 doi:10.20944/preprints202106.0066.v1

Effects of Coulomb Central Potential Induced by Lorentz Symmetry Breaking on Relativistic Quantum Oscillator

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Abstract

In this paper, we consider the effects of a radial electric field and a constant magnetic field induced by Lorentz symmetry violation on a generalized relativistic quantum oscillator by choosing a function $f(r) = b_1\,r + \frac{b_2}{r}$ in the equation subject to a Cornell-type potential $S(r) = \eta_L\,r + \frac{\eta_c}{r}$ introduce by modifying the mass term in the equation. We show that the analytical solutions to the Klein-Gordon oscillator can be achieved, and a quantum effect is observed due to the dependence of the angular frequency of the oscillator on the quantum numbers of the system.

Keywords: Lorentz symmetry violation, Relativistic wave-equations, scalar potential, electric & magnetic field, biconfluent Heun equation

PACS Number(s): 03.65.Pm, 11.30.Cp, 11.30.Qc

1 Introduction

In this paper, we investigate the behaviour of a scalar particle by solving the generalized Klein-Gordon oscillator [1, 2, 3, 4, 5, 6, 7] in a possible scenario of anisotropy generated by a Lorentz symmetry breaking term defined by a tensor $(K_F)_{\mu\nu\alpha\beta}$ that governs the Lorentz symmetry violation out of the Standard Model Extension [8, 9]. The scenario of the violation of Lorentz

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 doi:10.20944/preprints202106.0066.v1

symmetry is determined by a field configuration of crossed electric and magnetic field that gives rise to a Coulomb-type Poynting vector. Then, we search for relativistic bound state solutions to the generalized KG-oscillator under the influence of a Cornell-type scalar potential in the background of Lorentz symmetry violation. In recent decades, studies of the violation of the Lorentz symmetry have been made in several branches of physics [10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31].

Inspired by Refs. [8, 9, 32, 33, 34, 35, 36, 37, 38, 39], effects of the Lorentz symmetry violation by introducing a nonminimal coupling into the generalized KG-oscillator given by

$$p^{\mu}\,p_{\mu} \to (p^{\mu} + i\,M\,\omega\,X^{\mu})\,(p_{\mu} - i\,M\,\omega\,X_{\mu}) + \frac{\alpha}{4}\,(K_F)_{\mu\nu\alpha\beta}\,F^{\mu\nu}(x)\,F^{\alpha\beta}(x), \quad (1)$$

where α is a constant, $F_{\mu\nu}(x)$ is the electromagnetic field tensor, $(K_F)_{\mu\nu\alpha\beta}$ corresponds to a tensor that governs the Lorentz symmetry violation out of the Standard Model Extension, ω is the oscillator frequency, and $X_{\mu} = (0, f(r), 0, 0)$ is a function. For Klein-Gordon oscillator we have $f(r) = r$.

2 Generalized KG-oscillator Under the Effects of Lorentz Symmetry Violation

We consider the Minkowski flat space-time

$$ds^2 = -dt^2 + dr^2 + r^2\,d\phi^2 + dz^2. \quad (2)$$

The generalized KG-oscillator under the effects of Lorentz symmetry violation described earlier is given by

$$\left[-\frac{\partial^2}{\partial t^2} + \frac{1}{r}\left(\frac{\partial}{\partial r} + M\,\omega\,f(r)\right)\left(r\,\frac{\partial}{\partial r} - M\,\omega\,r\,f(r)\right) + \frac{\partial^2}{\partial z^2} + \frac{1}{r^2}\,\frac{\partial^2}{\partial \phi^2}\right]\Psi$$
$$+\frac{\alpha}{4}\,(K_F)_{\mu\nu\alpha\beta}\,F^{\mu\nu}(x)\,F^{\alpha\beta}(x)\,\Psi = (M + S(r))^2\,\Psi, \quad (3)$$

where $S(r)$ is the scalar potential.

