Fibonacci Power Sums

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Abstract

We evaluate various sums involving the powers of Fibonacci and Lucas numbers.

1 Introduction

Our main goal in this paper is to evaluate the following sums of powers of Fibonacci and Lucas numbers:

$$\sum_{k=0}^n (\pm 1)^k F_{j(rk+s)}^{2m}, \quad \sum_{k=0}^n (\pm 1)^k L_{j(rk+s)}^{2m},$$

$$\sum_{k=0}^n (\pm 1)^k F_{j(2rk+s)}^{2m+1}, \quad \sum_{k=0}^n (\pm 1)^k L_{j(2rk+s)}^{2m+1};$$

thereby extending the work of Ozeki [4], Prodinger [5], Kiliç et al [2] and several previous researchers. Here n is any non-negative integer, j , m , r and s are any integers and F_t and L_t are the Fibonacci and Lucas numbers.

The Fibonacci numbers, F_n , and the Lucas numbers, L_n , are defined, for $n \in \mathbb{Z}$, through the recurrence relations

$$F_n = F_{n-1} + F_{n-2}, \quad (n \geq 2), \quad F_0 = 0, \quad F_1 = 1; \quad (1)$$

and

$$L_n = L_{n-1} + L_{n-2}, \quad (n \geq 2), \quad L_0 = 2, \quad L_1 = 1; \quad (2)$$

with

$$F_{-n} = (-1)^{n-1} F_n, \quad L_{-n} = (-1)^n L_n. \quad (3)$$

Throughout this paper, we denote the golden ratio, $(1 + \sqrt{5})/2$, by α and write $\beta = (1 - \sqrt{5})/2 = -1/\alpha$, so that $\alpha\beta = -1$ and $\alpha + \beta = 1$.

Explicit formulas (Binet formulas) for the Fibonacci and Lucas numbers are

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \quad L_n = \alpha^n + \beta^n, \quad n \in \mathbb{Z}. \quad (4)$$

Koshy [3] and Vajda [6] have written excellent books dealing with Fibonacci and Lucas numbers.

2 Required identities

Lemma 1 (Adegoke [1, Lemma 1]). *For real or complex z , let a given well-behaved function $h(z)$ have, in its domain, the representation $h(z) = \sum_{k=c_1}^{c_2} g(k)z^{f(k)}$ where $f(k)$ and $g(k)$ are given real sequences and $c_1, c_2 \in [-\infty, \infty]$. Let j be an integer. Then,*

$$\sum_{k=c_1}^{c_2} g(k)z^{f(k)} F_{jf(k)}^m = \frac{1}{(\sqrt{5})^m} \sum_{i=0}^m (-1)^i \binom{m}{i} h(\beta^{ij} \alpha^{(m-i)j} z), \quad (F)$$

$$\sum_{k=c_1}^{c_2} g(k)z^{f(k)} L_{jf(k)}^m = \sum_{i=0}^m \binom{m}{i} h(\beta^{ij} \alpha^{(m-i)j} z). \quad (L)$$

Since $\beta^i \alpha^{m-i} = (-1)^i \alpha^{m-2i}$, identities (F) and (L) can also be written as

$$\sum_{k=c_1}^{c_2} g(k)z^{f(k)} F_{jf(k)}^m = \frac{1}{(\sqrt{5})^m} \sum_{i=0}^m (-1)^i \binom{m}{i} h((-1)^{ij} \alpha^{(m-2i)j} z), \quad (F')$$

$$\sum_{k=c_1}^{c_2} g(k)z^{f(k)} L_{jf(k)}^m = \sum_{i=0}^m \binom{m}{i} h((-1)^{ij} \alpha^{(m-2i)j} z). \quad (L')$$

Lemma 2. *For a real or complex variable z , integers $m \geq 0$, j , r , s , n ,*

$$\sum_{k=0}^n z^k F_{j(rk+s)}^m = \frac{1}{(\sqrt{5})^m} \sum_{i=0}^m (-1)^{i(js+1)} \binom{m}{i} \alpha^{(m-2i)js} \left(\frac{(-1)^{jri(n+1)} \alpha^{jr(m-2i)(n+1)} z^{n+1} - 1}{(-1)^{jri} \alpha^{jr(m-2i)} z - 1} \right), \quad (PF)$$

$$\sum_{k=0}^n z^k L_{j(rk+s)}^m = \sum_{i=0}^m (-1)^{ijs} \binom{m}{i} \alpha^{(m-2i)js} \left(\frac{(-1)^{jri(n+1)} \alpha^{jr(m-2i)(n+1)} z^{n+1} - 1}{(-1)^{jri} \alpha^{jr(m-2i)} z - 1} \right). \quad (PL)$$

