position $x_{t+1} = 1$. To ease notation for any value p we set $\bar{p} := 1 - p$, $\bar{q}_1 := 1 - q_1$. Besides δ_{nk} is Kronecker delta.

Further, we suppose that the random walk $(x_t), t = 0, 1, \ldots \infty$ is a Markov chain where if $x_t \equiv n \neq 0$ the only allowed transitions are either to site $n +$ sign (n) if no reset occurs, which happens with probability q_{n+1}; or else to $\{0\}$, when a reset occurs, with probability $1 - q_{n+1}$. Here the sequence (q_n) satisfies that $0 \leq q_n \leq 1$ for all n. It follows that the chain has transition rules

$$\Pr\left(x_{t+1} = m | x_t = n\right) = \begin{cases} q_{n+1}, m = n+1 \\ 1 - q_{n+1},\ m = 0 \end{cases} , t \geq n > 0 \tag{2}$$

$$\Pr\left(x_{t+1} = m | x_t = n\right) = \begin{cases} q_{|n|+1}, m = n-1 \\ 1 - q_{|n|+1},\ m = 0 \end{cases} , t \geq -n > 0 \tag{3}$$

$$\Pr\left(x_{t+1} = m | x_t = 0\right) = \begin{cases} \rho q_1, m = 1 \\ \bar{\rho} q_1, m = -1 \\ 1 - q_1,\ m = 0 \end{cases} \tag{4}$$

and 0 otherwise. We also suppose that the infinite product with general term q_n satisfies

$$\lim_{n\to\infty} \prod_{j=1}^{n} q_j = 0; \text{ alternatively } \sum_{j=1}^{\infty} (1 - q_j) = \infty \tag{5}$$

This mild requirement does not imply that $\lim_{n\to\infty} q_n = 0$ (see (12) below).

The model considered in [1] is recovered assuming $\rho = 1$ and that the jump-probability is constant: $q_1 = q_2 = \ldots q_n = \ldots$.

2.0.1. Reset times

We denote as $\mathbf{t}_1$ the random time at which the first reset happens. Here we consider its distribution probability $p_n := \Pr(\mathbf{t}_1 = n), n = 1, \ldots, \infty$ and other peculiarities of the Sisyphus random walk. Similarly we denote as $\mathbf{t}_k$ the random time at which the $k-^{\text{th}}$ reset happens. To this end note that for $n = 1, 2, \ldots$ the reset takes place at time n if in all previous times no reset has occurred–and so $|x_1| = 1, \ldots, |x_{n-1}| = n - 1$ and $x_n = 0$. Thus we have transitions $\{0\} \mapsto \{1\} \cdots \mapsto \{n-1\} \mapsto \{0\}$ and the correspondnig probability

$$p_n := \Pr(\mathbf{t}_1 = n) = q_1 \ldots q_{n-1} \bar{q}_n \tag{6}$$

which is proper random variable, in view of (5).

The following representation clarifies the meaning and different roles of (p_n) and (q_n)

$$p_n := \Pr(\mathbf{t}_1 = n) = \Pr(\mathbf{t}_t = n) := \Pr\left(x_{t+n} = 0, x_{t+j} \neq 0, 0 < j < n | x_t = 0\right) \tag{7}$$

and

$$\bar{q}_n = \Pr\left(x_{t+n} = 0 | x_t = 0, x_{t+j} \neq 0, 0 < j < n\right) \tag{8}$$

We relate both probabilities. We introduce recursively a sequence (β_n) via $\beta_0 \equiv 1$ and $\beta_n := q_1 \dots q_n, n = 1, 2, \dots$. Note then that

$$p_n = q_1 \dots q_{n-1}\bar{q}_n = \beta_{n-1} - \beta_n$$

This can be inverted as

$$\beta_n = p_{n+1} + p_{n+2} + \dots = \bar{F}_{\mathbf{t}_1}(n) = 1 - F_{\mathbf{t}_1}(n) \tag{9}$$

where$F_{\mathbf{t}_1} \equiv F$ is the cumulative distribution function (cdf) of $\mathbf{t}_1$. Recalling that $\beta_n := q_1 \dots q_n$, we finally have that (6) can be inverted as

