doi:10.20944/preprints202106.0036.v1

ON ALL REAL ZEROS FOR A CLASS OF EVEN ENTIRE FUNCTIONS

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ABSTRACT. The present paper deals with a class of even entire functions of order $\rho = 1$ and genus $\vartheta = 0$ of the polynomials form,

$$\sum_{m=0}^{\infty} \frac{(-1)^m \Phi^{(2m)}(0)}{\Gamma(2m+1)} x^{2m} = \Phi(0) \prod_{k=1}^{\infty} \left(1 - \frac{x}{\ell_k}\right),$$

where $\Phi(0) \neq 0$, real numbers x, nonnegative integers m, and $\ell_k \neq 0$ are all of the nonzero roots with $\sum_{k=1}^{\infty} 1/|\ell_k| < \infty$ and natural numbers k. We provide an efficient criterion for the polynomials with only real zeros. We also prove that the conjecture of Jensen is our special case.

CONTENTS

1. INTRODUCTION

The theory of entire functions has played an important role in solving the zeros of the real and complex variables functions (see, for example, [1, 2] and the references therein). The polynomials of entire functions with the distributions of their zeros [2], which is associated with the Taylor coefficients of entire functions, converge locally

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1991 *Mathematics Subject Classification.* 26C10.

Key words and phrases. entire function, zeros of polynomials, polynomials, conjecture of Jensen.

uniformly to them since they are expressed by the Weierstrass primary factors (see [3], p.18; [4], p.25). Laguerre in 1882 [5] and Pólya in 1913 [6] proposed the class of the Laguerre–Pólya class [7]. The theory of entire functions in Laguerre–Pólya class has been a increasing interest for finding their real zeros of entire functions [8, 9]. As an example of the progress made, the integral transforms of the entire functions in the Laguerre-Pólya class were reported in [10]. A fundamental paper makes a nice progress in study of an analog of the linear finite difference operators [11]. Moreover, another work on an entire function with the increasing Taylor coefficients was discussed in [12]. To discover the zeros of them, the sign regularity of Maclaurin coefficients of entire functions was also considered in [13].

1.1. **The statement of the problem.** Let $\mathbb{R}$, $\mathbb{N}$ and $\mathbb{C}$ denote the sets of real, natural and complex numbers, respectively.

We now consider the theory of the product of the cosine and hyperbolic cosine, which were proposed by Euler ([14], p.127–128).

Euler [14] suggested that the cosine can be expressed by the Taylor series and the product:

$$\sum_{m=0}^{\infty} \frac{\cos^{(2k)}(0)}{\Gamma(2m+1)} x^{2m} = \cos(0) \prod_{k=1}^{\infty} \left(1 - \frac{x^2}{\varphi_k^2}\right), \tag{1}$$

where $x \in \mathbb{R}$, $\varphi_k = \left(k - \frac{1}{2}\right)\pi$, $k \in \mathbb{N}$ and $m \in \mathbb{N} \cup \{0\}$ ([15], 4.3.66, p.74, 4.3.90, p.75).

Euler [14] also suggested that the hyperbolic cosine can be expressed by the Taylor series and the product:

$$\sum_{m=0}^{\infty} \frac{\cosh^{(2k)}(0)}{\Gamma(2m+1)} x^{2m} = \cosh(0) \prod_{k=1}^{\infty} \left(1 + \frac{x^2}{\psi_k^2}\right), \tag{2}$$

where $x \in \mathbb{R}$, $\psi_k = i\left(k - \frac{1}{2}\right)\pi$, $k \in \mathbb{N}$ and $m \in \mathbb{N} \cup \{0\}$ ([15], 4.5.63, 4.5.68, p.85).

It is know that the cosine and hyperbolic cosine are the even entire functions of order $\rho = 1$ and genus $\vartheta = 0$ (for the definitions of the order and genus of the even entire functions, see in Section 1).