Proof. Consider the geometric progression summation identity

$$h(z) = \sum_{k=0}^n x^k z^{rk+s} = \frac{x^{n+1} z^{rn+r+s} - z^s}{z^r x - 1}.$$

Identify

$$g(k) = x^k, \quad f(k) = rk + s, \quad c_1 = 0, \quad c_2 = n;$$

and note that

$$h((-1)^{ij} \alpha^{(m-2i)j} z) = \frac{x^{n+1} (-1)^{ij(rn+r+s)} \alpha^{j(m-2i)(rn+r+s)} z^{rn+r+s} - (-1)^{ijs} \alpha^{(m-2i)js} z^s}{x(-1)^{ijr} \alpha^{(m-2i)jr} z^r - 1}.$$

Use these in (F') to obtain

$$\begin{aligned} & \sum_{k=0}^n x^k z^{rk} F_{j(rk+s)}^m \\ &= \frac{1}{(\sqrt{5})^m} \sum_{i=0}^m \frac{(-1)^i \binom{m}{i} (-1)^{ijs} \alpha^{(m-2i)js} ((-1)^{ij(rn+r)} x^{n+1} z^{rn+r} \alpha^{j(m-2i)(rn+r)} - 1)}{x(-1)^{ijr} \alpha^{(m-2i)jr} z^r - 1}; \end{aligned}$$

from which identity (PF) follows when we write z for xz^r . The proof of (PL) is similar. $\square$

It is sometimes convenient to use the (α vs β) version of identities (PF) and (PL):

$$\sum_{k=0}^n z^k F_{j(rk+s)}^m = \frac{1}{(\sqrt{5})^m} \sum_{i=0}^m \frac{(-1)^i \binom{m}{i} \beta^{ijs} \alpha^{(m-i)js} (\beta^{ijr(n+1)} \alpha^{jr(m-i)(n+1)} z^{n+1} - 1)}{\beta^{ijr} \alpha^{jr(m-i)} z - 1}, \quad (\text{PF}')$$

$$\sum_{k=0}^n z^k L_{j(rk+s)}^m = \sum_{i=0}^m \frac{\binom{m}{i} \beta^{ijs} \alpha^{(m-i)js} (\beta^{ijr(n+1)} \alpha^{jr(m-i)(n+1)} z^{n+1} - 1)}{\beta^{ijr} \alpha^{jr(m-i)} z - 1}. \quad (\text{PL}')$$

Lemma 3. Let a, b, c and d be rational numbers and λ an irrational number. Then,

$$a + \lambda b = c + \lambda d \iff a = c, \quad b = d.$$

Lemma 4. If m is an integer and $(f(i))$ a real sequence, then,

$$\sum_{i=0}^{2m} f(i) = f(m) + \sum_{i=0}^{m-1} (f(i) + f(2m-i)), \quad (5)$$

$$\sum_{i=0}^{2m+1} f(i) = f(2m+1) - f(m) + \sum_{i=0}^m (f(i) + f(2m-i)). \quad (6)$$

In particular, if $f(2m-i) = f(i)$, then,

$$\sum_{i=0}^{2m} f(i) = f(m) + 2 \sum_{i=0}^{m-1} f(i), \quad (7)$$

$$\sum_{i=0}^{2m+1} f(i) = f(2m+1) - f(m) + 2 \sum_{i=0}^m f(i). \quad (8)$$

Lemma 5. For p and q integers,

$$1 + (-1)^p \alpha^{2q} = \begin{cases} (-1)^p \alpha^q F_q \sqrt{5}, & \text{if } p \text{ and } q \text{ have different parity;} \\ (-1)^p \alpha^q L_q, & \text{if } p \text{ and } q \text{ have the same parity.} \end{cases} \quad (9)$$

$$1 - (-1)^p \alpha^{2q} = \begin{cases} (-1)^{p-1} \alpha^q L_q, & \text{if } p \text{ and } q \text{ have different parity;} \\ (-1)^{p-1} \alpha^q F_q \sqrt{5}, & \text{if } p \text{ and } q \text{ have the same parity.} \end{cases} \quad (10)$$

Proof. We have

$$\begin{aligned} (-1)^{p+q} + (-1)^p \alpha^{2q} &= \alpha^{p+q} \beta^{p+q} + \alpha^{p+2q} \beta^p \\ &= \alpha^{p+q} \beta^p (\alpha^q + \beta^q) \\ &= (-1)^p \alpha^q L_q. \end{aligned} \quad (11)$$