Using the the properties of the tensor $(K_F)_{\mu\nu\alpha\beta}$ [8, 9, 32, 33, 34, 35, 36, 37, 38, 39], we can rewrite (3) in the form :

$$\left[-\frac{\partial^2}{\partial t^2}+\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial \phi^2}+\frac{\partial^2}{\partial z^2}-M^2\,\omega^2\,f^2(r)-M\,\omega\left(f'(r)+\frac{f(r)}{r}\right)\right]\Psi$$
$$+\left[-\frac{\alpha}{2}\,(\kappa_{DE})_{ij}\,E^i\,E^j+\frac{\alpha}{2}\,(\kappa_{HB})_{jk}\,B^i\,B^j-\alpha\,(\kappa_{DB})_{jk}\,E^i\,B^j\right]\Psi$$
$$=(M+S(r))^2\,\Psi. \tag{4}$$

Let us consider a possible scenario of Lorentz symmetry violation determined by only one non-null component of the tensor $(\kappa_{DB})_{jk}$ as being $(\kappa_{DB})_{13}=\kappa=const$ and by a field configuration given by [37, 38]:

$$\vec{B}=B_0\,\hat{z}\quad,\quad \vec{E}=\frac{\lambda}{r}\,\hat{r} \tag{5}$$

where B_0 is a constant, $\hat{z}$ is a unit vector in the z-direction and λ is a constant associated with a linear distribution of electric charge along the axial direction. Hence Equation (4) becomes

$$\left[-\frac{\partial^2}{\partial t^2}+\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial \phi^2}+\frac{\partial^2}{\partial z^2}-M^2\,\omega^2\,f^2(r)-M\,\omega\left(f'(r)+\frac{f(r)}{r}\right)\right]\Psi$$
$$-\frac{\alpha\,\lambda\,B_0\,\kappa}{r}\,\Psi=(M+S(r))^2\,\Psi. \tag{6}$$

Since the metric is independent of time and symmetrical by translations along the z-axis, as well by rotations. It is reasonable to write the solution to Eq. (6) as

$$\Psi(t,r,\phi,z)=e^{i\,(-E\,t+l\,\phi+k\,z)}\,\psi(r), \tag{7}$$

where E is the energy of the particle, $l=0,\pm1,\pm2,....$ are the eigenvalues of the z-component of the angular momentum operator, and k is a constant.

Substituting the solution (7) into the Eq. (6), we obtain the following radial wave-equation for $\psi(r)$:

$$\psi''(r)+\frac{1}{r}\,\psi'(r)+\left[E^2-k^2-\frac{l^2}{r^2}-M^2\,\omega^2\,f^2-M\,\omega\left(f'+\frac{f}{r}\right)\right]\psi(r)$$
$$-\frac{\alpha\,\lambda\,B_0\,\kappa}{r}\,\psi(r)=(M+S(r))^2\,\psi(r). \tag{8}$$

In this work, we are mainly interest on Cornell-type potential which is given by [2, 3, 40, 41, 42]

$$S(r) = \eta_L\, r + \frac{\eta_c}{r}. \tag{9}$$

To study the generalized KG-oscillator, we have chosen the function $f(r) = b_1\, r + \frac{b_2}{r}$ [1, 2, 3, 4, 5, 6, 7], where $b_1 > 0, b_2 > 0$ are arbitrary constants. Substituting potential (9) and using the function $f(r)$ into the Eq. (8), we obtain the following radial wave-equation :

$$\left[\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} + \Lambda - \Omega^2\, r^2 - \frac{j^2}{r^2} - \frac{a}{r} - b\, r\right] \psi(r) = 0, \tag{10}$$

where

$$\begin{aligned}
&\Lambda = E^2 - M^2 - k^2 - 2\, M\, \omega\, b_1 - 2\, M^2\, \omega^2\, b_1\, b_2 - 2\, \eta_L\, \eta_c,\\
&\Omega = \sqrt{M^2\, \omega^2\, b_1^2 + \eta_L^2},\\
&j = \sqrt{l^2 + M^2\, \omega^2\, b_2^2 + \eta_c^2},\\
&a = \alpha\, \lambda\, B_0\, \kappa + 2\, M\, \eta_c,\\
&b = 2\, M\, \eta_L.
\end{aligned} \tag{11}$$

Transforming $x = \sqrt{\Omega}\, r$ into the Eq. (13), we obtain the following equation:

$$\left[\frac{d^2}{dx^2} + \frac{1}{x}\frac{d}{dx} + \zeta - x^2 - \frac{j^2}{x^2} - \frac{\eta}{x} - \theta\, x\right] \psi(x) = 0, \tag{12}$$

where

$$\zeta = \frac{\Lambda}{\Omega} \quad , \quad \eta = \frac{a}{\sqrt{\Omega}} \quad , \quad \theta = \frac{b}{\Omega^{\frac{3}{2}}}. \tag{13}$$

