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Binomial Fibonacci Power Sums

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Abstract

We evaluate various binomial sums involving the powers of Fibonacci and Lucas numbers.

1 Introduction

Our main goal in this paper is to evaluate the following sums of powers of Fibonacci and Lucas numbers involving the binomial coefficients:

$$\sum_{k=0}^{n}(\pm 1)^k\binom{n}{k}F_{j(rk+s)}^{2m},\quad \sum_{k=0}^{n}(\pm 1)^k\binom{n}{k}L_{j(rk+s)}^{2m},$$

$$\sum_{k=0}^{n}(\pm 1)^k\binom{n}{k}F_{j(2rk+s)}^{2m+1},\quad \sum_{k=0}^{n}(\pm 1)^k\binom{n}{k}L_{j(2rk+s)}^{2m+1};$$

thereby extending the work of Wessner [11], Hoggatt and Bicknell [3, 4], Long [6], Kiliç et al [7] and several previous researchers. Here n is any non-negative integer, j, m, r and s are any integers and F_t and L_t are the Fibonacci and Lucas numbers.

There is a dearth of binomial cubic Fibonacci and Lucas identities. We will show that, for any non-negative integer n and any integer s,

$$\sum_{k=0}^{n}\binom{n}{k}F_{k+s}^3=\frac{1}{5}(2^nF_{2n+3s}+3F_{n-s}),$$

$$\sum_{k=0}^{n}\binom{n}{k}L_{k+s}^3=2^nL_{2n+3s}+3L_{n-s},$$

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^kF_{k+s}^3=\frac{1}{5}((-1)^n2^nF_{n+3s}-(-1)^s3F_{2n+s}),$$

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}L_{k+s}^3=(-1)^n2^nL_{n+3s}+(-1)^s3L_{2n+s},$$

$$\sum_{k=0}^{n}\binom{n}{k}2^kF_{k+s}^3=\begin{cases}5^{n/2-1}(F_{3n+3s}-(-1)^s3F_s), & n\text{ even;}\\ 5^{(n-3)/2}(L_{3n+3s}+(-1)^s3L_s) & n\text{ odd,}\end{cases}$$

$$\sum_{k=0}^{n}\binom{n}{k}2^kL_{k+s}^3=\begin{cases}5^{n/2}(L_{3n+3s}+(-1)^s3L_s), & n\text{ even;}\\ 5^{(n+1)/2}(F_{3n+3s}-(-1)^s3F_s) & n\text{ odd.}\end{cases}$$

We will also derive the following binomial summation identities which we believe are new:

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}F_{2jr+p}^{n-k}F_p^kF_{j(rk+s)}^2=\frac{1}{5}(F_{2jr}^nL_{pn-2js}-(-1)^{js}2F_{jr}^nL_{jr+p}^n),$$

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}F_{2jr+p}^{n-k}F_p^kL_{j(rk+s)}^2=F_{2jr}^nL_{pn-2js}+(-1)^{js}2F_{jr}^nL_{jr+p}^n,$$

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}L_{2jr+p}^{n-k}L_p^kF_{j(rk+s)}^2=\begin{cases}5^{n/2-1}F_{2jr}^nL_{pn-2js}-(-1)^{js}5^{n-1}2F_{jr}^nF_{jr+p}^n, & n\text{ even;}\\ 5^{(n-1)/2}F_{2jr}^nF_{pn-2js}-(-1)^{js}5^{n-1}2F_{jr}^nF_{jr+p}^n, & n\text{ odd,}\end{cases}$$

and

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}L_{2jr+p}^{n-k}L_p^kL_{j(rk+s)}^2=\begin{cases}5^{n/2}F_{2jr}^nL_{pn-2js}+(-1)^{js}5^n2F_{jr}^nF_{jr+p}^n, & n\text{ even;}\\ 5^{(n+1)/2}F_{2jr}^nF_{pn-2js}+(-1)^{js}5^n2F_{jr}^nF_{jr+p}^n, & n\text{ odd.}\end{cases}$$

The Fibonacci numbers, F_n, and the Lucas numbers, L_n, are defined, for $n\in\mathbb{Z}$, through the recurrence relations

$$F_n=F_{n-1}+F_{n-2},\,(n\geq 2),\quad F_0=0,\ F_1=1; \tag{1}$$

and

$$L_n=L_{n-1}+L_{n-2},\,(n\geq 2),\quad L_0=2,\ L_1=1; \tag{2}$$

with

$$F_{-n}=(-1)^{n-1}F_n,\quad L_{-n}=(-1)^nL_n. \tag{3}$$

Throughout this paper, we denote the golden ratio, $(1+\sqrt{5})/2$, by α and write $\beta=(1-\sqrt{5})/2=-1/\alpha$, so that $\alpha\beta=-1$ and $\alpha+\beta=1$.

Explicit formulas (Binet formulas) for the Fibonacci and Lucas numbers are

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \quad L_n = \alpha^n + \beta^n, \quad n \in \mathbb{Z}. \tag{4}$$

Koshy [8] and Vajda [10] have written excellent books dealing with Fibonacci and Lucas numbers.