Similarly,

$$(-1)^{p+q} - (-1)^p \alpha^{2q} = (-1)^{p-1} \alpha^q F_q \sqrt{5}. \quad (12)$$

□

Corresponding to (11) and (12) we have

$$(-1)^{p+q} + (-1)^p \beta^{2q} = (-1)^p \beta^q L_q \quad (13)$$

and

$$(-1)^{p+q} - (-1)^p \beta^{2q} = (-1)^p \beta^q F_q \sqrt{5}. \quad (14)$$

Identities (11), (12), (13) and (14) imply

$$(-1)^q + \alpha^{2q} = \alpha^q L_q, \quad (15)$$

$$(-1)^q - \alpha^{2q} = -\alpha^q F_q \sqrt{5}, \quad (16)$$

$$(-1)^q + \beta^{2q} = \beta^q L_q, \quad (17)$$

$$(-1)^q - \beta^{2q} = \beta^q F_q \sqrt{5}. \quad (18)$$

3 Results

Theorem 1. For m, j, r, s, n integers, m non-negative,

$$\sum_{k=0}^n F_{j(rk+s)}^{2m} = \begin{cases} 5^{-m} \sum_{i=0}^{m-1} (-1)^{i(jrn+js+1)} \binom{2m}{i} \frac{F_{jr(m-i)(n+1)}}{F_{jr(m-i)}} L_{(m-i)(jrn+2js)} & (jmr \text{ even}); \\ \quad + (-1)^{m(js+1)} 5^{-m} \binom{2m}{m} (n+1) \\ 5^{-m} \sum_{i=0}^{m-1} (-1)^{i(s+1)} \binom{2m}{i} \frac{L_{jr(m-i)(n+1)}}{L_{jr(m-i)}} L_{(m-i)(jrn+2js)} & (jmr \text{ odd, } n \text{ even}); \\ \quad + (-1)^{(s+1)} 5^{-m} \binom{2m}{m} \\ 5^{-m+1} \sum_{i=0}^{m-1} (-1)^{ijs} \binom{2m}{i} \frac{F_{jr(m-i)(n+1)}}{L_{jr(m-i)}} F_{(m-i)(jrn+2js)} & (jmr \text{ odd, } n \text{ odd}); \end{cases} \quad (M1)$$

$$\sum_{k=0}^n L_{j(rk+s)}^{2m} = \begin{cases} \sum_{i=0}^{m-1} (-1)^{i(jrn+js)} \binom{2m}{i} \frac{F_{jr(m-i)(n+1)}}{F_{jr(m-i)}} L_{(m-i)(jrn+2js)} & (jmr \text{ even}); \\ \quad + (-1)^{mjs} \binom{2m}{m} (n+1) \\ \sum_{i=0}^{m-1} (-1)^{is} \binom{2m}{i} \frac{L_{jr(m-i)(n+1)}}{L_{jr(m-i)}} L_{(m-i)(jrn+2js)} & (jmr \text{ odd}, n \text{ even}); \\ \quad + (-1)^{(m+s+1)} \binom{2m}{m} \\ 5 \sum_{i=0}^{m-1} (-1)^{i(js+1)} \binom{2m}{i} \frac{F_{jr(m-i)(n+1)}}{L_{jr(m-i)}} F_{(m-i)(jrn+2js)} & (jmr \text{ odd}, n \text{ odd}). \end{cases} \quad (\text{M2})$$

Proof. Set $z = 1$ and write $2m$ for m in identity (PF) to obtain

$$5^m \sum_{k=0}^n F_{j(rk+s)}^{2m} = \sum_{i=0}^{2m} \frac{(-1)^i \binom{2m}{i} (-1)^{ijs} \alpha^{(m-i)2js} ((-1)^{jri(n+1)} \alpha^{2jr(m-i)(n+1)} - 1)}{(-1)^{ijr} \alpha^{(m-i)2jr} - 1}.$$

If jr is even or m is even, then, on account of identity (10), Lemma 5, we have

$$(-1)^{jri(n+1)} \alpha^{2jr(m-i)(n+1)} - 1 = (-1)^{jri(n+1)} \alpha^{jr(m-i)(n+1)} F_{jr(m-i)(n+1)} \sqrt{5},$$

$$(-1)^{jri} \alpha^{2jr(m-i)} - 1 = (-1)^{jri} \alpha^{jr(m-i)} F_{jr(m-i)} \sqrt{5}.$$

