Type II Vacuum Spacetime Admitting Closed Timelike Curves

Faizuddin Ahmed[1]

Department of Physics, Cotton College, Guwahati-781001, India

Current Affiliation : National Academy Gauripur, Assam, India, 783331

Abstract

We present a cyclic symmetry type II vacuum spacetime admitting closed timelike curves (CTCs) which appear after a certain instant of time, *i.e.,* a time-machine spacetime. The various authors in past have considered the 2D and 4D flat generalization of Misner space, but in the present work, we have considered the curved spacetime generalzations of 4D Misner space, and is asymptotically flat radially.

Keywords: closed timelike curves, Misner space, vacuum spacetime

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1 Introduction

One of the most interesting aspects of Einstein's theory of gravitation is that there are solutions of the Field Equations admitting closed time-like curves (CTCs). The first known space-time with such causality violating behaviour is the Gödel universe [1] and, in fact, there are a considerable number of space-times in the literature that admit CTCs. The presence of CTCs, or equivalently, closed null or time-like geodesics in certain solutions of the field equations lead to the theoretical possibility of time-travel. A

[1] **faizuddinahmed15@gmail.com ; faiz4U.enter@rediffmil.com**

small sample of solutions that allow such causality violating behaviour are [2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22]. One way of classifying such causality violating space-times would be to categorize the metrics as either eternal time-machine in which CTCs always exist (in this class would be [1] or [2]), or as time-machine space-times in which CTCs appear after a certain instant of time. In the latter category would be the ones discussed in [14, 15, 16, 17]. The most natural and simplest situation would, of course, be a vacuum solution. In fact, several vacuum space-times with CTCs are known. Some well known ones are the NUT-Taub metric [23], the space-times found by Ori [16, 17] and the Kerr-Newmann black holes [21, 22].

Many of the above models, however, suffer from one or more severe drawbacks as possible candidates for a working time-machine. For instance, in some of these solutions, the weak energy condition (WEC) is violated indicating unrealistic matter-energy content [5]. The WEC states that for any physical (time-like) observer the energy density is non-negative, which is the case for all known types of (classical) matter fields. We note that the eternal time-machine space-times are unrealistic models for a putative time machine. A workable model of a time machine must be a space-time where CTCs appear at a definite instant of time.

Among the time machine space-times, we mention two: the first being Ori's compact core [16] which is represented by a vacuum metric locally isometric to pp waves and second, which is more relevant to the present work, the Misner space [24]. This is essentially a two dimensional metric (hence flat) with peculiar identifications. The Misner space is interesting in the context of CTCs as it is an example of a space-time where CTCs evolve from causally well-behaved initial conditions.

The metric for the Misner space [24]

$$ds^2 = -2\,dt\,dx - t\,dx^2 \quad . \tag{1}$$

where $-\infty < t < \infty$ but the co-ordinate x periodic. The metric (1) is regular everywhere as $detg = -1$ including at $t = 0$. The curves $t = t_0$, where t_0 is a constant, are closed since x is periodic. The curves $t < 0$ are spacelike, but $t > 0$ are timelike and the null curves $t = t_0 = 0$ form the chronology horizon. The second type of curves, namely, $t = t_0 > 0$ are closed time-like curves (CTCs). This metric has been the subject of intense study and quite recently, Levanony and Ori [25] have studied the motion of extended bodies in the 2D Misner space and its flat 4D generalizations. However, the major problem with the Misner space as a feasible time machine, namely, that the space-time is not physically relevant, since basically, it is a cut and paste version of the Minkowski spacetime, remains, and this lacunae has been identified in [25]. A non-flat 4D space-time, satisfying all the energy conditions, but with causality violating properties of the Misner space, primarily that CTCs evolve smoothly from a initially causally well-behaved stage, would be physically more acceptable as a time-machine space-time.

In this paper, we shall attempt to show that causality violating curves may appear in vacuum space-times (which automatically satisfy the energy conditions) with comparatively simple structure.

2 Analysis of the space-time

Consider the following metric

$$ds^2 = e^{3\,r}\,dr^2 + e^{-r}\,dz^2 - e^{2\,r}\,\left(t\,d\phi^2 + 2\,dt\,d\phi + 2\,\beta\,dr\,d\phi\right), \tag{2}$$

where ϕ co-ordinate is assumed periodic $0 \leq \phi \leq \phi_0$, with $\phi_0 > 0$ and $\beta > 0$ is a real number. We have used co-ordinates $x^1 = r$, $x^2 = \phi$, $x^3 = z$ and $x^4 = t$. The ranges of the other co-ordinates are $t, z \in (-\infty, \infty)$ and $0 < r < \infty$. The metric has signature $(+,+,+,-)$ and the determinant of the corresponding metric tensor $g_{\mu\nu}$, $det\ g = -e^{6\,r}$. The space-time (3) is a vacuum solution of the field equations with the Ricci tensor $R_{\mu\nu} = 0$.

