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$3x + 1$ problem: Iterative operations neither cause loops nor diverge to infinity

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Abstract The 3x+1 problem is a problem of continuous iteration for integers. According to the original description of the problem, we derive a formula that can perform continuous iterative operations on odd numbers. We can convert this formula into a linear equation. The process of solving this equation shows that the relationship between the iteration result and the odd number being iterated is linear. In addition, we can construct a loop iteration equation by the formula and obtain the result of the equation: the equation has only one positive integer solution. Extending this result to all positive even numbers, we get the answer to the $3x + 1$ question.
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Key words $3x + 1$ problem; Collatz conjecture; Ulams problem; Syracuse problem

1 Introduction

The $3x + 1$ problem is currently an unsolved problem in number theory. According to the description of this problem: take a positive integer, if it is an even number, divide it by 2, and if it is an odd number, multiply it by 3 and add 1 to get a positive integer. Repeat this operation, and then get a string of natural numbers, and finally return 1.

People guess that same result will be obtained for all natural number iterations.[2]−[5]

This problem is also known as the Collatz conjecture, Syracuse problem or Ulams problem. Since this issue was raised, no substantial progress has been made in proving this issue. As of 2017, more than 2^{64} natural numbers have been verified by the computer. As a result, all tested integers will eventually return 1 without exception. Until September 2019, Terence Tao proved that this conjecture was almost correct for all integers.

This article, without violating the original intent of the problem, improves the original description of the problem and analyzes the three possible results of successive iterations of odd numbers, thereby solving this problem.

2 Iterative formula for positive odd integers

It is known that the number of factors equal to 2 contained in a positive integer is constant. Let x be a positive integer, then $3x + 1$ is an even. After dividing by all factors equal to 2, the quotient is a positive odd integer.

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Let x be a positive odd integer, $T = \dfrac{3x+1}{2^{\beta}}$ (β is a positive integer). If T is a positive odd integer, then call it an *iteration* of x. Let $T(x)$ be the iteration result of x, β be the *iterative exponent*.

Let $T^{(n)}(x)$ be the result of the n-th iteration of x.
If $T^{(n)}(x) = 1$ $(x > 1)$ then x is *convergence*.
Let β_n be the iterative exponent of the n-th iteration of x.
Let $s(0) = 0$, $s(j) = \sum_{i=1}^{j} \beta_i$. $(j \in Z^+)$

Theorem 1. *The n-th iteration formula of x is as follows.*

$$T^{(n)}(x) = \frac{3^n x + \sum_{i=1}^{n} 2^{s(i-1)} 3^{n-i}}{2^{s(n)}}. \tag{2.1}$$

Proof. For $n = 1$. Since $2^{s(1-n)}3^{n-1} = 1$, so, according to (2.1), we have

$$T^{(1)}(x) = \frac{3x+1}{2^{\beta_1}}.$$

If (2.1) holds, then for $n+1$ we have

$$\begin{aligned} T^{(n+1)}(x) &= \frac{3T^{(n)}(x)+1}{2^{\beta_{n+1}}} \\ &= \frac{3\dfrac{3^n x + \sum_{i=1}^{n} 2^{s(i-1)} 3^{n-i}}{2^{s(n)}} + 1}{2^{\beta_{n+1}}} \\ &= \frac{3^{n+1} x + \sum_{i=1}^{n+1} 2^{s(i-1)} 3^{n+1-i}}{2^{s(n+1)}}. \end{aligned}$$

3 $T^{(n)}(x)$ tends to infinity of possibilities

Let $[A]$ be the integer part of the real number A.
If $T^{(n)}(x)$ tends to infinity when $n \to \infty$, then x is *divergent*.

Theorem 2. *No matter how big n is, If x is a finite value, then $T^{(n)}(x)$ is also a finite value.*

Proof. Let $A_n = 2^{s(n)}$, $B_n = 3^n$, $C_n = \sum_{i=1}^{n} 2^{s(i-1)} 3^{n-i}$.