$$q_n = \frac{\bar{F}(n)}{\bar{F}(n-1)}, n = 1, 2 \dots \tag{10}$$

2.1. *Reset averse and reset-inclined systems*

One of the most defining traits in the random walk (2)-(4) is what we call *propensity* towards resetting, a measure of how likely is that the resetting mechanism is triggered as the time from the last reset increases. We say that a system is *inclined* towards resetting if such probability grows as the distance to the origin increases: $\bar{q}_n < \bar{q}_{n+1}$, for all n. Intuitively, for a reset-inclined system, the random walk becomes more anxious to return to the origin the greatest the time since the last visit or, alternatively, the farthest off it is. If this probability decreases (respectively, remains unchanged) we say that the system is reset-averse or reset-neutral. Reset-neutral chains correspond to having $q_n = q_{n-1} \equiv q_1 \in (0, 1)$ for all n. This is the choice considered in [1]. In this case

$$\Pr(\mathbf{t}_1 = n) = q_1^{n-1}(1 - q_1), \; F(n) = 1 - q_1^n \tag{11}$$

Actually, we are interested in this property for large n. We say that a system is *ultimately averse, neutral or, respectively, inclined* towards resetting if as the time from the last reset tends to infinity the reset probability (q_n) satisfies

$$\lim_{n\to\infty} q_n := \lim_{n\to\infty} \Pr\left(x_{t+n} \neq 0 | x_t = 0, x_{t+j} \neq 0, 0 < j < n\right) = \begin{cases} 0, \text{ (inclined)} \\ q_\infty \in (0,1) \text{ (neutral)} \\ 1, \text{(averse)} \end{cases} \tag{12}$$

The election $q_n = q_1/n$, corresponds to an ultimately reset-inclined system. Here we have $\lim_{n\to\infty} q_n = 0$ and

$$p_n = \frac{q_1^{n-1}}{(n-1)!} - \frac{q_1^n}{n!}, \; \bar{F}(n) = \frac{q_1^n}{n!}, \; n = 1, 2 \dots \tag{13}$$

A simple calculation yields $< \mathbf{t}_1 >= e^{q_1} \leq e$, which is bounded respect the parameter q_1.

Finally, the choice $q_n = n/(n+1)$ corresponds to a reset-averse system. Here the chain has power law decay tails:

$$p_n = \frac{1}{n(n+1)} \text{ and } \bar{F}(n) = \frac{1}{n+1} \tag{14}$$

The elections (13) and (14) are natural modifications of (11) and reflect that the probability to commit an error that sends the walker to square one diminishes (increases) with every step. This may be put down to a capability to learn or, in contrast, to forget or grow tired with the distance to the origin. (14) corresponds to $q_n = q_{n-1}(1 + \frac{1}{n^2-1})$ –and hence to learning– while if $q_n = q_{n-1}(1 - \frac{1}{n})$ Zipf law (13): $q_n = q_1/n$, follows.

(14) may also account for uncertainty in the relevant parameters. Suppose we accept the basic model (11) to hold but *are ignorant of the value of parameter* q_1. *Besides we accept that all values for* q_1 *are equally likely*; in this situation parameter q_1 should be assumed to have Uniform $(0,1)$ distribution. Bayes theorem implies that the distribution at posteriori of $\mathbf{t}_1$ must be given by (14):

$$\Pr(\mathbf{t}_1 = n) = \int_0^1 \Pr(\mathbf{t}_1 = n|q_1)dq_1 = \int_0^1 dq_1 q_1^{n-1}(1-q_1) = \frac{1}{n(n+1)} \tag{15}$$

We next show that the above behavior is ubiquitous so the reset propensity is directly *related with the tail's behavior*. Indeed, since the sequence $\bar{F}(n)$ is strictly monotone and $\bar{F}(n) \downarrow 0$ as $n \to \infty$ the Stolz-Cesáro theorem gives

$$q_\infty := \lim_{n\to\infty} q_n = \lim_{n\to\infty} \frac{\bar{F}(n)}{\bar{F}(n-1)} = \lim_{n\to\infty} \frac{p_n}{p_{n-1}} \tag{16}$$

Requiring $q_\infty \equiv e^{-\lambda} \in (0,1)$ we obtain that asymptotically (p_n) must grow as

$$p_n \approx ce^{-\lambda n}, c, \lambda > 0,\ n \to \infty \tag{17}$$

which is the paradigmatic example of ultimately neutral systems. Note that such (p_n) has medium tails. By contrast tails of the form

$$p_n \approx ce^{-\lambda n^\alpha},\ n \to \infty \text{ where } c > 0, \lambda > 0, \alpha > 0 \tag{18}$$

give $q_\infty = 1$ if $0 < \alpha < 1$ and $q_\infty = 0$ if $\alpha > 1$. The exponential case $\alpha = 1$, i.e. the geometric distribution, marks the crossover between these cases.