Our results emanate from the following general Fibonacci and Lucas summation identities (Lemma 2):

$$\sum_{k=0}^{n} \binom{n}{k} x^{n-k} z^k F_{j(rk+s)}^m = \frac{1}{(\sqrt{5})^m} \sum_{i=0}^{m} (-1)^{i(js+1)} \binom{m}{i} \alpha^{(m-2i)js} \left(x + (-1)^{ijr} \alpha^{(m-2i)jr} z\right)^n, \tag{BF}$$

$$\sum_{k=0}^{n} \binom{n}{k} x^{n-k} z^k L_{j(rk+s)}^m = \sum_{i=0}^{m} (-1)^{ijs} \binom{m}{i} \alpha^{(m-2i)js} \left(x + (-1)^{ijr} \alpha^{(m-2i)jr} z\right)^n. \tag{BL}$$

For lower m, the identities (BF) and (BL) are more useful in the equivalent form

$$\sum_{k=0}^{n} \binom{n}{k} x^{n-k} z^k F_{j(rk+s)}^m = \frac{1}{(\sqrt{5})^m} \sum_{i=0}^{m} (-1)^{i} \binom{m}{i} \beta^{ijs} \alpha^{(m-i)js} \left(x + \beta^{ijr} \alpha^{(m-i)jr} z\right)^n, \tag{BF$'$}$$

$$\sum_{k=0}^{n} \binom{n}{k} x^{n-k} z^k L_{j(rk+s)}^m = \sum_{i=0}^{m} \binom{m}{i} \beta^{ijs} \alpha^{(m-i)js} \left(x + \beta^{ijr} \alpha^{(m-i)jr} z\right)^n. \tag{BL$'$}$$

When $m = 1$, we have the weighted linear binomial identities:

$$\sum_{k=0}^{n} \binom{n}{k} x^{n-k} z^k F_{j(rk+s)} = \frac{1}{\sqrt{5}} \left(\alpha^{js}(x + \alpha^{jr} z)^n - \beta^{js}(x + \beta^{jr} z)^n\right), \tag{F1}$$

$$\sum_{k=0}^{n} \binom{n}{k} x^{n-k} z^k L_{j(rk+s)} = \alpha^{js}(x + \alpha^{jr} z)^n + \beta^{js}(x + \beta^{jr} z)^n, \tag{L1}$$

which are valid for n a non-negative integer, j, r, s integers and real or complex x and z. Most linear binomial Fibonacci identities can be obtained from identities (F1) and (L1) by substituting appropriate choices of n, j, r, s, x and z. For example, if we write $2r$ for r and set $x = (-1)^{jr}$, $z = 1$ in (F1) and (L1), we obtain

$$\sum_{k=0}^{n} (-1)^{jrk} \binom{n}{k} F_{j(2rk+s)} = (-1)^{jrn} L_{jr}^n F_{j(rn+s)}, \tag{5}$$

$$\sum_{k=0}^{n} (-1)^{jrk} \binom{n}{k} L_{j(2rk+s)} = (-1)^{jrn} L_{jr}^n L_{j(rn+s)}; \tag{6}$$

which are valid for n a non-negative integer and integers r, s and j. The special case ($s=0$) of identity (6) was also derived by Layman [9]. As another example of linear binomial Fibonacci identities that may be derived from (F1) and (L1), write $2r$ for r and set $x=(-1)^{jr}$, $z=-1$. This gives

$$\sum_{k=0}^{n}(-1)^{(jr+1)k}\binom{n}{k}F_{j(2rk+s)}=\begin{cases}5^{n/2}F_{jr}^{n}F_{j(rn+s)}, & n \text{ even};\\ (-1)^{jr+1}5^{(n-1)/2}F_{jr}^{n}L_{j(rn+s)}, & n \text{ odd};\end{cases} \tag{7}$$

$$\sum_{k=0}^{n}(-1)^{(jr+1)k}\binom{n}{k}L_{j(2rk+s)}=\begin{cases}5^{n/2}F_{jr}^{n}L_{j(rn+s)}, & n \text{ even};\\ (-1)^{jr+1}5^{(n+1)/2}F_{jr}^{n}F_{j(rn+s)}, & n \text{ odd}.\end{cases} \tag{8}$$

Setting ($x=F_{p+jr}$, $z=-F_p$) and also ($x=L_{p+jr}$, $z=-L_p$) and making use of the identities of Hoggat et al, (see Lemma 3), where p is any integer, we find

$$\sum_{k=0}^{n}(-1)^{k}\binom{n}{k}F_{p+jr}^{n-k}F_p^kF_{j(rk+s)}=(-1)^{js+1}F_{jr}^{n}F_{pn-js}; \tag{9}$$

$$\sum_{k=0}^{n}(-1)^{k}\binom{n}{k}F_{p+jr}^{n-k}F_p^kL_{j(rk+s)}=(-1)^{js+1}F_{jr}^{n}L_{pn-js}; \tag{10}$$