Consider closed orbits of constant $t = t_0$, $r = r_0$ and $z = z_0$, the metric

(2) reduces to one-dimensional form

$$ds^2 = -t\, e^{2\,r}\, d\phi^2 \tag{3}$$

These orbits are null curves for $t = 0$, spacelike throughout for $t = t_0 < 0$, but become timelike for $t = t_0 > 0$, which indicates the presence of closed time-like curves (CTCs).

In order that the above analysis is valid, the CTCs evolve from an space-like $t = constant$ hypersurface (and thus t is a time coordinate)[16]. This can be ascertained by calculating the norm of the vector $\nabla_\mu t$ (or by determining the sign of the component g^{tt} in the inverse metric tensor $g^{\mu\nu}$[16]). We find from (2) that

$$g^{tt} = e^{-3\,r}\,\left(t\, e^{r} + \beta^2\right) \tag{4}$$

Thus a hypersurface $t = constant$ is spacelike provided $g^{tt} < 0$ for $t = t_0 < 0$, but become timelike provided $g^{tt} > 0$ for sufficiently large $t = t_0 > 0$. Here we choose r_0 ($r = r_0$, a constant) and β is sufficiently small positive number such that the above condition is satisfied.

For cyclic symmetry metric, consider the Killing vector $\eta = \partial_\phi$ which has the normal form

$$\eta^\mu = (0, 1, 0, 0) \quad . \tag{5}$$

Its co-vector is

$$\eta_\mu = e^{2\,r}\,(-\beta, -t, 0, -1) \quad . \tag{6}$$

(5) satisfies the Killing equation $\eta_{\mu\,;\,\nu} + \eta_{\nu\,;\,\mu} = 0$. For a cyclically symmetric metric, the norm $\eta_\mu\, \eta^\mu$ of the Killing vector form spacelike, closed orbits [26, 27]. We note that

$$\eta^\mu\, \eta_\mu = -t\, e^{2\,r} \quad , \tag{7}$$

which is spacelike for $t < 0$, and closed. As $r = 0$ does not a represent symmetry axis, the spacetime is multiple-connected like the Misner space. Hence the spacetime (2) is cyclically symmetric without a symmetry axis.

3 Classification of the space-time and the kinematical properties

For classification of the spacetime (2), we can construct the following set of null tetrads $(k, l, m, \bar{m})$ where l, n are real and $m, \bar{m}$ are complex in the form

$$\begin{aligned} k_\mu &= (0, 1, 0, 0)\,, \\ l_\mu &= e^{2\,r}\left(\beta, \frac{t}{2}, 0, 1\right), \\ m_\mu &= \frac{1}{\sqrt{2}}\left(e^{\frac{3\,r}{2}}\,0, i\,e^{-\frac{r}{2}}, 0\right), \\ \bar{m}_\mu &= \frac{1}{\sqrt{2}}\left(e^{\frac{3\,r}{2}}\,0, -i\,e^{-\frac{r}{2}}, 0\right), \end{aligned} \tag{8}$$

where $i = \sqrt{-1}$. The set of null tetrads above are such that the metric tensor for the line element (2) can be expressed as

$$g_{\mu\nu} = -k_\mu\,l_\nu - l_\mu\,k_\nu + m_\mu\,\bar{m}_\nu + \bar{m}_\mu\,m_\nu \quad . \tag{9}$$

The vectors (8) are null vector and are orthogonal

$$\begin{aligned} k^\mu\,k_\mu = l^\mu\,l_\mu = m^\mu\,m_\mu = \bar{m}^\mu\,\bar{m}_\mu = 0, \\ k^\mu\,m_\mu = k^\mu\,\bar{m}_\mu = l^\mu\,m_\mu = l^\mu\,\bar{m}_\mu = 0 \end{aligned} \tag{10}$$

except for $k_\mu\,l^\mu = -1$ and $m_\mu\,\bar{m}^\mu = 1$. Using this null tetrad above we have calculated the five Weyl scalars

$$\Psi_2 = \frac{e^{-3\,r}}{2} \qquad \text{and} \qquad \Psi_4 = -\frac{\beta}{4}\,e^{-r} \tag{11}$$

are non-vanishing, while $\Psi_0 = \Psi_1 = \Psi_3 = 0$. This clearly shows that the metric (2) is of type II in the Petrov classification scheme. The metric (2) is free-from curvature singularities. The curvature invariant known as Kretchsmann scalar is given by

$$R^{\mu\nu\rho\sigma}\,R_{\mu\nu\rho\sigma} = 12\,e^{-6\,r} \tag{12}$$

The Kretschmann scalar, the Weyl scalar Ψ_2, Ψ_4 vanish at $r \to \infty$ indicating asymptotic flatness of the spacetime (2).