Note that slowly, power law decaying sequences as

$$p_n \approx c/n^\alpha, c > 0, \alpha > 1,\ n \to \infty \tag{19}$$

also correspond to ultimately reset-averse systems. Thus heavy tails of the sequence (p_n) correspond to reset averse systems while the opposite holds with medium and light (super-exponential) tails like those in (11) and (18).

More complicated tails can be handled noting the behavior of ultimately averse, neutral or inclined reset systems under sums and products. We use $q_\infty := \vartheta$ to denote that $\lim_{n\to\infty} q_n \in (0,1)$ (thus $\lim_{n\to\infty} q_n = 0, \vartheta$ or 1). Hence,with obvious notation the sums and product rules for $q_\infty^{(1)}, q_\infty^{(2)}$, say, read

$$0+0=0; 0+\vartheta=\vartheta; 0+1=1; \vartheta+\vartheta=\vartheta; \vartheta+1=1+1=1;$$

$$0\cdot 0 = 0\cdot\vartheta = 0\cdot 1 = 0; \quad \vartheta\cdot\vartheta = \vartheta\cdot 1 = \vartheta; 1\cdot 1 = 1$$

where the symbol $\vartheta\cdot\vartheta = \vartheta$ is used to mean that if

$$\lim_{n\to\infty} q_n^{(1)} \in (0,1),\ \lim_{n\to\infty} q_n^{(2)} \in (0,1),\ \text{then}\ \lim_{n\to\infty} q_n^{(1)} q_n^{(2)} \in (0,1)$$

As an example, for $0 < c < 1$ consider the hybrid system

$$p_n = \frac{n\bar{\nu}+1}{n(n+1)}\nu^{n-1} = O(\frac{e^{-\lambda n}}{n})$$

where $\nu := e^{-\lambda}, \lambda > 0$. Here $p_n \equiv p_n^{(1)} p_n^{(2)}$ and tails display mixed exponential and power-law decay. Hence $q_\infty = q_\infty^{(1)} \cdot q_\infty^{(2)} = \vartheta \cdot 1 = \vartheta$ corresponding to an ultimately neutral system. This corroborated by exact evaluation of q_n. Equation (10) yields that

$$q_n = n\nu/(n+1) \text{ and } \lim_{n\to\infty} q_n = \nu \in (0,1)$$

Table 1: The table summarizes the propensity to resetting in terms of the deacy of $p_n := \Pr(\mathbf{t}_1 = n)$ and the equilibrium distribution. In all cases $\lambda > 0$.

$p_{n\to\infty}$	$F_{\mathbf{t}_1}(n)$	$q_{n\to\infty}$	Propensity	Tails	$\mathbb{E}\mathbf{t}_1$	π_{n}
$O(e^{-\lambda n^\alpha}), \alpha > 1$	$O(e^{-\lambda n^\alpha})$	0	inclined	Super-exp.	$< \infty$	$O($
$O((\frac{e^{-\lambda}}{n})^n,$	$O((\frac{e^{-\lambda}}{n})^n,$	0	inclined	Super-exp.	$< \infty$	$O($
$O(e^{-\lambda n})$	$O(e^{-\lambda n})$	$\in (0,1)$	neutral	exp.	$< \infty$	$O($
$O(e^{-\lambda n^\alpha}), 0 < \alpha < 1$	$O(e^{-\lambda n^\alpha}), 0 < \alpha < 1$	1	averse	Sub-exp.	$< \infty$	$O($
$O(1/n^\alpha),\ \alpha > 2$	$O(1/n^{\alpha-1}),\ \alpha > 2$	1	averse	Power-law	$< \infty$	$O($
$O(1/n^\alpha),\ 1 < \alpha \le 2$	$O(1/n^{\alpha-1}),\ 1 < \alpha \le 2$	1	averse	Power-law	$= \infty$	—