Thus,

$$5^m \sum_{k=0}^n F_{j(rk+s)}^{2m} = \sum_{i=0}^{2m} (-1)^{i(jrn+js+1)} \binom{2m}{i} \frac{F_{jr(m-i)(n+1)}}{F_{jr(m-i)}} \alpha^{j(m-i)(rn+2s)},$$

and hence,

$$\begin{aligned} 5^m 2 \sum_{k=0}^n F_{j(rk+s)}^{2m} &= \sum_{i=0}^{2m} (-1)^{i(jrn+js+1)} \binom{2m}{i} \frac{F_{jr(m-i)(n+1)}}{F_{jr(m-i)}} L_{j(m-i)(rn+2s)} \\ &\quad + \sqrt{5} \sum_{i=0}^{2m} (-1)^{i(jrn+js+1)} \binom{2m}{i} \frac{F_{jr(m-i)(n+1)}}{F_{jr(m-i)}} F_{j(m-i)(rn+2s)}; \end{aligned}$$

since

$$\alpha^{j(m-i)(rn+2s)} = \frac{1}{2} L_{j(m-i)(rn+2s)} + \frac{\sqrt{5}}{2} F_{j(m-i)(rn+2s)}. \quad (19)$$

Thus, invoking Lemma 3 with $\lambda = \sqrt{5}$, we have

$$5^m 2 \sum_{k=0}^n F_{j(rk+s)}^{2m} = \sum_{i=0}^{2m} (-1)^{i(jrn+js+1)} \binom{2m}{i} \frac{F_{jr(m-i)(n+1)}}{F_{jr(m-i)}} L_{j(m-i)(rn+2s)}, \quad (20)$$

and

$$0 = \sum_{i=0}^{2m} (-1)^{i(jrn+js+1)} \binom{2m}{i} \frac{F_{jr(m-i)(n+1)}}{F_{jr(m-i)}} F_{j(m-i)(rn+2s)}. \quad (21)$$

Note that the sum on the right hand side of each of identities (20) and (21) has a removable singularity at $i = m$. Use of L'Hospital's rule and the Binet formula gives

$$\left. \frac{F_{jr(m-i)(n+1)}}{F_{jr(m-i)}} \right|_{i=m} = n + 1. \quad (22)$$

Using identity (5), Lemma 4 to re-write identity (20) while taking note of (22) gives the first identity in (M1).

If j , r and m are odd integers, then, in view of identity (10), Lemma 5, we have

$$(-1)^{jri(n+1)} \alpha^{2jr(m-i)(n+1)} - 1 = \begin{cases} (-1)^i \alpha^{jr(m-i)(n+1)} L_{jr(m-i)(n+1)}, & n \text{ even}; \\ \alpha^{jr(m-i)(n+1)} F_{jr(m-i)(n+1)} \sqrt{5}, & n \text{ odd}; \end{cases}$$

$$(-1)^{jri} \alpha^{2jr(m-i)} - 1 = (-1)^{jri} \alpha^{jr(m-i)} L_{jr(m-i)}.$$

Thus,

$$5^m \sum_{k=0}^n F_{j(rk+s)}^{2m} = \begin{cases} \sum_{i=0}^{2m} (-1)^{i(s+1)} \binom{2m}{i} \frac{L_{jr(m-i)(n+1)}}{L_{jr(m-i)}} \alpha^{j(m-i)(rn+2s)}, & n \text{ even}; \\ \sum_{i=0}^{2m} (-1)^{ijs} \binom{2m}{i} \frac{F_{jr(m-i)(n+1)} \sqrt{5}}{L_{jr(m-i)}} \alpha^{j(m-i)(rn+2s)}, & n \text{ odd}. \end{cases}$$

The second and third identities of (M1) now follow upon use of identity (19) and invocation of Lemma 3. The proof of (M2) proceeds in the same way as that of (M1) but now we use (PL). □

Since the proof of each of Theorems 2 – 4 is similar to that of Theorem 1, we only indicate in each case the z , m and r choices to be made.