1.2. **A class of even entire functions.** By the observation of the above works of Euler, we now suggest a class of even entire functions of order $\rho = 1$ and genus $\vartheta = 0$, which is structured as follows:

Definition 1. *A real even entire function of order* $\rho = 1$ *and genus* $\vartheta = 0$*,*

$$\Phi(x) = \sum_{m=1}^{\infty} \frac{(-1)^m \Phi^{(2m)}(0)}{\Gamma(2m+1)} x^{2m}, \tag{3}$$

is said to defined in a class, written $\Phi(x) \in \mathbb{Y}$*, if* $\Phi(x)$ *can be expressed in the form*

$$\Phi(x) = \Phi(0) \prod_{k=1}^{\infty} \left(1 - \frac{x}{\ell_k}\right), \tag{4}$$

where $\Phi(0) \neq 0$*,* $m \in \mathbb{N} \cup \{0\}$*,* $k \in \mathbb{N}$*,* $\sum_{k=1}^{\infty} 1/|\ell_k| < \infty$*,* $\ell_k \neq 0$ *and* $x \in \mathbb{R}$*.*

With (4) we know that $\ell_k \neq 0$ are all zeros of $\Phi(x) \in \mathbb{Y}$ by applying the theory of entire functions. The behavior of $\Phi(x) \in \mathbb{Y}$ as one of subclasses of entire functions of real and complex variables in the Laguerre–Pólya class is considered in the present paper.

1.3. **The conjecture of Jensen.** Riemann in 1859 [16] proposed the Riemann Ξ function $\Xi(x)$ by

$$\log \Xi(x) = \log \Xi(0) + \sum_{k=1}^{\infty} \left(1 - \frac{x^2}{\widetilde{\rho}_k^2}\right), \tag{5}$$

which leads to the product

$$\Xi(x) = \Xi(0) \prod_{k=1}^{\infty} \left(1 - \frac{x^2}{\rho_k^2}\right), \tag{6}$$

which was discovered by Cahen [17], Landau [18] and Titchmarsh [19], where run all of the positive real roots of $\Xi(x) = 0$ for $k \in \mathbb{N}$. It is easy to verity that (6) can be also derived from the product of Hadamard [20]

$$\Xi(x) = \xi(0) \prod_{k=1}^{\infty} \left(1 - \frac{\frac{1}{2} + ix}{\frac{1}{2} + i\rho_k}\right), \tag{7}$$

where ρ_k run all of the real roots of $\Xi(t) = 0$ with $k \in \mathbb{N}$.

Based on the work of Jensen [21], Pólya in 1927 [22] considered that $\Xi(x)$ is represented as the polynomials associated to its Taylor expansion, i.e.,

$$\Xi(x) = \sum_{m=0}^{\infty} \frac{(-1)^m \Xi^{(2m)}(0)}{\Gamma(2m+1)} x^{2m}, \tag{8}$$

where $x \in \mathbb{R}$ and $m \in \mathbb{N}$.

By connection with the product of Hadamard for the entire Riemann zeta-function $\xi(x)$ [20], Eq. (7) was rewritten by Edwards [23] as

$$\Xi(x) = \xi(0) \prod_{k=1}^{\infty} \left(1 - \frac{\frac{1}{2} + ix}{\frac{1}{2} + i\rho_k}\right) = \Xi(0) \prod_{k=1}^{\infty} \left(1 - \frac{x^2}{\widetilde{\rho}_k^2}\right). \tag{9}$$

Jensen in 1913 [21] proposed the following assert:

The conjecture of Jensen is that the roots ρ_k of the polynomials associated to its Taylor expansion of $\Xi(t)$ are all real, where $k \in \mathbb{N}$.

The conjecture of Jensen for the zeta-function was studied, completed, and expanded by Pólya in 1927 [22] and further discussed by Titchmarsh [24]. Recently, a breakthrough for new progress on conjecture of Jensen for the zeta-function was have made by Griffin et al. [25]. Up to now, a interesting paper by Bombieri [26] makes a progress report in the conjecture of Jensen that remains a unsolved problem in analytic number theory and mathematical physics.

1.4. **The main targets of this paper.** In this paper we mainly plan to prove the following theorems:

Theorem 1. *Let* $\Phi(x) \in \mathbb{Y}$ *and* $x \in \mathbb{R}$*. If* $\Phi(x)$ *has the critical line* $Im(x) = 0$*, then all of its zeros are real.*

Theorem 2. *The conjecture of Jensen is true.*

The structure of this paper is designed as follows. In Section 2 we introduce the theory of the entire functions. In Section 3 we give the proof of *Theorem 1*. In Section 4 we present the proof of *Theorem 2*.

2. PRELIMINARIES

In this section we introduce some results in the theory of the entire functions.

Let $s \in \mathbb{C}$. We now start with the definition of the Weierstrass primary factors.