3. Equilibrium distribution

Here we consider the large time or equilibrium distribution of the random walk. Call $x_\infty \equiv \lim_{t\to\infty} x_t$ the limit of the process and $\pi_n := \Pr(x_\infty = n), n \in \mathbb{Z}$ its distribution. When it exists (π_n) has the remarkable porperty that it is an equilibrium state, in the sense that it has initially this distribution then it will not abandon it. (π_m) satisfies the system

$$\sum_{n\in\mathbb{Z}} g_{nm}\pi_n = \pi_m,\ m \in \mathbb{Z} \tag{20}$$

where (g_{nm}) is the transition probability matrix defined in (3):

$$g_{nm} := \Pr\left(x_{t+1} = m | x_t = n\right) \tag{21}$$

To handle this we divide the matrix in upper and lower parts, connected only by the column and rows with index 0, i.e.

$$G = \begin{pmatrix} G_- & 0 \\ 0 & G_+ \end{pmatrix} \tag{22}$$

where G_- is essentially obtained from G_+ by reflection and $G_{+,nm}, n, m = 0, \infty$ reads (including the $0-$ column)

$$G_+ = \begin{pmatrix} \bar{q}_1 & \rho q_1 & 0 & 0 & 0 & \dots \\ \bar{q}_2 & 0 & q_2 & 0 & 0 & \dots \\ \bar{q}_3 & 0 & 0 & q_3 & 0 & \dots \\ \dots & & \dots & & & \\ \bar{q}_n & 0 & \dots & 0 & 0 & q_n \end{pmatrix} \tag{23}$$

By insertion we find

$$\pi_1 = \rho q_1 \pi_0,\ \pi_{-1} = \bar{\rho} q_1 \pi_0$$

along with the recursive system

$$\pi_{n+1} = q_{n+1}\pi_n, n \geq 1 \text{ and } \pi_{n-1} = q_{|n|-1}\pi_n, n \leq -1 \tag{24}$$

Solving recursively we find

$$\pi_n = \rho\pi_0 q_1 q_2 \dots q_n = \pi_0\rho\bar{F}(n) \text{ and } \pi_{-n} = \pi_0\bar{\rho}\bar{F}(n), n \geq 1 \tag{25}$$

Normalization gives $1/\pi_0 = \sum_{n=0}^{\infty} np_n \equiv < \mathbf{t}_1 > \equiv \mu$. which requires $< \mathbf{t}_1 > \equiv \mu < \infty$, i.e. (p_n) must decay at least as $p_n \approx 1/n^r, r > 2$. In this case, letting $\rho_n \epsilon \rho \mathbf{1}_{n>0} + \bar{\rho}\mathbf{1}_{n<0} + \delta_{n0}$, the stationary distribution is

$$\pi_n = (\rho_n/\mu)\bar{F}(|n|), n = -\infty, \dots, \infty \tag{26}$$

The probability that the random walk has drifted to site n for large time decreases as $\bar{F}(n)$ does, see Table (1). Note that in no case our Markov process satisfies the "detailed balance" condition for the stationary distribution: $\pi_m g_{nm} \neq \pi_n g_{mn}$. This was to be expected since detailed balance guarantees time-reversibility, a trait that the system at hand clearly does not exhibit.

For the cases (11), (13) we have

$$\pi_n = \rho_n(1-q_1)q_1^{|n|} \text{ and } \pi_n = \rho_n e^{q_1}\frac{q_1^{|n|}}{|n|!}, \; n = -\infty, \dots 0, \dots, \infty \tag{27}$$

Finally, for system (14) there is neither equilibrium nor stationary distribution, indicating that the chain spreads out far from the origin and it does not settle to an equilibrium.

4. Escape probabilities

In a classical study W. Feller [37] showed that most recurrent properties of general diffusion processes can be codified in terms of two of the functions that define escape probabilities from an interval $(c,d), c < d$. Given that the process has started from a general $x_0, c < x_0 < d$. Feller considers the "scale and speed functions", defined as

$$s(x_0) = \Pr(\tau_d > \tau_c) \text{ and } m(x_0) \equiv < \tau_{c,d} > \tag{28}$$

and shows that they solve certain differential equations (see [37] for an overview). Here, for any $a \in \mathbb{R}$ we introduce the "hitting time" $\tau^a = \inf\{t > 0 : x_t = a\}$ which represents the lapse of time necessary to travel from the starting value to a; besides $\tau_{c,d} := \min\{\tau_c, \tau_d\}$ is the escape time from the interval (c,d).

We perform a similar study here and determine, for given levels $a, b \in \mathbb{N}, -b < 0 < a$, the probability that the random walk (x_t) reaches $a > 0$ before having reached $-b$. Note that by translational invariance the case when (x) starts from general x_0 is immediately reduced to that with $x_0 = 0$.