Now, we use the appropriate boundary conditions that the wave-functions is regular both at $x \to 0$ and $x \to \infty$. Suppose the possible solution to the Eq. (12) is

$$\psi(x) = x^j\, e^{-\frac{1}{2}\,(x+\theta)\, x}\, H(x). \tag{14}$$

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Substituting the solution (14) into the Eq. (12), we obtain the following equation

$$H''(x)+\left[\frac{1+2\,j}{x}-2\,x-\theta\right]\,H'(x)+\left[-\frac{\beta}{x}+\Theta\right]\,H(x)=0, \tag{15}$$

where

$$\Theta=\zeta+\frac{\theta^2}{4}-2\,(1+j)\quad,\quad\beta=\eta+\frac{\theta}{2}\,(1+2\,j). \tag{16}$$

Equation (15) is the biconfluent Heun's differential equation [40, 2, 3, 6, 7, 43, 44] with $H(x)$ is the Heun polynomials function.

The above equation (15) can be solved by the Frobenius method. Writing the solution as a power series expansion around the origin [45]:

$$H(x)=\sum_{i=0}^{\infty}d_i\,x^i. \tag{17}$$

Substituting the power series solution into the Eq. (15), we obtain the following recurrence relation

$$d_{n+2}=\frac{1}{(n+2)(n+2+2\,j)}\,[\{\beta+\theta\,(n+1)\}\,d_{n+1}-(\Theta-2\,n)\,d_n]. \tag{18}$$

With few coefficients are

$$\begin{aligned} d_1&=\left[\frac{\eta}{1+2\,j}+\frac{\theta}{2}\right]\,d_0,\\ d_2&=\frac{1}{4\,(1+j)}\,[(\beta+\theta)\,d_1-\Theta\,d_0]. \end{aligned} \tag{19}$$

The power series expansion $H(x)$ becomes a polynomial of degree n by imposing the following two conditions [40, 2, 3, 6, 7]

$$\begin{aligned} \Theta&=2\,n,\quad(n=1,2,...)\\ d_{n+1}&=0. \end{aligned} \tag{20}$$

By analyzing the first condition, we obtain following energy eigenvalue $E_{n,l}$:

$$\begin{aligned} E^2_{n,l}&=M^2+k^2+2\,M\,\omega\,b_1+2\,M^2\,\omega^2\,b_1\,b_2+2\,\eta_L\,\eta_c+2\,\Omega\left(n+1+\sqrt{l^2+M^2\,\omega^2\,b_2^2+\eta_c^2}\right)\\ &-\frac{M^2\,\eta_L^2}{\Omega^2}. \end{aligned} \tag{21}$$

Note that Eq. (21) is not the general expression of the relativistic energy eigenvalues of a generalized KG-oscillator. One can obtain the individual energy levels $E_{1,l}, E_{2,l}, E_{3,l}, ...$ and wave-function $\Psi_{1,l}, \Psi_{2,l}, ...$ one by one by imposing the additional recurrence condition $c_{n+1} = 0$ on the eigenvalue problem. It should be noted here that for $b_2 \to 0$, $\eta_c \to 0$, the energy eigenvalues (21) reduces to the result obtained in [38] (see Eq. (32) in the Ref. [38]). Thus we can see that the presence of an extra Coulomb-like scalar potential and the generalized function $f(r)$ modifies the energy spectrum of a generalized KG-oscillator field and shifted the energy levels.

The corresponding wave-functions are given by

$$\psi_{n,l}(x) = x^{\sqrt{l^2+M^2\,\omega^2\,b_2^2+\eta_c^2}}\, e^{-\frac{1}{2}\left[x+\frac{2\,M\,\eta_L}{\Omega^{\frac{3}{2}}}\right]x}\, H(x). \tag{22}$$

Now, we evaluate the individual energy levels and eigenfunctions one by one as in [40, 2, 3, 6, 7]. For example, $n = 1$, we have $\Theta = 2$ and $c_2 = 0$ which implies

$$\Rightarrow \frac{2}{\beta+\theta}\, d_0 = \left(\frac{\eta}{1+2\,j} + \frac{\theta}{2}\right) d_0$$

$$\Rightarrow \Omega_{1,l}^3 - \left[\frac{a^2}{4\,\left(\frac{1}{2}+j\right)}\right] \Omega_{1,l}^2 - \left[\frac{M\,a\,\eta_L\,(1+j)}{\left(\frac{1}{2}+j\right)}\right] \Omega_{1,l} - \left(\frac{3}{2}+j\right) M^2\,\eta_L^2 = 0 \tag{23}$$

a constraint on the parameter $\Omega_{1,l}$ that is on the angular frequency of the oscillator. Note that its values changes for each quantum number n and l, so we have labeled $\Omega \to \Omega_{n,l}$. This third degree algebraic equation has at least one real root and it is exactly this real solution that gives a first degree polynomial to the function $H(x)$.