See the following indefinite equation

$$A_n Y = B_n X + C_n. \tag{3.1}$$

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Since A_n, B_n and C_n are both positive integers and $(A_n, B_n) = 1$, there are infinitely many integer solutions to this equation:

$$\begin{cases} Y = Y_1 + B_n t, \\ X = X_1 + A_n t. \end{cases} \qquad (t \in \mathbb{Z})$$

The following is the process of getting Y_1 and X_1:[1]
(i). For $B_n > A_n$, the following equation can be constructed

$$\begin{cases} A_n \left[\dfrac{B_n}{A_n}\right] - B_n = -q_1, \\ A_n \left(\left[\dfrac{B_n}{A_n}\right] + 1\right) - B_n = q_2, \end{cases} \tag{3.2}$$

Since $(q_1, q_2) = 1$, so, there are integers θ and λ such that the following formula holds

$$\lambda q_2 - \theta q_1 = C_n. \tag{3.3}$$

Through (3.2) and (3.3), we get a special solution of the equation

$$\begin{cases} X_1 = \lambda + \theta, \\ Y_1 = \lambda \left(\left[\dfrac{B_n}{A_n}\right] + 1\right) + \theta \left[\dfrac{B_n}{A_n}\right]. \end{cases}$$

and

$$\lambda = \frac{C_n + \theta(A_n - q_2)}{q_2}.$$

Thus, we obtain

$$\begin{cases} X_1 = \dfrac{C_n + \theta A_n}{q_2}, \\ Y_1 = X_1 \left(\left[\dfrac{B_n}{A_n}\right] + 1\right) - \theta. \end{cases}$$

For all $n > 4$,

$$C_n > 0.8 \times 3^n + \sum_{i=5}^{n} 2^{s(i-1)} 3^{n-i} > 0.8 B_n. \tag{3.4}$$

Since $q_2 < A_n$, so, $X_1 > \dfrac{C_n}{A_n} + \theta$. So, if X_1 is a finite, then $\dfrac{C_n}{A_n}$ is also a finite. In this case, according to (3.4), $\dfrac{B_n}{A_n}$ can only be a finite value, as is Y_1.
(ii). For $A_n > B_n$, same as above, the following equation can be constructed

$$\begin{cases} A_n - B_n \left[\dfrac{A_n}{B_n}\right] = q_1, \\ A_n - B_n \left(\left[\dfrac{A_n}{B_n}\right] + 1\right) = -q_2, \\ \lambda q_1 - \theta q_2 = C_n. \end{cases} \tag{3.5}$$

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Through (3.5), we get a special solution of the equation

$$\begin{cases} Y_1 = \theta + \lambda, \\ X_1 = Y_1 \left[\dfrac{A_n}{B_n}\right] + \theta. \end{cases}$$

Since $Y_1 = \dfrac{X_1 - \theta}{[A_n/B_n]}$, so, if X_1 is finite, then Y_1 is also finite.
Both cases show that when X is a finite value, Y can only be a finite value.

If both ends of the equal sign are multiplied by $2^{s(n)}$ at the same time, then formula (2.1) can be written as

$$2^{s(n)} T^{(n)}(x) = 3^n x + \sum_{i=1}^{n} 2^{s(i-1)} 3^{n-i}. \tag{3.6}$$

Let $T^{(n)}(x) = Y$ and $x = X$. Formulas (3.6) and (3.1) are the same. The previous analysis indicates that when $n \to \infty$, if x is finite, then $T^{(n)}(x)$ is also finite.