We start noting that when resets are switched off the only source of randomness lies in the first displacement of the random walk away from $x = 0$; hence, $x_n = n$ for all n if $x_1 = 1$. In this case $\tau_{a,b}$–the minimum time to hit either $a > 0$ or $-b < 0$- is a binary random variable that takes values a and b with probabilities ρ and $\bar{\rho}$. Besides $\Pr_0(\tau_a < \tau_b) = \rho$.

Obviously $\tau_{a,b}$ will increase when a reset mechanism is introduced; it is tempting to think that however *resets do not affect the escape probabilities*, namely $\Pr_0(\tau_a < \tau_b) = \rho$ still holds. However this is not correct!. To dispel such misinterpretation note tath *resets introduce a bias which favors the closest barrier* against the farthest one. This is similar to the classical waiting time paradox where cycles with very large inter-reset times have a greater probability than smaller ones. Intuitively, if restarts occur very often the possibility to reach the farthest barrier diminishes. We now determine this probability.

A very simple argument goes as follows. Consider the probability ℓ_a that the random walk (x) reaches $a > 0$ before having reached $-b$ *when we know that* (x_t) *hits* a *or* $-b$ *in the given cycle*. The event that escape occurs at a given cycle, the first, say, is

$$E := \{x_1 > 0, \mathbf{t}_1 > a\} \cup \{x_1 < 0, \mathbf{t}_1 > b\} \equiv E_1 \cup E_2 \tag{29}$$

with probability

$$\kappa \equiv \Pr(E) = \rho\bar{F}(a) + \bar{\rho}\bar{F}(b) \tag{30}$$

Hence the probability that escape occurs via the upper barrier is can be evaluated as the probability of E_1 conditional on E having happened.

$$\ell_a = \Pr(\text{ escape via } a|E) = \Pr(E_1|E_1 \cup E_2) = \frac{\rho\bar{F}(a)}{\kappa} \tag{31}$$

The reasoning when escape occurs at a general given cycle is a bit more involved but does not change the result.

Denote $\ell_a^0 \equiv \rho$ the corresponding probability when no resets are introduced. Then

$$\ell_a \geq \ell_a^0 \Leftrightarrow \bar{F}(b) \leq \bar{F}(a) \Leftrightarrow b \geq a \tag{32}$$

which means that *resets increase the probability to hit first the closest barrier*, as expected. Further, when $a = b$ (31) yields $\ell_a = \ell_a^0$.

We thus have for the neutral chain (11), the reset-averse chain (13) and the reset-inclined chain (14), respectively

$$\ell_a = \begin{cases} \rho\left(\rho + \bar{\rho}q_1^{b-a}\right)^{-1} \\ \rho b!\left(b!\rho + \bar{\rho}a!q_1^{b-a}\right)^{-1} \\ \left(1 + \frac{\bar{\rho}(a+1)}{\rho(b+1)}\right)^{-1} \end{cases} = \begin{cases} \rho\left(\rho + \bar{\rho}q_1^{d}\right)^{-1} \\ \rho\left(\rho + \bar{\rho}q_1^{d}a!/(a+d)!\right)^{-1} \\ \left(1 + \frac{\bar{\rho}(a+1)}{\rho(a+d+1)}\right)^{-1} \end{cases} \tag{33}$$

In the second equality we introduce $d := b - a$, which measures the departure from symmetry of the problem and suppose $b \geq a$ for ease of notation. Figure 2 plots $\Pr(A)$ versus d.

5. Escape times

Here we consider the distribution of $\tau_{a,b} \equiv \min\{\tau_b, \tau_a\}$ for general levels a, b and a general model (2)-(4).

5.1. Symmetry properties of First passage times

Denote for a moment as $\tau_{a,b}^{\rho}$ the FPT to either a or $-b$ when $\Pr(x_{t+1} = 1|x_t = 0, x_{t+1} \neq 0) = \rho$. This quantity $\tau_{a,b}$ has a nice interpretation. Suppose the model (11) holds. Say *a success has been scored every time a reset does not happen*. Then $\tau_{n,n}^1 = \tau_n^1$ is the time that takes to get $n \geq 1$ successes in a row provided the probability of individual success is q_1 a classical problem in probability. Obviously, if $n = 1$ then τ_n^1–the first time to reach level 1– must have a geometric distribution with parameter q_1. However, even with $n = 2$ this problem has no easy solution, not even for the mean times.