Definition 2. *The Weierstrass primary factors are defined by ([4], Lecture 4, p.25)*

$$F(s,0) = 1 - s, \tag{10}$$

where $\vartheta = 0$*, and*

$$F(s,\vartheta) = (1-s)\exp\left(s + \frac{1}{2}s^2 + \cdots + \frac{1}{\vartheta}s^{\vartheta}\right), \tag{11}$$

where $\vartheta > 1$ *and* $\vartheta \in \mathbb{N}$*.*

Definition 3. *Let* $\Gamma = \{\mu_k\}_{k=1}^{\infty}$ *be a sequence of complex numbers and* $k \in \mathbb{N}$ *such that*

$$|\mu_1| < |\mu_2| < |\mu_3| < \cdots < |\mu_k| < |\mu_{k+1}| < \cdots \tag{12}$$

and

$$\lim_{k\to\infty} \mu_k = \infty. \tag{13}$$

A canonical product of genus ϑ is defined by ([4], Lecture 4, p.28)

$$\mathbb{F}\left(s\right)=\prod_{k=1}^{\infty}F\left(s,\vartheta\right). \tag{14}$$

Definition 4. *The maximum modulus of $\mathbb{F}\left(s\right)$ on a disk of radius y is defined as (See the book of Boas ([3], p.1)*

$$\mathbb{MV}\left(y\right)=\max_{|s|=y}\left|\mathbb{F}\left(s\right)\right|. \tag{15}$$

Definition 5. *The order ρ of $\mathbb{F}\left(s\right)$ is defined by (See the book of Boas ([3], p.8)*

$$\rho=\lim_{y\to\infty}\sup\frac{\log\log\mathbb{MV}\left(y\right)}{\log y}. \tag{16}$$

Definition 6. *The exponent of convergence λ for $\mathbb{F}\left(s\right)$ (or called the convergence exponent of its zeros) is defined by (See the book of Boas ([3], p.14)*

$$\lambda=\inf\left\{\beta\left|\left|\mu_k\right|^{-\beta}<\infty\right.,\mathbb{F}\left(\mu_k\right)=0\right\}, \tag{17}$$

where $k\in\mathbb{N}$.

Lemma 1. *Let $\Gamma=\{\mu_k\}_{k=1}^{\infty}$ be a sequence of complex numbers and $k\in\mathbb{N}$. If*

$$\sum_{k=1}^{\infty}1/\left|\mu_k\right|^{\vartheta+1}<\infty, \tag{18}$$

then the product

$$\mathbb{G}\left(s\right)=\prod_{k=1}^{\infty}F\left(s/\mu_k,\vartheta\right), \tag{19}$$

converges uniformly on every compact set Δ.

Proof. See the book of Levin ([4], Theorem 2, p.29). □

Lemma 2. *Let $\Gamma=\{\mu_k\}_{k=1}^{\infty}$ be a sequence of complex numbers and $k\in\mathbb{N}$. If*

$$\sum_{k=1}^{\infty}1/\left|\mu_k\right|<\infty, \tag{20}$$

then an even entire function

$$\mathbb{G}\left(s\right)=\prod_{k=1}^{\infty}F\left(s/\mu_k,0\right), \tag{21}$$

converges uniformly on every compact set Δ.

Proof. See the book of Boas ([3], (2.12.6) and (2.12.7), p.35). ☐

Lemma 3. *(Borel) A canonical product* $\mathbb{F}(s)$ *of genus* ϑ *is an entire function of order equal to the convergence exponent of its zeros.*

Proof. See the book of Boas ([3], Theorem 2.6.5., p.19). ☐

3. THE PROOF OF THEOREM 1

We now present the proof of Theorem 1. In order to prove it, we give two hypothesis tests and we prove that if they are false, our result is true.

Since $\Phi(x) \in \mathbb{Y}$, we have

$$\Phi(x) = \sum_{m=1}^{\infty} \frac{(-1)^m \Phi^{(2m)}(0)}{\Gamma(2m+1)} x^{2m} = \Phi(0) \prod_{k=1}^{\infty} \left(1 - \frac{x}{\ell_k}\right) \tag{22}$$

with

$$\Phi(x) = \Phi(-x), \tag{23}$$

where $x \in \mathbb{C}$.

Here, all of its zeros are ℓ_k for $k \in \mathbb{N}$.

Now, we consider two cases as follows:

Case 1. Now, we assume that

$$\lambda_k = \sigma_k + i\hbar_k, \tag{24}$$

run the zeros of $\Phi(x)$, where $\sigma_k \in \mathbb{R}\setminus\{0\}$ and $\hbar_k \in \mathbb{R}\setminus\{0\}$.