The angular frequency of the oscillator for the radial mode $n = 1$ is

$$\omega_{1,l} = \frac{1}{M\,b_1}\sqrt{\Omega_{1,l}^2 - \eta_L^2}. \tag{24}$$

We can see, Eq. (24) gives us the possible values of the oscillator frequency for the radial mode $n = 1$ and it depends on the quantum numbers $\{n, l\}$

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of the system, and the Lorentz symmetry breaking parameters $(\alpha, B_0, \lambda, \kappa)$. Further, for each relativistic energy level and wave-function, we have a different relation of this angular frequency of the oscillator with these parameters.

The ground state energy level for the radial mode $n = 1$ is given by

$$\begin{aligned} E^2_{1,l} &= M^2 + k^2 + 2\,M\,\omega_{1,l}\,b_1 + 2\,M^2\,\omega^2_{1,l}\,b_1\,b_2 + 2\,\eta_L\,\eta_c \\ &+2\,\Omega_{1,l}\left(2 + \sqrt{l^2 + M^2\,\omega^2_{1,l}\,b_2^2 + \eta_c^2}\right) - \left(\frac{M\,\eta_L}{\Omega_{1,l}}\right)^2. \end{aligned} \tag{25}$$

And the ground state eigenfunction is

$$\psi_{1,l}(x) = x^{\sqrt{l^2+M^2\,\omega^2_{1,l}\,b_2^2+\eta_c^2}}\, e^{-\frac{1}{2}\left[x+\frac{2\,M\,\eta_L}{\Omega^{\frac{3}{2}}_{1,l}}\right]x}\,(d_0 + d_1\,x). \tag{26}$$

where

$$d_1 = \frac{1}{\sqrt{\Omega_{1,l}}}\left[\frac{2\,M\,\eta_c + \alpha\,\lambda\,B_0\,\kappa}{\left(2 + \sqrt{l^2 + M^2\,\omega^2_{1,l}\,b_2^2 + \eta_c^2}\right)} + \frac{M\,\eta_L}{\Omega_{1,l}}\right] d_0. \tag{27}$$

We can see that the lowest energy state (25) plus the ground state wave-function (26)–(27) with the restriction on the angular frequency of the oscillator (24) is defined for the radial mode $n = 1$. The presence of the Cornell-type scalar potential $S(r)$, the generalized function $f(r)$ and the Lorentz symmetry breaking parameters $(\alpha, B_0, \lambda, \kappa)$ modified the energy spectrum and the wave-function.

3 Conclusions

We have analysed the behaviour of a generalized KG-oscillator by choosing the function $f(r)$ in the equation under the effects of a central potential induced by Lorentz symmetry violation. We have chosen the function $f(r) = b_1\,r + \frac{b_2}{r}$ and derived the radial wave-equation in the presence of a Cornell-type scalar potential introduce by modifying the mass term

$M \to M + S(r)$ in the equation. For a suitable wave-function, we have obtained the Heun's biconfluent differential equation and finally truncating the power series solution, the non-compact expression of the energy eigenvalues (21), and the radial wave-function (22) is obtained. By imposing the truncation condition $d_{n+1} = 0$ on the eigenvalue problem, one can obtained the individual energy level and the wave-function one by one. As for example, we have obtained the energy level Eq. (25), and the radial wave-function Eqs. (26)–(27) along with the condition (23) imposed on the angular frequency of the oscillator for the lowest state of the quantum system defined by the radial mode $n = 1$. Noted that for $b_2 \to 0$, and $\eta_c \to 0$, the energy eigenvalues (21) reduces to the result in Ref. [38] (see Eq. (32) in the Ref. [38]). Thus the result presented in this work in comparison to those in Ref. [38] get modified due to the presence of an extra Coulomb-type potential form $\frac{\eta_c}{r}$ in the scalar potential $S(r)$, and the generalized function $f(r)$ where an additional term $\frac{b_2}{r}$ is present. Hence, the result presented in this work is more suitable than those result obtained in Ref. [38]. We see that there is a restriction (24) on the angular frequency of the oscillator $\omega_{1,l}$ for the radial mode $n = 1$ that depends on the Lorentz symmetry breaking parameters $(\alpha, B_0, \lambda, \kappa)$ and the potential parameters which gives us a first degree polynomial solution to the Heun function $H(x)$.

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