We note the interesting relation between the asymmetric and symmetric cases.

1. If l_a is defined in in (31) and $l_a + l_b = 1$ and $\mathbb{E}X \equiv < X >$ indicates the expected value of the random variable X we have

$$\mathbb{E}\tau_{a,b}^{\rho} = l_a\mathbb{E}\tau_{a,a}^{\rho} + l_b\mathbb{E}\tau_{b,b}^{\rho} \tag{34}$$

2. $\tau^{\rho}_{a,a}$ is independent of ρ. Besides the distributions in the symmetric case and one sided case are equal, namely, for any b

$$\tau^{\rho}_{a,a} = \tau^{1}_{a,b} = \tau^{1}_{a,\infty} \equiv \tau^{1}_{a} \,; \tau^{\rho}_{a,\infty} = \tau^{\rho}_{a} \tag{35}$$

Indeed when the interval is symmetric the escape time will not be influenced by whether resets favor upward or downward flights; hence (35) must hold. For the sake of comparison we see that (38) overestimates the time that takes to reach the boundaries.

A first approximation is given by $< \tau_{a,b} >\approx< \mathbf{N} > \times < \mathbf{t}_1 >$ where $\mathbf{N}$ is the number of cycles until escape. To warm up we consider first the distribution of $\mathbf{N}$.

By independence of cycles $\mathbf{N}$ has a geometric distribution with exit parameter $\kappa := \Pr(E)$ where E, κ are defined in (29), (30). Thus we have $\mathbf{N}$ ∼Geom (κ):

$$\Pr(\mathbf{N} = n) = \kappa(1-\kappa)^{n-1}, n = 1, 2 \ldots \text{ and} \tag{36}$$

$$< \mathbf{N} >= (1/\kappa), < \tau_{a,b} >\approx< \mathbf{N} > \times < \mathbf{t}_1 >= \frac{\sum_{n=1}^{\infty} n p_n}{\rho \bar{F}(a) + \bar{\rho}\bar{F}(b)} \tag{37}$$

In particular for the symmetric case $a = b$

$$< \tau_{a,b} >\approx \Big(\sum_{n=1}^{\infty} n p_n\Big)/(\sum_{n=a+1}^{\infty} p_n) \geq a + \Big(\sum_{n=1}^{a} n p_n\Big)/(\sum_{n=a+1}^{\infty} p_n) \tag{38}$$

Clearly this approximation is only reasonable when the system needs a large number of cycles to exit the interval, i.e. $\kappa \approx 0$.

5.2. Mean exit time

To study the exact time to hit a or b we note that depending of what happens at the the first reset $\mathbf{t}_1$ there are five excluding and exhausting possibilities. These scenarios are

- (S1) $x_1 > 0$ and $\mathbf{t}_1 > a$
- (S2) $x_1 < 0$ and $\mathbf{t}_1 > b$.
- (S3) $x_1 > 0$ and $\mathbf{t}_1 \leq a$.
- (S4) corresponds to having $x_1 < 0$ and $\mathbf{t}_1 \leq b$.
- (S5) corresponds to $x_1 = 0$.

Under scenario (S1) (x_t) hits a before it hits b with $\tau_{a,b} = a$. Under scenario (S2) (x_t) hits b before a and $\tau_{a,b} = b$. Scenarios $(S3)$ to $(S5)$ refresh (x_t) to the origin and the "race" starts again from scratch, so $\tau_{a,b} = \mathbf{t}_1 + \tau'_{a,b}$ where $\tau'_{a,b}$ is the time that remains until exit once the new cycle starts. This implies that

$$\tau_{a,b} = \begin{cases} a \text{ if } x_1 > 0, \mathbf{t}_1 > a \\ b \text{ if } x_1 < 0, \mathbf{t}_1 > b \\ \mathbf{t}_1 + \tau'_{a,b} \text{ if } 2 \leq \mathbf{t}_1 \leq a, x_1 > 0 \text{ or } 2 \leq \mathbf{t}_1 \leq b, x_1 < 0 \text{ or } \mathbf{t}_1 = 1 \end{cases} \tag{39}$$

$$\text{and } \mathbb{E}\tau_{a,b} = a\rho\bar{F}(a) + b\bar{\rho}\bar{F}(b) +$$
$$\rho\mathbb{E}(\mathbf{t}_1 \mathbf{1}_{\mathbf{t}_1 \leq a}) + \bar{\rho}\mathbb{E}(\mathbf{t}_1 \mathbf{1}_{\mathbf{t}_1 \leq b}) + (\bar{\rho}F(b) + \rho F(a))\mathbb{E}\tau'_{a,b}$$