Putting (24) into (22), we show that for $k \in \mathbb{N}$,

$$\begin{aligned} &\textstyle\sum_{m=1}^{\infty} \frac{(-1)^m \Phi^{(2m)}(0)}{\Gamma(2m+1)} \lambda_k^{2m} \\ &= \textstyle\sum_{m=1}^{\infty} \frac{(-1)^m \Phi^{(2m)}(0)}{\Gamma(2m+1)} (\sigma_k + i\hbar_k)^{2m} \\ &= \Phi(0) \textstyle\prod_{k=1}^{\infty} \left[1 - \frac{\lambda_k}{\ell_k}\right] \\ &= \Phi(0) \textstyle\prod_{k=1}^{\infty} \left[1 - \frac{\sigma_k + i\hbar_k}{\ell_k}\right] = 0. \end{aligned} \tag{25}$$

With (25) we obtain

$$1 - \frac{\sigma_k + i\hbar_k}{\ell_k} = 0, \tag{26}$$

where $k \in \mathbb{N}$.

From (26) we get

$$\ell_k = \sigma_k + i\hbar_k. \tag{27}$$

From (27) it follows that $\Phi(x)$ has the critical line $Im(\ell_k) = \hbar_k$, where $\hbar_k \in \mathbb{R}\setminus\{0\}$.

This is contradicted against the fact $\Phi(x)$ has the critical line $Im(x) = 0$.

Case 2.

Now, we assume that

$$\lambda_k = i\gamma_k, \tag{28}$$

run the zeros of $\Phi(x)$, where $\gamma_k \in \mathbb{R}\setminus\{0\}$.

By (28), we have

$$\Phi(\gamma_k) = \sum_{m=1}^{\infty} \frac{(-1)^m \Phi^{(2m)}(0)}{\Gamma(2m+1)} \gamma_k^{2m} = \Phi(0) \prod_{k=1}^{\infty} \left(1 - \frac{i\gamma}{\ell_k}\right) = 0 \tag{29}$$

such that

$$1 - \frac{i\gamma}{\ell_k} = 0, \tag{30}$$

where $k \in \mathbb{N}$.

From (30) we show that

$$\ell_k = i\gamma, \tag{31}$$

and we find that $\Phi(x)$ has the critical line $Im(\ell_k) = \gamma$, where $\gamma_k \in \mathbb{R}\setminus\{0\}$.

This implies that (31) is contradicted against the fact $\Phi(x)$ has the critical line $Im(x) = 0$.

To sum up, two cases are contradicted against the fact $\Phi(x)$ has the critical line $Im(x) = 0$.

Hence, we complete the proof of *Theorem 1*.

We now introduce an alternative method to study the of the real zeros of $\Phi(x) \in \mathbb{Y}$ as follows:

Corollary 1. *Let $\Phi(x) \in \mathbb{Y}$ and $x \in \mathbb{R}$. If $\Phi(x)$ has a real zero, then all of its zeros are real.*

Proof. Since $\Phi(x) \in \mathbb{Y}$, we have

$$\Phi(x) = \sum_{m=1}^{\infty} \frac{(-1)^m \Phi^{(2m)}(0)}{\Gamma(2m+1)} x^{2m} = \Phi(0) \prod_{k=1}^{\infty} \left(1 - \frac{x}{\ell_k}\right). \tag{32}$$

Let us consider that $\Phi(x)$ has a real zero $\eta \in \mathbb{R}$.

Then we obtain

$$\Phi(\eta) = 0 \tag{33}$$

such that

$$\Phi\left(\eta\right)=\sum_{m=1}^{\infty}\frac{\left(-1\right)^{m}\Phi^{\left(2m\right)}\left(0\right)}{\Gamma\left(2m+1\right)}\eta^{2m}=\Phi\left(0\right)\prod_{k=1}^{\infty}\left(1-\frac{\eta}{\ell_{k}}\right)=0. \tag{34}$$

From (34) we give

$$\ell_{k}-\eta=0. \tag{35}$$

Hence,

$$Im\left(\ell_{k}\right)=0, \tag{36}$$

which implies that $\Phi\left(x\right)$ has the critical line $Im\left(x\right)=0$.