Theorem 2. For m , j , r , s , n integers, m non-negative,

$$\sum_{k=0}^n (-1)^k F_{j(rk+s)}^{2m} = \begin{cases} 5^{-m} (-1)^n \sum_{i=0}^{m-1} (-1)^{i(s+n+1)} \binom{2m}{i} \frac{F_{jr(m-i)(n+1)}}{F_{jr(m-i)}} L_{(m-i)(jrn+2js)} \\ \quad + (-1)^{(s+1)} 5^{-m} \binom{2m}{m} (n+1) & (jmr \text{ odd}); \\ 5^{-m} \sum_{i=0}^{m-1} (-1)^{i(js+1)} \binom{2m}{i} \frac{L_{jr(m-i)(n+1)}}{L_{jr(m-i)}} L_{(m-i)(jrn+2js)} \\ \quad + (-1)^{m(js+1)} 5^{-m} \binom{2m}{m} & (jmr \text{ even}, n \text{ even}); \\ -5^{-m+1} \sum_{i=0}^{m-1} (-1)^{i(jr+js+1)} \binom{2m}{i} \frac{F_{jr(m-i)(n+1)}}{L_{jr(m-i)}} F_{(m-i)(jrn+2js)} & (jmr \text{ even}, n \text{ odd}); \end{cases} \quad (M3)$$

$$\sum_{k=0}^n (-1)^k L_{j(rk+s)}^{2m} = \begin{cases} (-1)^n \sum_{i=0}^{m-1} (-1)^{i(s+n)} \binom{2m}{i} \frac{F_{jr(m-i)(n+1)}}{F_{jr(m-i)}} L_{(m-i)(jrn+2js)} & (jmr \text{ odd}); \\ \quad + (-1)^s \binom{2m}{m} (n+1) \\ \sum_{i=0}^{m-1} (-1)^{ijs} \binom{2m}{i} \frac{L_{jr(m-i)(n+1)}}{L_{jr(m-i)}} L_{(m-i)(jrn+2js)} & (jmr \text{ even}, n \text{ even}); \\ \quad + (-1)^m js \binom{2m}{m} \\ -5 \sum_{i=0}^{m-1} (-1)^{i(jr+js)} \binom{2m}{i} \frac{F_{jr(m-i)(n+1)}}{L_{jr(m-i)}} F_{(m-i)(jrn+2js)} & (jmr \text{ even}, n \text{ odd}); \end{cases} \quad (\text{M4})$$

Proof. Use $z = -1$ and write $2m$ for m in identities (PF) and (PL). $\square$

Theorem 3. For m, j, r, s, n integers, m non-negative,

$$\sum_{k=0}^n F_{j(2rk+s)}^{2m+1} = \begin{cases} 5^{-m} \sum_{i=0}^{m-1} (-1)^{i(js+1)} \binom{2m+1}{i} \frac{F_{jr(2m+1-2i)(n+1)}}{F_{jr(2m+1-2i)}} F_{j(2m+1-2i)(rn+s)} & (jr \text{ even}); \\ \quad + (-1)^{m(js+1)} 5^{-m} \binom{2m+1}{m} \frac{F_{jr(n+1)}}{F_{jr}} F_{j(rn+s)} \\ 5^{-m} \sum_{i=0}^{m-1} (-1)^{i(s+1)} \binom{2m+1}{i} \frac{L_{jr(2m+1-2i)(n+1)}}{L_{jr(2m+1-2i)}} F_{j(2m+1-2i)(rn+s)} & (jr \text{ odd}, n \text{ even}); \\ \quad + (-1)^{m(s+1)} 5^{-m} \binom{2m+1}{m} \frac{L_{jr(n+1)}}{L_{jr}} F_{j(rn+s)} \\ 5^{-m} \sum_{i=0}^{m-1} (-1)^{i(s+1)} \binom{2m+1}{i} \frac{F_{jr(2m+1-2i)(n+1)}}{L_{jr(2m+1-2i)}} L_{j(2m+1-2i)(rn+s)} & (jr \text{ odd}, n \text{ odd}); \\ \quad + (-1)^{m(s+1)} 5^{-m} \binom{2m+1}{m} \frac{F_{jr(n+1)}}{L_{jr}} L_{j(rn+s)} \end{cases} \quad (\text{M5})$$