Thus we finally get

$$\mathbb{E}\tau_{a,b} = \frac{\rho\big(a\bar{F}(a) + \mathbb{E}(\mathbf{t}_1 \mathbf{1}_{\mathbf{t}_1 \leq a})\big) + \bar{\rho}\big(b\bar{F}(b) + \mathbb{E}((\mathbf{t}_1 \mathbf{1}_{\mathbf{t}_1 \leq b})\big)}{\bar{\rho}\bar{F}(b) + \rho\bar{F}(a)} \tag{40}$$

If $b \to \infty$ then $b\bar{F}(b) \to 0$ and we recover the mean hitting time to level a as

$$\mathbb{E}\tau_a = a + \frac{1}{\rho\bar{F}(a)}\left(\mathbb{E}(\mathbf{t}_1) - \rho\mathbb{E}(\mathbf{t}_1 \mathbf{1}_{\mathbf{t}_1 > a})\right) \tag{41}$$

Particularly interesting is the symmetric case $a = b$. Here

$$\mathbb{E}\tau^{\rho}_{a,a} = \mathbb{E}\tau^{1}_{a} = a + \frac{\mathbb{E}(\mathbf{t}_1 \mathbf{1}_{\mathbf{t}_1 \le a})}{\bar{F}(a)} = a + \left(\sum_{n=1}^{a} n p_n\right)/\left(\sum_{n=a+1}^{\infty} p_n\right) \tag{42}$$

Note how this implies (34).

5.3. Distribution of the exit time

Finally we consider the distribution of $\tau_{a,b}$. We evaluate its generating function

$$G(z) = \sum_{n=1}^{\infty} z^n \Pr(\tau_{a,b} = n) \tag{43}$$

by using (39). Here $z \in \mathbb{C}, |z| \le 1$. Recall that

$$\tau_{a,b} = a\mathbf{1}_{x_1>0,\mathbf{t}_1>a} + b\mathbf{1}_{x_1<0,\mathbf{t}_1>b} + \left(\mathbf{t}_1 + \tau'_{a,b}\right)$$
$$\left(\mathbf{1}_{2\le\mathbf{t}_1\le a, x_1>0} + \mathbf{1}_{2\le\mathbf{t}_1\le b, x_1<0} + \mathbf{1}_{\mathbf{t}_1=1}\right) \tag{44}$$

Note also that

$$\mathbb{E}\left(z^{\mathbf{t}_1+\tau'_{a,b}}\mathbf{1}_{2\le\mathbf{t}_1\le a, x_1>0}\right) = \mathbb{E}\left(z^{\mathbf{t}_1}\mathbf{1}_{2\le\mathbf{t}_1\le a, x_1>0}\right)\mathbb{E}\left(z^{\tau_{a,b}}\right) = \rho\hat{p}^a(z)G_{\tau_{a,b}}(z)$$

where we define the truncated generating function $\hat{p}^a(z) = \sum_{k=1}^{a} z^k p_k$.

It follows from (39) that $G_{\tau_{a,b}}(z)$ is the sum of the following terms

$$G_{\tau_{a,b}}(z) = E_1 + E_2 G_{\tau'_{a,b}}(z) \tag{45}$$

where

$$E_1 := z^a \Pr(x_1 > 0, \mathbf{t}_1 > a) + z^b \Pr(x_1 < 0, \mathbf{t}_1 > b) = z^a\rho\bar{F}(a) + z^b\bar{\rho}\bar{F}(b),$$
$$E_2 := \mathbb{E}\left(z^{\mathbf{t}_1}\mathbf{1}_{\mathbf{t}_1\le a, x_1>0}\right) + \mathbb{E}\left(z^{\mathbf{t}_1}\mathbf{1}_{\mathbf{t}_1\le b, x_1<0}\right) + \mathbb{E}\left(\mathbf{1}_{\mathbf{t}_1=1}\right)$$

Thus finally, in Laplace space, the generating functions reads

$$G_{\tau_{a,b}}(z) = \frac{z^a\rho\bar{F}(a) + z^b\bar{\rho}\bar{F}(b)}{1 - \rho\hat{p}^a(z) - \bar{\rho}\hat{p}^b(z)} \tag{46}$$