By *Theorem 1*, we deduce that all of the zeros of $\Phi\left(x\right)$ are real. ☐

It is easy to give the following result:

Corollary 2. *Let* $\Phi\left(x\right)\in\mathbb{Y}$ *and* $x\in\mathbb{R}$. *Then, under the condition of the truth of Theorem 1 (or Corollary 1), we have*

$$\Phi\left(x\right)=\Phi\left(0\right)\prod_{k=1}^{\infty}\left(1-\frac{x^{2}}{\widetilde{\ell}_{k}^{2}}\right), \tag{37}$$

where $\widetilde{\ell}_{k}>0$ *run all of the positive roots of* $\Phi\left(t\right)=0$ *with* $k\in\mathbb{N}$.

Proof. From *Theorem 1* we deduce that all of the zeros of $\Phi\left(x\right)$ are real.

Hence,

$$\begin{aligned}
&\Phi\left(x\right)\\
&=\Phi\left(0\right)\prod_{k=1}^{\infty}\left(1-\tfrac{x}{\ell_{k}}\right)\\
&=\Phi\left(0\right)\prod_{k=1}^{\infty}\left(1-\tfrac{x}{\widetilde{\ell}_{k}}\right)\prod_{k=1}^{\infty}\left(1+\tfrac{x}{\widetilde{\ell}_{k}}\right)\\
&=\Phi\left(0\right)\prod_{k=1}^{\infty}\left(1-\tfrac{x}{\widetilde{\ell}_{k}}\right)\left(1+\tfrac{x}{\widetilde{\ell}_{k}}\right)\\
&=\Phi\left(0\right)\prod_{k=1}^{\infty}\left(1-\tfrac{x^{2}}{\widetilde{\ell}_{k}^{2}}\right),
\end{aligned} \tag{38}$$

where $\widetilde{\ell}_{k}=\left|\ell_{k}\right|>0$ run all of the positive roots of $\Phi\left(t\right)=0$ with $k\in\mathbb{N}$.

In a similar way for *Corollary 1*, we obtain the same result.

Hence, the desired result follows. ☐

Remark. As a direct result, applying (4), $\Phi\left(x\right)\in\mathbb{Y}$ has the followings:

H1: $\ell_{k}\neq0$ are all zeros of $\Phi\left(x\right)\in\mathbb{Y}$;

H2: $\Phi\left(x\right)\in\mathbb{Y}$ is a class of even entire functions of order $\rho=1$ and genus $\vartheta=0$;

H3: $\Phi(x) \in \mathbb{Y}$ is of the exponent of convergence $\lambda = 1$;

H4: $\sum\limits_{k=1}^{\infty} 1/|\ell_k| < \infty$;

H5: $\Phi(x) \in \mathbb{Y}$ converges uniformly on every compact set.

4. THE PROOF OF THEOREM 2

In order to prove the conjecture of Jensen, we at first prove $\Xi(x) \in \mathbb{Y}$ and by *Theorem 1*, we give the proof of *Theorem 2*, in other words that all of its zeros are real.

Now, we suggest the following result for the behavior of $\Xi(x)$:

Theorem 3. *Let* $x \in \mathbb{R}$. *Then*

$$\Xi(x) \in \mathbb{Y}. \tag{39}$$

Proof. From the work of Hadamard [20], we have

$$\Xi(x) = \xi(0) \prod_{k=1}^{\infty} \left(1 - \frac{\frac{1}{2} + ix}{\frac{1}{2} + i\rho_k}\right) \tag{40}$$

such that

$$\begin{aligned}
&\Xi(x) \\
&= \xi(0) \prod_{k=1}^{\infty} \left(1 - \tfrac{\frac{1}{2}+ix}{\frac{1}{2}+i\rho_k}\right) \\
&= \xi(0) \prod_{k=1}^{\infty} \tfrac{i(\rho_n - x)}{\frac{1}{2}+i\rho_k} = \xi(0) \prod_{k=1}^{\infty} \left[\tfrac{i\rho_k}{i\rho_k} \cdot \tfrac{i(\rho_k - x)}{\frac{1}{2}+i\rho_k}\right] \\
&= \xi(0) \prod_{k=1}^{\infty} \left[\tfrac{i\rho_k}{\frac{1}{2}+i\rho_k} \cdot \tfrac{i(\rho_k - x)}{i\rho_k}\right] \\
&= \xi(0) \left(\prod_{k=1}^{\infty} \tfrac{i\rho_k}{\frac{1}{2}+i\rho_k}\right) \cdot \left(\prod_{k=1}^{\infty} \tfrac{\rho_k - x}{\rho_k}\right) \\
&= \xi(0) \left(\prod_{k=1}^{\infty} \tfrac{i\rho_n}{\frac{1}{2}+i\rho_k}\right) \cdot \left[\prod_{k=1}^{\infty} \left(1 - \tfrac{x}{\rho_k}\right)\right] \\
&= \Xi(0) \prod_{k=1}^{\infty} \left(1 - \tfrac{x}{\rho_k}\right),
\end{aligned} \tag{41}$$