Using the null tetrad (8), we have calculated the *optical* scalars [27], namely, the *expansion*, the *rotation* and the *shear* and they are found to be

$$\begin{aligned}
\boldsymbol{\theta} &= \frac{1}{2}\,k^{\mu}_{\;;\,\mu} = 0 \quad , \\
\boldsymbol{\omega}^2 &= \frac{1}{2}\,k_{[\mu\,;\,\nu]}\,k^{\mu\,;\,\nu} = 0 \quad , \\
\boldsymbol{\sigma}\,\bar{\boldsymbol{\sigma}} &= \frac{1}{2}\,k_{(\mu\,;\,\nu)}\,k^{\mu\,;\,\nu} - \boldsymbol{\theta}^2 = 0 \quad ,
\end{aligned} \tag{13}$$

and the null vector (8) satisfy the geodesics equation

$$k_{\mu\,;\,\nu}\,k^{\nu} = 0 \tag{14}$$

Hence the spacetime is non-diverging, have shear-free null geodesics congruence. We have determine the Newmann-Penrose spin co-efficients [27] and they are

$$\begin{aligned}
\nu &= -\frac{\beta}{2\sqrt{2}}\,e^{\frac{r}{2}}, \\
-\tau &= \pi = \frac{e^{-\frac{3r}{2}}}{\sqrt{2}}, \\
\beta &= 3\,\alpha = -\frac{3}{4\sqrt{2}}\,e^{-\frac{3r}{2}}, \\
\gamma &= -\frac{1}{4}, \\
\kappa &= \rho = \sigma = \tau = \mu = \lambda = \pi = \epsilon = 0,
\end{aligned} \tag{15}$$

where the symbols are same as in [27].

4 Conclusion

Our primary motivation in this paper is to write down a metric for a spacetime that incorporates the Misner space and its causality violating properties and to classify it. The solution presented here is Ricci flat, cyclicly symmetric metric (2) and serves as a model of a time-machine spacetime in the sense that CTCs appear at a definite instant and the energy conditions are satisfied. Most well-known CTC spacetimes violate one or more of these caveats and the spacetime discussed here thus falls in the category of a miniscule number of those which may be termed as true time-machine space-times.

Appendix A

The considered metric (2) can be transformed to a simple form as follows. Let us perform a transformation

$$r \to \frac{1}{2}\ln\varrho \tag{1}$$

into the metric (2), we have

$$ds^2 = \frac{1}{\sqrt{\varrho}}\left(\frac{d\varrho^2}{4} + dz^2\right) + \varrho\,(-t\,d\phi^2 - 2\,dt\,d\phi) - \beta\,d\varrho\,d\phi. \tag{2}$$

Recently, an axially symmetric Petrov type II space-time was seen (Bidyut B. Hazarika, Int. J. Mod. Phys. **A 36** (03), 2150017 (2021) ; DOI: 10.1142/S0217751X21500172) with the following metric form

$$\begin{aligned} ds^2 \;&=\; \frac{4\,\sinh^4 r\,\cosh^2 r\,dr^2}{\alpha^2\,f(r)} + \operatorname{csch}^2 r\,f(r)\,dz^2 - \sinh^4 r\,\sinh t\,d\phi^2 \\ &\quad -2\,\sinh^4 r\,\cosh t\,dt\,d\phi + \frac{2\,\beta\,\sinh^3 r\,\cosh r}{f(r)}\,dr\,d\phi, \end{aligned} \tag{3}$$

where α, β are nonzero real numbers.

One can perform the following transformations

$$r \to \sinh^{-1}\rho \quad , \quad t \to \sinh^{-1} T \tag{4}$$

into the metric (3), one will get

$$ds^2 = \frac{4\,\rho^4\,d\rho^2}{\alpha^2\,f(\rho)} + \frac{f(\rho)}{\rho^2}\,dz^2 + \rho^4\,(-T\,d\phi^2 - 2\,dT\,d\phi) + \frac{2\,\beta\,\rho^3}{f(\rho)}\,d\rho\,d\phi. \tag{5}$$