and

$$\sum_{k=0}^{n}(-1)^{k}\binom{n}{k}L_{p+jr}^{n-k}L_p^kF_{j(rk+s)}=\begin{cases}(-1)^{js+1}5^{\frac{n}{2}}F_{jr}^{n}F_{pn-js}, & n \text{ even};\\ (-1)^{js+1}5^{\frac{n-1}{2}}F_{jr}^{n}L_{pn-js}, & n \text{ odd};\end{cases} \tag{11}$$

$$\sum_{k=0}^{n}(-1)^{k}\binom{n}{k}L_{p+jr}^{n-k}L_p^kL_{j(rk+s)}=\begin{cases}(-1)^{js}5^{\frac{n}{2}}F_{jr}^{n}L_{pn-js}, & n \text{ even};\\ (-1)^{js}5^{\frac{n+1}{2}}F_{jr}^{n}F_{pn-js}, & n \text{ odd}.\end{cases} \tag{12}$$

Identities (9), (10) were obtained by Carlitz [1] while these and (11) and (12) may be found in Dresel [2]. The special case ($s=0$) of (9) was also derived by Layman [9].

2 Required identities and preliminary results

Lemma 1. *For real or complex z, let a given well-behaved function $h(z)$ have, in its domain, the representation $h(z)=\sum_{k=c_1}^{c_2}g(k)z^{f(k)}$ where $f(k)$ and $g(k)$ are given real sequences and $c_1,c_2\in[-\infty,\infty]$. Let j be an integer. Then,*

$$\sum_{k=c_1}^{c_2}g(k)z^{f(k)}F_{jf(k)}^{m}=\frac{1}{(\sqrt{5})^m}\sum_{i=0}^{m}(-1)^i\binom{m}{i}h\left(\beta^{ij}\alpha^{(m-i)j}z\right), \tag{F}$$

$$\sum_{k=c_1}^{c_2}g(k)z^{f(k)}L_{jf(k)}^{m}=\sum_{i=0}^{m}\binom{m}{i}h\left(\beta^{ij}\alpha^{(m-i)j}z\right). \tag{L}$$

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Proof. We have

$$\begin{aligned}\sum_{k=c_1}^{c_2} g(k)z^{f(k)}F_{jf(k)}^m &= \sum_{k=c_1}^{c_2} g(k)z^{f(k)}\frac{\left(\alpha^{jf(k)}-\beta^{jf(k)}\right)^m}{(\sqrt{5})^m}\\ &= \frac{1}{(\sqrt{5})^m}\sum_{k=c_1}^{c_2} g(k)z^{f(k)}\sum_{i=0}^{m}(-1)^i\binom{m}{i}\beta^{ijf(k)}\alpha^{(m-i)jf(k)}\\ &= \frac{1}{(\sqrt{5})^m}\sum_{i=0}^{m}(-1)^i\binom{m}{i}\sum_{k=c_1}^{c_2} g(k)\left(\beta^{ij}\alpha^{(m-i)j}z\right)^{f(k)}\\ &= \frac{1}{(\sqrt{5})^m}\sum_{i=0}^{m}(-1)^i\binom{m}{i}h\left(\beta^{ij}\alpha^{(m-i)j}z\right).\end{aligned}$$

The proof of (L) is similar. □

Since $\beta^i\alpha^{m-i} = (-1)^i\alpha^{m-2i}$, identities (F) and (L) can also be written as

$$\sum_{k=c_1}^{c_2} g(k)z^{f(k)}F_{jf(k)}^m = \frac{1}{(\sqrt{5})^m}\sum_{i=0}^{m}(-1)^i\binom{m}{i}h\left((-1)^{ij}\alpha^{(m-2i)j}z\right), \tag{F$'$}$$

$$\sum_{k=c_1}^{c_2} g(k)z^{f(k)}L_{jf(k)}^m = \sum_{i=0}^{m}\binom{m}{i}h\left((-1)^{ij}\alpha^{(m-2i)j}z\right). \tag{L$'$}$$

Lemma 2. *For non-negative integers m and n, integers j, r and s and real or complex x and z,*

$$\sum_{k=0}^{n}\binom{n}{k}x^{n-k}z^kF_{j(rk+s)}^m = \frac{1}{(\sqrt{5})^m}\sum_{i=0}^{m}(-1)^{i(js+1)}\binom{m}{i}\alpha^{(m-2i)js}\left(x+(-1)^{ijr}\alpha^{(m-2i)jr}z\right)^n, \tag{BF}$$

$$\sum_{k=0}^{n}\binom{n}{k}x^{n-k}z^kL_{j(rk+s)}^m = \sum_{i=0}^{m}(-1)^{ijs}\binom{m}{i}\alpha^{(m-2i)js}\left(x+(-1)^{ijr}\alpha^{(m-2i)jr}z\right)^n. \tag{BL}$$

Proof. Consider the binomial identity

$$h(z) = \sum_{k=0}^{n} g(k)z^{f(k)} = z^s(x+z^r)^n, \tag{13}$$

where

$$f(k) = rk+s, \quad g(k) = \binom{n}{k}x^{n-k}. \tag{14}$$

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Thus,

$$h\left((-1)^{ij}\alpha^{(m-2i)j}z\right) = (-1)^{ijs}\alpha^{(m-2i)js}z^s(x + (-1)^{ijr}\alpha^{(m-2i)jr}z^r)^n. \tag{15}$$