where

$$\Xi(0) = \xi(0) \prod_{k=1}^{\infty} \left(1 - \frac{\frac{1}{2}}{\frac{1}{2} + i\rho_k}\right) = \xi(0) \prod_{k=1}^{\infty} \frac{i\rho_k}{\frac{1}{2} + i\rho_k}. \tag{42}$$

Let us recall the Maclaurin formula of $\Xi(x)$,

$$\Xi(x) = \sum_{m=0}^{\infty} \frac{(-1)^m \, \Xi^{(2m)}(0)}{\Gamma(2m+1)} x^{2m}, \tag{43}$$

where $x \in \mathbb{R}$ and $m \in \mathbb{N}$.

By (41) and (43), we arrive at

$$\Xi(x) = \sum_{m=0}^{\infty} \frac{(-1)^m \Xi^{(2m)}(0)}{\Gamma(2m+1)} x^{2m} = \Xi(0) \prod_{k=1}^{\infty} \left(1 - \frac{x}{\rho_k}\right). \tag{44}$$

If by *Lemma 2*, we write the Weierstrass primary factor

$$F(x, 0) = 1 - x, \tag{45}$$

by

$$\Xi(x) / \Xi(0) = \prod_{k=1}^{\infty} \left(1 - \frac{x}{\rho_k}\right) = \mathbb{F}(x/\rho_k, 0), \tag{46}$$

then $\Xi(x)/\Xi(0)$ converges uniformly on every compact set Δ since $\Xi(0)$ is a constant [18].

Hence, $\Xi(x)$ converges uniformly on every compact set Δ.

By *Lemma 2*, the genus of $\Xi(x)$ is

$$\vartheta = 0. \tag{47}$$

According to Titchmarsh ([24], Theorem 2.12., p.29), we show that the order of $\Xi(x)$ is $\rho = 1$.

By *Lemma 3*, the exponent of convergence of $\Xi(x)$ is ([24], p.30)

$$\lambda = 1. \tag{48}$$

By (17) and (48),

$$\sum_{k=1}^{\infty} 1/|\rho_k| < \infty. \tag{49}$$

Hence, it is easy to see that $\Xi(x)$ is a real even entire function of order $\rho = 1$, exponent of convergence $\lambda = 1$ and genus $\vartheta = 0$.

By above results and (44), we obtain $\Xi(x) \in \mathbb{Y}$, which is the desired result. ☐

We now give the proof of Theorem 2.

First proof.

By the theorem of Hardy [27], the Riemann zeta-function $\zeta(1/2+ix)$ and entire Riemann zeta-function $\xi(1/2+ix)$ have infinite many zeros ([23], p.226; [24], p.256).

It implies that $\Xi(x)$ has infinitely many zeros on the critical line $Im(x) = 0$.

By *Theorem 3*, we have

$$\Xi(x) \in \mathbb{Y}. \tag{50}$$

By *Theorem 1*, we obtain the desired result.

Second proof.

doi:10.20944/preprints202106.0036.v1

By the reported result of the numerical computation ([23], p.96; ([24], p.30)), we have a first zero ℓ_0 of $\Xi(x)$.

By *Corollary 1*, all of the zeros for $\Xi(x)$ are real.

To sum up, by *Corollary 2* and (44), we have

$$\Xi(x) = \xi(0)\prod_{k=1}^{\infty}\left(1-\frac{\frac{1}{2}+ix}{\frac{1}{2}+i\rho_k}\right) = \Xi(0)\prod_{k=1}^{\infty}\left(1-\frac{x}{\rho_k}\right) = \Xi(0)\prod_{k=1}^{\infty}\left(1-\frac{x^2}{\widetilde{\rho_k}^2}\right), \tag{51}$$

where $\widetilde{\rho_k} = |\rho_k|$ run all of the positive roots of $\Xi(t) = 0$ with $k \in \mathbb{N}$.

Hence, the desired result follows.

By the above proofs, the conjecture of Jensen holds.

Acknowledgements. This work is supported by the Yue-Qi Scholar of the China University of Mining and Technology (No. 102504180004).

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