$$\sum_{k=0}^n L_{j(2rk+s)}^{2m+1} = \begin{cases} \sum_{i=0}^{m-1} (-1)^{ijs} \binom{2m+1}{i} \frac{F_{jr(2m+1-2i)(n+1)}}{F_{jr(2m+1-2i)}} L_{j(2m+1-2i)(rn+s)} & (jr \text{ even}); \\ \quad + (-1)^{mjs} \binom{2m+1}{m} \frac{F_{jr(n+1)}}{F_{jr}} L_{j(rn+s)} \\ \sum_{i=0}^{m-1} (-1)^{is} \binom{2m+1}{i} \frac{L_{jr(2m+1-2i)(n+1)}}{L_{jr(2m+1-2i)}} L_{j(2m+1-2i)(rn+s)} & (jr \text{ odd, } n \text{ even}); \\ \quad + (-1)^{ms} \binom{2m+1}{m} \frac{L_{jr(n+1)}}{L_{jr}} L_{j(rn+s)} \\ 5 \sum_{i=0}^{m-1} (-1)^{is} \binom{2m+1}{i} \frac{F_{jr(2m+1-2i)(n+1)}}{L_{jr(2m+1-2i)}} F_{j(2m+1-2i)(rn+s)} & (jr \text{ odd, } n \text{ odd}). \\ \quad + (-1)^{ms} 5 \binom{2m+1}{m} \frac{F_{jr(n+1)}}{L_{jr}} F_{j(rn+s)} \end{cases} \quad (\text{M6})$$

Proof. Use $z = 1$ and write $2m + 1$ for m and $2r$ for r in identities (PF) and (PL). $\square$

Theorem 4. For m, j, r, s, n integers, m non-negative,

$$\sum_{k=0}^n (-1)^k F_{j(2rk+s)}^{2m+1} = \begin{cases} 5^{-m} (-1)^n \sum_{i=0}^{m-1} (-1)^{i(js+1)} \binom{2m+1}{i} \frac{F_{jr(2m+1-2i)(n+1)}}{F_{jr(2m+1-2i)}} F_{j(2m+1-2i)(rn+s)} & (jr \text{ odd}); \\ \quad + (-1)^{m(js+1)} 5^{-m} (-1)^n \binom{2m+1}{m} \frac{F_{jr(n+1)}}{F_{jr}} F_{j(rn+s)} \\ 5^{-m} \sum_{i=0}^{m-1} (-1)^{i(js+1)} \binom{2m+1}{i} \frac{L_{jr(2m+1-2i)(n+1)}}{L_{jr(2m+1-2i)}} F_{j(2m+1-2i)(rn+s)} & (jr \text{ even, } n \text{ even}); \\ \quad + (-1)^{m(js+1)} 5^{-m} \binom{2m+1}{m} \frac{L_{jr(n+1)}}{L_{jr}} F_{j(rn+s)} \\ -5^{-m} \sum_{i=0}^{m-1} (-1)^{i(js+1)} \binom{2m+1}{i} \frac{F_{jr(2m+1-2i)(n+1)}}{L_{jr(2m+1-2i)}} L_{j(2m+1-2i)(rn+s)} & (jr \text{ even, } n \text{ odd}); \\ \quad - (-1)^{m(js+1)} 5^{-m} \binom{2m+1}{m} \frac{F_{jr(n+1)}}{L_{jr}} L_{j(rn+s)} \end{cases} \quad (\text{M7})$$

$$\sum_{k=0}^n (-1)^k L_{j(2rk+s)}^{2m+1} = \begin{cases} (-1)^n \sum_{i=0}^{m-1} (-1)^{ijs} \binom{2m+1}{i} \frac{F_{jr(2m+1-2i)(n+1)}}{F_{jr(2m+1-2i)}} L_{j(2m+1-2i)(rn+s)} & (jr \text{ odd}); \\ \quad + (-1)^{mjs} (-1)^n \binom{2m+1}{m} \frac{F_{jr(n+1)}}{F_{jr}} L_{j(rn+s)} \\ \sum_{i=0}^{m-1} (-1)^{ijs} \binom{2m+1}{i} \frac{L_{jr(2m+1-2i)(n+1)}}{L_{jr(2m+1-2i)}} L_{j(2m+1-2i)(rn+s)} & (jr \text{ even}, n \text{ even}); \\ \quad + (-1)^{mjs} \binom{2m+1}{m} \frac{L_{jr(n+1)}}{L_{jr}} L_{j(rn+s)} \\ -5 \sum_{i=0}^{m-1} (-1)^{ijs} \binom{2m+1}{i} \frac{F_{jr(2m+1-2i)(n+1)}}{L_{jr(2m+1-2i)}} F_{j(2m+1-2i)(rn+s)} & (jr \text{ even}, n \text{ odd}); \\ \quad - (-1)^{mjs} 5 \binom{2m+1}{m} \frac{F_{jr(n+1)}}{L_{jr}} F_{j(rn+s)} \end{cases} \quad (\text{M8})$$

Proof. Use $z = -1$ and write $2m + 1$ for m and $2r$ for r in identities (PF) and (PL) $\square$

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