Use of (14) and (15) in identity (F$'$), with $c_1 = 0$, $c_2 = n$ gives

$$\sum_{k=0}^{n}\binom{n}{k}x^{n-k}z^{rk}F_{j(rk+s)}^m = \frac{1}{(\sqrt{5})^m}\sum_{i=0}^{m}(-1)^{i(js+1)}\binom{m}{i}\alpha^{(m-2i)js}\left(x + (-1)^{ijr}\alpha^{(m-2i)jr}z^r\right)^n,$$

from which identity (BF) follows when we write $z^{1/r}$ for z. To prove (BL), use (14) and (15) in identity (L$'$). □

It is sometimes convenient to use the (α vs β) version of identities (BF) and (BL):

$$\sum_{k=0}^{n}\binom{n}{k}x^{n-k}z^{k}F_{j(rk+s)}^m = \frac{1}{(\sqrt{5})^m}\sum_{i=0}^{m}(-1)^{i}\binom{m}{i}\beta^{ijs}\alpha^{(m-i)js}\left(x + \beta^{ijr}\alpha^{(m-i)jr}z\right)^n, \tag{BF$'$}$$

$$\sum_{k=0}^{n}\binom{n}{k}x^{n-k}z^{k}L_{j(rk+s)}^m = \sum_{i=0}^{m}\binom{m}{i}\beta^{ijs}\alpha^{(m-i)js}\left(x + \beta^{ijr}\alpha^{(m-i)jr}z\right)^n. \tag{BL$'$}$$

Lemma 3 (Hoggatt et al [5]). *For p and q integers,*

$$L_{p+q} - L_p\alpha^q = -\beta^pF_q\sqrt{5}, \tag{16}$$
$$L_{p+q} - L_p\beta^q = \alpha^pF_q\sqrt{5}, \tag{17}$$
$$F_{p+q} - F_p\alpha^q = \beta^pF_q, \tag{18}$$
$$F_{p+q} - F_p\beta^q = \alpha^pF_q. \tag{19}$$

Quadratic binomial Fibonacci identities may be obtained from $m = 2$ in (BF$'$) and (BL$'$):

$$\begin{aligned}5\sum_{k=0}^{n}\binom{n}{k}x^{n-k}z^{k}F_{j(rk+s)}^2 &= \alpha^{2js}(x + \alpha^{2jr}z)^n + \beta^{2js}(x + \beta^{2jr}z)^n \\ &\quad - 2(-1)^{js}(x + (-1)^{jr}z)^n,\end{aligned} \tag{F2}$$

$$\begin{aligned}\sum_{k=0}^{n}\binom{n}{k}x^{n-k}z^{k}L_{j(rk+s)}^2 &= \alpha^{2js}(x + \alpha^{2jr}z)^n + \beta^{2js}(x + \beta^{2jr}z)^n \\ &\quad + 2(-1)^{js}(x + (-1)^{jr}z)^n.\end{aligned} \tag{L2}$$

Theorem 1. *For non-negative integer n and integers j, r, s, p,*

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}F_{2jr+p}^{n-k}F_p^kF_{j(rk+s)}^2 = \frac{1}{5}(F_{2jr}^nL_{pn-2js} - (-1)^{js}2F_{jr}^nL_{jr+p}^n), \quad p \neq 0, \tag{20}$$

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}F_{2jr+p}^{n-k}F_p^kL_{j(rk+s)}^2=F_{2jr}^nL_{pn-2js}+(-1)^{js}2F_{jr}^nL_{jr+p}^n,\quad p\neq 0,\tag{21}$$

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}L_{2jr+p}^{n-k}L_p^kF_{j(rk+s)}^2=\begin{cases}5^{n/2-1}F_{2jr}^nL_{pn-2js}-(-1)^{js}5^{n-1}2F_{jr}^nF_{jr+p}^n, & n\ even;\\ 5^{(n-1)/2}F_{2jr}^nF_{pn-2js}-(-1)^{js}5^{n-1}2F_{jr}^nF_{jr+p}^n, & n\ odd,\end{cases}\tag{22}$$

and

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}L_{2jr+p}^{n-k}L_p^kL_{j(rk+s)}^2=\begin{cases}5^{n/2}F_{2jr}^nL_{pn-2js}+(-1)^{js}5^n2F_{jr}^nF_{jr+p}^n, & n\ even;\\ 5^{(n+1)/2}F_{2jr}^nF_{pn-2js}+(-1)^{js}5^n2F_{jr}^nF_{jr+p}^n, & n\ odd.\end{cases}\tag{23}$$

Proof. Choose $x=F_{2jr+p}$, $z=-F_p$ in (F2), noting Lemma 3, to obtain

$$\begin{aligned}5\sum_{k=0}^{n}(-1)^k\binom{n}{k}F_{2jr+p}^{n-k}F_p^kF_{j(rk+s)}^2&=F_{2jr}^n(\alpha^{2js}\beta^{pn}+\alpha^{pn}\beta^{2js})\\&\quad-2(-1)^{js}(F_{2jr+p}-(-1)^{jr}F_p)^n,\end{aligned}$$

from which identity (20) follows. The same (x,z) choice in (L2) produces identity (21).