Hence the mass function of $\tau_{a,b}$ is

$$\mathbb{P}(\tau_{a,b} = n) = \frac{\bar{F}(a)}{2\pi i}\oint dz \frac{G_{\tau_{a,b}}(z)}{z^{n+1}}, n \ge 1 \tag{47}$$

If either $b = a$ (symmetric case) or $\rho = 1, b = 1$ (one sided case) it simplifies to

$$G_{\tau_{a,a}}(z) = \frac{z^a\bar{F}(a)}{1 - \hat{p}^a(z)} \tag{48}$$

$$\mathbb{P}(\tau_{a,a} = n) = \frac{\bar{F}(a)}{2\pi i}\oint dz \frac{dz}{z^{n+1-a}(1 - \hat{p}^a(z))}, n \ge a \tag{49}$$

The FPT to a is recovered letting $b \to \infty$; then $\hat{p}^b(z) \to \hat{p}(z) := \sum_{n=1}^{\infty} z^n p_n$ and

$$G_{\tau_a}(z) = \frac{z^a \rho \bar{F}(a)}{1 - \rho \hat{p}^a(z) - \bar{\rho}\hat{p}(z)} \tag{50}$$

5.4. FPT under the model (11)

If equation(11) holds the distribution of $\tau_{a,a}$ simplifies. The generating function and distribution of the exit time read $\hat{p}^a(z) = \bar{q}_1 z(1-(zq_1)^a)/(1-zq_1)$ and

$$G_{\tau_{a,a}}(z) = \frac{(q_1 z)^a(1-q_1 z)}{1 - z + q_1^a \bar{q}_1 z^{a+1}} \tag{51}$$

Hence when $a = 1$ we recover $G_{\tau_{1,1}}(z) = q_1 z/(1-\bar{q}_1 z)$ corresponding to a geometric distribution. Note

$$\Pr(\mathbf{t}_1 = n) = q_1^{n-1}(1-q_1), \ \Pr(\tau_{1,1} = n) = \bar{q}_1^{n-1} q_1 \tag{52}$$

For $a = 2$ we have

$$G_{\tau_{2,2}}(z) = \frac{(q_1 z)^2}{1 - \bar{q}_1 z - q_1 \bar{q}_1 z^2} \tag{53}$$

If $s_\pm := \bar{q}_1 \pm \sqrt{\bar{q}_1^2 + 4q_1\bar{q}_1}$ this can be inverted as

$$\mathbb{P}(\tau_{2,2} = n) = q_1^2 \sum_{j=0}^{n-2} \binom{n-2-j}{j} q_1^j \bar{q}_1^{n-2-j} = \frac{q_1^2\left(s_+^{n-1} - s_-^{n-1}\right)}{2^{n-2}(s_+ - s_-)} \tag{54}$$

Hence summing an arithmetic-geometric series we find if $\ell = 1/q_1$

$$\mathbb{E}\tau_{a,a} = a + \frac{1}{q_1^a \bar{q}_1^2}\left(q_1(1 - q_1^{a+1} - \bar{q}_1(a+1)q_1^{a+1}\right) = \frac{\ell^a - 1}{\ell - 1} \tag{55}$$

Let ξ the number of trials until the first consecutive a successes occur in a sequence of of Bernouilli trials with probability of individual success q_1. This problem does not have a simple answer except when $a = 1$. Here $\xi \sim \text{Geom}(q_1)$.

To handle the case $a \geq 2$ we note that the distribution of ξ is that of the FPT to a with $\rho = 1$ and is recovered letting $b \to \infty$ i.e. (see (35)) and using (51)

$$\xi = \tau^1_{a,\infty} = \tau^1_a = \tau^\rho_{a,a} \text{ and } G_\xi(z) = \frac{(q_1 z)^a(1-q_1 z)}{1 - z + q_1^a \bar{q}_1 z^{a+1}} \tag{56}$$

6. Discussion

We have considered a discrete-time random walk (x_t) which at random times is reset to the starting position and performs a deterministic motion between them. We have discussed how to interpret the property that the system be averse, neutral or inclined towards resetting. We show that such behavior is critical for the existence and properties of the stationary distribution. We obtained double barrier probabilities, first passage times and the distribution of the escape time from intervals. We pointed out that the distribution of the FPT to level $k \geq 1$ solves a , a classical problem in probability, namely that of the number of trials required in an unfair coin-toss or Bernuilli trial to obtain k successes in a row.

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