Finally doing another transformation as

$$\rho \to \varrho^{\frac{1}{4}} \tag{6}$$

into the metric (5), one will get

$$ds^2 = \frac{1}{\sqrt{\varrho}} \left(\frac{d\varrho^2}{4\,\alpha^2\, f(\varrho)} + f(\varrho)\, dz^2 \right) + \varrho\, (-T\, d\phi^2 - 2\, dT\, d\phi) + \frac{\beta}{2\, f(\varrho)}\, d\varrho\, d\phi. \tag{7}$$

One can choose $f(\varrho) = 1$ as considered in *section 5* (Int. J. Mod. Phys. **A 36** (03), 2150017 (2021)). So the line element (7) becomes

$$ds^2 = \frac{1}{\sqrt{\varrho}} \left(\frac{d\varrho^2}{4\,\alpha^2} + dz^2 \right) + \varrho\, (-T\, d\phi^2 - 2\, dT\, d\phi) + \frac{\beta}{2}\, d\varrho\, d\phi, \tag{8}$$

a Petrov type II Ricci flat or vacuum space-time, where α, β are nonzero real numbers.

Since the metric (8) is now a Ricci flat ($R_{\mu\nu} = 0$) space-time, one can choose for simplicity $\alpha \to 1$ and $\beta \to -2\,\beta$. Therefore, the vacuum space-time (8) finally becomes

$$ds^2 = \frac{1}{\sqrt{\varrho}} \left(\frac{d\varrho^2}{4} + dz^2 \right) + \varrho\, (-T\, d\phi^2 - 2\, dT\, d\phi) - \beta\, d\varrho\, d\phi \tag{9}$$

which is similar to Petrov II vacuum metric (2).

References

[1] K. Gödel, Rev. Mod. Phy. **21**, 447 (1949).

[2] W. J. van Stockum, Proc. R. Soc. Edin. **57**, 135 (1937).

[3] F. J. Tipler, Phys. Rev. **D 9**, 2203 (1974).

[4] J. R. Gott, Phys. Rev. Lett., **66**, 1126 (1991).

[5] M. S. Morris, K. S. Thorne and U. Yurtsever, Phys. Rev. **D 49**, 3990 (1988) ; Phys. Rev. Lett. **61**, 1446 (1988).

[6] M. Gürses, A. Karasu and Ö. Sarioğlu, Class. Quan. Grav., **22**, 1527 (2005).

[7] M. Alcubierre, Class. Quan. Grav. **11**, L73 (1994).

[8] D. Sarma, M. Patgiri and F. U. Ahmed, Ger. Rel. Grav., **46**, 1633 (2014).

[9] D. Sarma, M. Patgiri and F. U. Ahmed, Ann. of Phys., **329**, 179 (2013).

[10] B. R. Steadmann, Gen. Rel. Grav., **35**, 1721 (2003).

[11] V. M. Rosa and P. S. Letelier, `Gödel-type spacetimes: History and new developments` by M. Plaue and M. Scherfner, Eds., Kurt Gödel Society Collegium Logicum V **X**, 129 (2010).

[12] V. M. Rosa and P. S. Letelier, Gen. Rel. Grav. **39**, 1419 (2007).

[13] V. M. Rosa and P. S. Letelier, Phys. Lett. A370, 99 (2007).

[14] A. Ori, Y. Soen, Phys. Rev. **D49**, 3990 (1994).

[15] A. Ori, Y. Soen, Phys. Rev. **D54**, 4858 (1996).

[16] A. Ori, Phys. Rev. Lett. **95**, 021101 (2005).

[17] A. Ori, Phys. Rev. **D 76**, 044002 (2007).

[18] I. D. Soares, J. Math. Phys. **21**, 521 (1980).

[19] Ø. Grøn and S. Johannesen, New Jour. Phys. **10**, 103025 (2008).

[20] Ø. Grøn and S. Johannesen, Nuovo Cimento, **B125**, 1215 (2010).

[21] R. P. Kerr, Phys. Rev. Lett. **11**, 237 (1963).

[22] B. Carter, Phys. Rev. **174**, 1559 (1968).

[23] C. W. Misner, A. H. Taub, Sov. Phys. JETP **28**, 122 (1969).

[24] C. W. Misner, `Relativity Theory and Astrophysics I : Relativity and Cosmology`, edited by J. Ehlers (`Lectures in Applied Mathematics`, **Vol. 8**), AMS (1967).

[25] D. Levanony and A. Ori, Phys. Rev. **D83**, 044043 (2011).

[26] M. Mars and J. M. M. Senovilla, Class. Quantum Grav. **10**, 1633(1993).

[27] H. Stephani, D. Kramer, M. MacCallum, C. Hoenselaers, E. Herlt, `Exact Solutions to Einstein's Field Equations`, Cambridge University Press, Cambridge (2003).