Set $x=L_{2jr+p}$, $z=-L_p$ in (F2), utilizing Lemma 3. This gives

$$\begin{aligned}5\sum_{k=0}^{n}(-1)^k\binom{n}{k}L_{2jr+p}^{n-k}L_p^kF_{j(rk+s)}^2&=F_{2jr}^n(\sqrt{5})^n(\alpha^{pn-2js}+(-1)^n\beta^{pn-2js})\\&\quad-2(-1)^{js}(L_{2jr+p}-(-1)^{jr}L_p)^n;\end{aligned}$$

and hence identity (22). The same (x,z) choice in (L2) produces identity (23). □

Cubic binomial Fibonacci identities may be obtained from $m=2$ in (BF$'$) and (BL$'$):

$$\begin{aligned}5\sqrt{5}\sum_{k=0}^{n}\binom{n}{k}x^{n-k}z^kF_{j(rk+s)}^3&=\alpha^{3js}(x+\alpha^{3jr}z)^n-\beta^{3js}(x+\beta^{3jr}z)^n\\&\quad-(-1)^{js}3\alpha^{js}(x+(-1)^{jr}\alpha^{jr}z)^n\\&\quad+(-1)^{js}3\beta^{js}(x+(-1)^{jr}\beta^{jr}z)^n,\end{aligned}\tag{F3}$$

$$\begin{aligned}\sum_{k=0}^{n}\binom{n}{k}x^{n-k}z^kL_{j(rk+s)}^3&=\alpha^{3js}(x+\alpha^{3jr}z)^n+\beta^{3js}(x+\beta^{3jr}z)^n\\&\quad+(-1)^{js}3\alpha^{js}(x+(-1)^{jr}\alpha^{jr}z)^n\\&\quad+(-1)^{js}3\beta^{js}(x+(-1)^{jr}\beta^{jr}z)^n.\end{aligned}\tag{L3}$$

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Theorem 2. *For non-negative integer n and any integer s,*

$$\sum_{k=0}^{n}\binom{n}{k}F_{k+s}^3=\frac{1}{5}(2^nF_{2n+3s}+3F_{n-s}),\tag{24}$$

$$\sum_{k=0}^{n}\binom{n}{k}L_{k+s}^3=2^nL_{2n+3s}+3L_{n-s},\tag{25}$$

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^kF_{k+s}^3=\frac{1}{5}((-1)^n2^nF_{n+3s}-(-1)^s3F_{2n+s}),\tag{26}$$

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}L_{k+s}^3=(-1)^n2^nL_{n+3s}+(-1)^s3L_{2n+s},\tag{27}$$

$$\sum_{k=0}^{n}\binom{n}{k}2^kF_{k+s}^3=\begin{cases}5^{n/2-1}(F_{3n+3s}-(-1)^s3F_s), & n\ \textit{even};\\ 5^{(n-3)/2}(L_{3n+3s}+(-1)^s3L_s) & n\ \textit{odd},\end{cases}\tag{28}$$

$$\sum_{k=0}^{n}\binom{n}{k}2^kL_{k+s}^3=\begin{cases}5^{n/2}(L_{3n+3s}+(-1)^s3L_s), & n\ \textit{even};\\ 5^{(n+1)/2}(F_{3n+3s}-(-1)^s3F_s) & n\ \textit{odd}.\end{cases}\tag{29}$$

Lemma 4. *Let a, b, c and d be rational numbers and λ an irrational number. Then,*

$$a+\lambda b=c+\lambda d \iff a=c,\quad b=d\,.$$

Lemma 5. *If m is an integer and $(f(i))$ a real sequence, then,*

$$\sum_{i=0}^{2m}f(i)=f(m)+\sum_{i=0}^{m-1}(f(i)+f(2m-i)),\tag{30}$$

$$\sum_{i=0}^{2m+1}f(i)=f(2m+1)-f(m)+\sum_{i=0}^{m}(f(i)+f(2m-i)).\tag{31}$$

In particular, if $f(2m-i)=f(i)$, then,

$$\sum_{i=0}^{2m}f(i)=f(m)+2\sum_{i=0}^{m-1}f(i),\tag{32}$$

$$\sum_{i=0}^{2m+1}f(i)=f(2m+1)-f(m)+2\sum_{i=0}^{m}f(i).\tag{33}$$

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Lemma 6. *For p and q integers,*

$$1+(-1)^p\alpha^{2q}=\begin{cases}(-1)^p\alpha^qF_q\sqrt{5}, & p \text{ and } q \text{ have different parity;}\\(-1)^p\alpha^qL_q, & p \text{ and } q \text{ have the same parity.}\end{cases} \tag{34}$$

$$1-(-1)^p\alpha^{2q}=\begin{cases}(-1)^{p-1}\alpha^qL_q, & p \text{ and } q \text{ have different parity;}\\(-1)^{p-1}\alpha^qF_q\sqrt{5}, & p \text{ and } q \text{ have the same parity.}\end{cases} \tag{35}$$