4 Possibility of $T^{(n)}(x) = x$

Theorem 3. *Let $r(i)$ be an integer, $r(i+1) > r(i)$ and $r(0) = 0$. If the following equation have positive integer solution, it must satisfy $r(i) = 2i$ $(i = 0, 1, 2, 3, \ldots\ldots, n)$.*

$$f(n) = \frac{\sum_{i=1}^{n} 2^{r(i-1)} 3^{n-i}}{2^{r(n)} - 3^n}. \quad (n \in Z^+) \tag{4.1}$$

Proof. For $n = 1$, we have

$$f(1) = \frac{1}{2^{r(1)} - 3} \begin{cases} = -1, & r(1) = 1. \\ = 1, & r(1) = 2. \\ \leq 1/5, & r(1) > 2. \end{cases}$$

The above formula shows that, only $f(1) = 1$ is a positive integer solution of this formula.
If the conditions required by the theorem is true for n, then for $n + 1$,

$$f(n+1) = \frac{\sum_{i=1}^{n+1} 2^{r(i-1)} 3^{n+1-i}}{2^{r(n+1)} - 3^{n+1}}. \tag{4.2}$$

where $r(i) = 2i$ $(i = 0, 1, 2, 3, \ldots\ldots, n)$.
According to the conditions, the formula (4.2) can be written as

$$f(n+1) = \frac{2^{2n+2} - 3^{n+1}}{2^{r(n+1)} - 3^{n+1}}.$$

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Since $r(n+1) > r(n)$ and $r(n) = 2n$, let $r(n+1) = 2n+d$ (d is a positive integer). Refer to the equation below

$$h(n) = \frac{2^{2n+2} - 3^{n+1}}{2^{2n+d} - 3^{n+1}}.$$

1. If $d = 1$, the above function can be written as

$$h(n) = \frac{2(2^{2n+1} - 3^{n+1}) + 3^{n+1}}{2^{2n+1} - 3^{n+1}} = 2 + \frac{3^{n+1}}{2^{2n+1} - 3^{n+1}}.$$

and
$$\frac{3^{(n+1)+1}}{2^{2(n+1)+1} - 3^{(n+1)+1}} \div \frac{3^{n+1}}{2^{2n+1} - 3^{n+1}} = \frac{3}{4 + \dfrac{3^{n+1}}{2^{2n+1} - 3^{n+1}}}.$$

The above results show that, for all $n > 1$, (i) $h(n) > 2$; (ii) $h(n)$ is monotonically decreasing. Since $h(2) = \frac{37}{5}$, $h(3) = \frac{175}{47}$ and $h(4) \doteq 2.9$, so, $h(n)$ is not a positive integer.

2. If $d = 2$, then $h(n) \equiv 1$;
3. If $d > 2$ then, for all $n > 1$, $h(n) < 1$.

Based on the above analysis, the theorem holds.

If $T^{(n)}(x) = x$ $(x > 1)$, then x is said to be *cyclic.*

After continuous n iterations for x_c, if $T^{(n)}(x_c) = x_c$, then, according to formula (2.1), we have

$$x_c = \frac{3^n x_c + \sum_{i=1}^{n} 2^{s(i-1)} 3^{n-i}}{2^{s(n)}}.$$

From the formula above we get

$$x_c = \frac{\sum_{i=1}^{n} 2^{s(i-1)} 3^{n-i}}{2^{s(n)} - 3^n}. \tag{4.3}$$

Let $s(n) = r(n)$ and $x_c = f(n)$. Then the above equation is the same as equation (4.1). According to Theorem 3, only if $s(i) = 2i$ $(0 \le i \le n)$ is satisfied, equation (4.3) has a positive integer solution, and except for $x_c = 1$, the equation has no other positive integer solutions. This indicates that when $x_c = 1$, for all $n \ge 1$, the equation $T^{(n)}(1) = 1$ is true; And for all $x > 1$, there is no $T^{(n)}(x_c) = x_c$.

Thus, for every odd number greater than 1, a finite number of iterations cannot constitute a loop.

5 Conclusion.

The above analysis shows that all positive integers are neither *cyclic* nor *divergent.* Therefore, no matter how many iterations, every odd integer will return 1, that is, all positive odd numbers *converge.*

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Because after dividing by all factors equal to 2, every positive even number is an odd integer, so this conclusion also applies to even numbers. Therefore, *all positive integers are Converge.*

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