Proof. We have

$$\begin{aligned}(-1)^{p+q}+(-1)^p\alpha^{2q}&=\alpha^{p+q}\beta^{p+q}+\alpha^{p+2q}\beta^p\\&=\alpha^{p+q}\beta^p(\alpha^q+\beta^q)\\&=(-1)^p\alpha^qL_q.\end{aligned} \tag{36}$$

Similarly,

$$(-1)^{p+q}-(-1)^p\alpha^{2q}=(-1)^{p-1}\alpha^qF_q\sqrt{5}. \tag{37}$$

Corresponding to (36) and (37) we have

$$(-1)^{p+q}+(-1)^p\beta^{2q}=(-1)^p\beta^qL_q \tag{38}$$

and

$$(-1)^{p+q}-(-1)^p\beta^{2q}=(-1)^p\beta^qF_q\sqrt{5}. \tag{39}$$

□

Identities (36), (37), (38) and (39) imply

$$(-1)^q+\alpha^{2q}=\alpha^qL_q, \tag{40}$$

$$(-1)^q-\alpha^{2q}=-\alpha^qF_q\sqrt{5}, \tag{41}$$

$$(-1)^q+\beta^{2q}=\beta^qL_q, \tag{42}$$

$$(-1)^q-\beta^{2q}=\beta^qF_q\sqrt{5}. \tag{43}$$

3 Main results

Theorem 3. *Let m and n be non-negative integers and let j, r and s be any integers. Then,*

$$\begin{aligned}&\sum_{k=0}^{n}\binom{n}{k}F_{j(rk+s)}^{2m}\\&=\begin{cases}5^{-m}\sum\limits_{i=0}^{m-1}(-1)^{i(js+jrn+1)}\binom{2m}{i}L_{(m-i)jr}^nL_{(m-i)(jrn+2js)}+(-1)^{m(js+1)}\binom{2m}{m}5^{-m}2^n, & jmr \text{ even;}\\5^{n/2-m}\sum\limits_{i=0}^{m-1}(-1)^{i(s+1)}\binom{2m}{i}F_{(m-i)jr}^nL_{(m-i)(jrn+2js)}, & jmr \text{ odd, } n \text{ even;}\\5^{(n+1)/2-m}\sum\limits_{i=0}^{m-1}(-1)^{is}\binom{2m}{i}F_{(m-i)jr}^nF_{(m-i)(jrn+2js)}, & jmr \text{ odd, } n \text{ odd.}\end{cases}\end{aligned} \tag{44}$$

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$$\sum_{k=0}^{n}\binom{n}{k}L_{j(rk+s)}^{2m}$$
$$=\begin{cases}\sum\limits_{i=0}^{m-1}(-1)^{i(js+jrn)}\binom{2m}{i}L_{(m-i)jr}^{n}L_{(m-i)(jrn+2js)}+(-1)^{mjs}\binom{2m}{m}2^{n}, & jmr\ \text{even};\\ 5^{n/2}\sum\limits_{i=0}^{m-1}(-1)^{is}\binom{2m}{i}F_{(m-i)jr}^{n}L_{(m-i)(jrn+2js)}, & jmr\ \text{odd},\ n\ \text{even};\\ 5^{(n+1)/2}\sum\limits_{i=0}^{m-1}(-1)^{i(s+1)}\binom{2m}{i}F_{(m-i)jr}^{n}F_{(m-i)(jrn+2js)}, & jmr\ \text{odd},\ n\ \text{odd}.\end{cases} \tag{45}$$

Proof. In (BF) write $2m$ for m and set $x=1$ and $z=1$. This gives

$$5^{m}\sum_{k=0}^{n}\binom{n}{k}F_{j(rk+s)}^{2m}=\sum_{i=0}^{2m}(-1)^{i(js+1)}\binom{2m}{i}\alpha^{(m-i)2js}\left(1+(-1)^{ijr}\alpha^{(m-i)2jr}\right)^{n}. \tag{46}$$

Now, on account of Lemma 6, identity (34), we have

$$1+(-1)^{ijr}\alpha^{(m-i)2jr}=\begin{cases}(-1)^{ijr}\alpha^{(m-i)jr}L_{(m-i)jr}, & jrm\ \text{even};\\ (-1)^{i}\alpha^{(m-i)jr}F_{(m-i)jr}\sqrt{5}, & jrm\ \text{odd}.\end{cases} \tag{47}$$

Thus, using (47) in (46), we have

$$5^{m}\sum_{k=0}^{n}\binom{n}{k}F_{j(rk+s)}^{2m}=\begin{cases}\sum\limits_{i=0}^{2m}(-1)^{i(js+jrn+1)}\binom{2m}{i}\alpha^{(m-i)(jrn+2js)}L_{(m-i)jr}^{n}, & jrm\ \text{even};\\ (\sqrt{5})^{n}\sum\limits_{i=0}^{2m}(-1)^{i(js+n+1)}\binom{2m}{i}\alpha^{(m-i)(jrn+2js)}F_{(m-i)jr}^{n}, & jrm\ \text{odd}.\end{cases} \tag{48}$$

Observe that the left side of (48) evaluates to a rational number since it is the finite sum of rational numbers. Since,

$$2\alpha^{(m-i)(jrn+2js)}=L_{(m-i)(jrn+2js)}+F_{(m-i)(jrn+2js)}\sqrt{5}, \tag{49}$$

identity (44) now follows by comparing both sides of identity (48) in each case of jmr even or jmr odd, invoking Lemma 4 with $\lambda=\sqrt{5}$. Note the use of Lemma 5, identity (30) to re-write the ($i=0$ to $2m$) sum. The proof of identity (45) is similar; set $x=1$ and $z=1$ in (BL) and write $2m$ for m. □

Theorem 4. *Let m and n be non-negative integers and let j, r and s be any integers. Then,*

$$\sum_{k=0}^{n}(-1)^{k}\binom{n}{k}F_{j(rk+s)}^{2m}$$
$$=\begin{cases}5^{-m}(-1)^{n}\sum\limits_{i=0}^{m-1}(-1)^{i(s+n+1)}\binom{2m}{i}L_{(m-i)jr}^{n}L_{(m-i)(jrn+2js)}+(-1)^{s+1}\binom{2m}{m}5^{-m}2^{n}, & jmr\ \text{odd};\\ 5^{n/2-m}\sum\limits_{i=0}^{m-1}(-1)^{i(js+1)}\binom{2m}{i}F_{(m-i)jr}^{n}L_{(m-i)(jrn+2js)}, & jmr\ \text{even},\ n\ \text{even};\\ -5^{(n+1)/2-m}\sum\limits_{i=0}^{m-1}(-1)^{i(js+jr+1)}\binom{2m}{i}F_{(m-i)jr}^{n}F_{(m-i)(jrn+2js)}, & jmr\ \text{even},\ n\ \text{odd}.\end{cases} \tag{50}$$

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}L_{j(rk+s)}^{2m}$$
$$=\begin{cases}(-1)^n\sum\limits_{i=0}^{m-1}(-1)^{i(s+n)}\binom{2m}{i}L_{(m-i)jr}^{n}L_{(m-i)(jrn+2js)}+(-1)^s\binom{2m}{m}2^n, & jmr\ odd;\\ 5^{n/2}\sum\limits_{i=0}^{m-1}(-1)^{ijs}\binom{2m}{i}F_{(m-i)jr}^{n}L_{(m-i)(jrn+2js)}, & jmr\ even,\ n\ even;\\ -5^{(n+1)/2}\sum\limits_{i=0}^{m-1}(-1)^{i(js+jr)}\binom{2m}{i}F_{(m-i)jr}^{n}F_{(m-i)(jrn+2js)}, & jmr\ even,\ n\ odd.\end{cases}\tag{51}$$

Proof. In (BF) write $2m$ for m and set $x=1$ and $z=-1$. This gives

$$5^m\sum_{k=0}^{n}(-1)^k\binom{n}{k}F_{j(rk+s)}^{2m}=\sum_{i=0}^{2m}(-1)^{i(js+1)}\binom{2m}{i}\alpha^{(m-i)2js}\left(1-(-1)^{ijr}\alpha^{(m-i)2jr}\right)^n.\tag{52}$$

Now, on account of Lemma 6, identity (35), we have

$$1-(-1)^{ijr}\alpha^{(m-i)2jr}=\begin{cases}(-1)^{ijr-1}\alpha^{(m-i)jr}F_{(m-i)jr}\sqrt{5}, & jrm\text{ even};\\ (-1)^{i-1}\alpha^{(m-i)jr}L_{(m-i)jr}, & jrm\text{ odd}.\end{cases}\tag{53}$$

Thus, using (53) in (52), we have

$$5^m\sum_{k=0}^{n}(-1)^k\binom{n}{k}F_{j(rk+s)}^{2m}=\begin{cases}\sum\limits_{i=0}^{2m}(-1)^{in+is+i-n}\binom{2m}{i}\alpha^{(m-i)(jrn+2js)}L_{(m-i)jr}^{n}, & jrm\text{ odd};\\ (\sqrt{5})^n\sum\limits_{i=0}^{2m}(-1)^{ijnr+ijs+i-n}\binom{2m}{i}\alpha^{(m-i)(jrn+2js)}F_{(m-i)jr}^{n}, & jrm\text{ even}.\end{cases}\tag{54}$$

The left side of (54) evaluates to a rational number since it is the finite sum of rational numbers. Making use of identity (49), identity (50) follows by comparing both sides of identity (54) in each case of jmr even or jmr odd, invoking Lemma 4 with $\lambda=\sqrt{5}$. The proof of identity (51) is similar; put $x=1$ and $z=-1$ in (BL) and write $2m$ for m. □

The proofs of Theorems 5 and 6 are similar to those of Theorems 3 and 4. We therefore omit the details and indicate only the appropriate choices of x, z, m and r to be made in identities (BF) and (BL) in each case.

Theorem 5. *Let m and n be non-negative integers and let j, r and s be any integers. Then,*

$$\sum_{k=0}^{n}\binom{n}{k}F_{j(2rk+s)}^{2m+1}$$
$$=\begin{cases}5^{-m}\sum\limits_{i=0}^{m}(-1)^{i(js+1)}\binom{2m+1}{i}L_{(2m+1-2i)jr}^{n}F_{(2m+1-2i)(jrn+js)}, & jr\ even;\\ 5^{n/2-m}\sum\limits_{i=0}^{m}(-1)^{i(js+1)}\binom{2m+1}{i}F_{(2m+1-2i)jr}^{n}F_{(2m+1-2i)(jrn+js)}, & jr\ odd,\ n\ even;\\ 5^{(n-1)/2-m}\sum\limits_{i=0}^{m}(-1)^{i(js+1)}\binom{2m+1}{i}F_{(2m+1-2i)jr}^{n}L_{(2m+1-2i)(jrn+js)}, & jr\ odd,\ n\ odd;\end{cases}\tag{55}$$

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$$\sum_{k=0}^{n}\binom{n}{k}L_{j(2rk+s)}^{2m+1}$$
$$=\begin{cases}\sum\limits_{i=0}^{m}(-1)^{ijs}\binom{2m+1}{i}L_{(2m+1-2i)jr}^{n}L_{(2m+1-2i)(jrn+js)}, & jr\ \textit{even};\\ 5^{n/2}\sum\limits_{i=0}^{m}(-1)^{ijs}\binom{2m+1}{i}F_{(2m+1-2i)jr}^{n}L_{(2m+1-2i)(jrn+js)}, & jr\ \textit{odd},\ n\ \textit{even};\\ 5^{(n+1)/2}\sum\limits_{i=0}^{m}(-1)^{ijs}\binom{2m+1}{i}F_{(2m+1-2i)jr}^{n}F_{(2m+1-2i)(jrn+js)}, & jr\ \textit{odd},\ n\ \textit{odd}.\end{cases} \tag{56}$$

Proof. Set $x=1$, $z=1$ and write $2m+1$ for m and $2r$ for r in identities (BF) and (BL). Note that

$$1+\alpha^{(2m+1-2i)2jr}=\begin{cases}\alpha^{(2m+1-2i)jr}L_{(2m+1-2i)jr}, & jr\text{ even};\\ \alpha^{(2m+1-2i)jr}F_{(2m+1-2i)jr}\sqrt{5}, & jr\text{ odd}.\end{cases}$$

□

Theorem 6. *Let m and n be non-negative integers and let j, r and s be any integers. Then,*

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}F_{j(2rk+s)}^{2m+1}$$
$$=\begin{cases}(-1)^n5^{-m}\sum\limits_{i=0}^{m}(-1)^{i(js+1)}\binom{2m+1}{i}L_{(2m+1-2i)jr}^{n}F_{(2m+1-2i)(jrn+js)}, & jr\ \textit{odd};\\ 5^{n/2-m}\sum\limits_{i=0}^{m}(-1)^{i(js+1)}\binom{2m+1}{i}F_{(2m+1-2i)jr}^{n}F_{(2m+1-2i)(jrn+js)}, & jr\ \textit{even},\ n\ \textit{even};\\ -5^{(n-1)/2-m}\sum\limits_{i=0}^{m}(-1)^{i(js+1)}\binom{2m+1}{i}F_{(2m+1-2i)jr}^{n}L_{(2m+1-2i)(jrn+js)}, & jr\ \textit{even},\ n\ \textit{odd};\end{cases} \tag{57}$$

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}L_{j(2rk+s)}^{2m+1}$$
$$=\begin{cases}(-1)^n\sum\limits_{i=0}^{m}(-1)^{ijs}\binom{2m+1}{i}L_{(2m+1-2i)jr}^{n}L_{(2m+1-2i)(jrn+js)}, & jr\ \textit{odd};\\ 5^{n/2}\sum\limits_{i=0}^{m}(-1)^{ijs}\binom{2m+1}{i}F_{(2m+1-2i)jr}^{n}L_{(2m+1-2i)(jrn+js)}, & jr\ \textit{even},\ n\ \textit{even};\\ -5^{(n+1)/2}\sum\limits_{i=0}^{m}(-1)^{ijs}\binom{2m+1}{i}F_{(2m+1-2i)jr}^{n}F_{(2m+1-2i)(jrn+js)}, & jr\ \textit{even},\ n\ \textit{odd}.\end{cases} \tag{58}$$

Proof. Put $x=1$, $z=-1$ and write $2m+1$ for m and $2r$ for r in identities (BF) and (BL). Note that

$$1-\alpha^{(2m+1-2i)2jr}=\begin{cases}-\alpha^{(2m+1-2i)jr}F_{(2m+1-2i)jr}\sqrt{5}, & jr\text{ even};\\ -\alpha^{(2m+1-2i)jr}L_{(2m+1-2i)jr}, & jr\text{ odd}.\end{cases}$$

□

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