Article On a New Class of Stancu-Kantorovich-Type Operators

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Abstract: The present paper introduces a new class of Stancu-Kantorovich operators constructed in the King sense. For this class of operators we establish some convergence results, error estimations theorems and graphical properties of approximation for the cases considered, namely, operators that preserve the test functions $e_0(x) = 1$ and $e_1(x) = x$, $e_0(x) = 1$ and $e_2(x) = x^2$, as well as $e_1(x) = x$ and $e_2(x) = x^2$. The class of operators that preserve the test functions $e_1(x) = x$ and $e_2(x) = x^2$ is a genuine generalization of the class introduced by Indrea et al. in paper [9].

Keywords: Stancu operators, Kantorovich operators, Stancu-Kantorovich operators, King-type operators, approximation by positive linear operators.

MSC: 41A36, 41A25

1. Introduction

By C[0, 1] we denote the space of continuous functions defined on [0, 1] and by $L_1[0, 1]$ the space of all functions defined on [0, 1] which are Lebesgue integrable. Let \mathbb{N} be the set of all positive integers.

We consider $e_i(t) = t^j$ for $t \in [0, 1], j \in \mathbb{N}$.

Let α , $\beta \ge 0$ and $\alpha \le \beta$. Stancu introduced in paper [17] the following operators, which are a generalization of the well-known Bernstein operators (see [2]), $S_m^{(\alpha,\beta)} : C[0,1] \to C[0,1]$ defined as:

$$S_m^{(\alpha,\beta)}(f,x) = \sum_{k=0}^m p_{m,k}(x) f\left(\frac{k+\alpha}{m+\beta}\right).$$

where $p_{m,k}(x) = {m \choose k} x^k (1-x)^{m-k}$, for every $m \in \mathbb{N}$, $f \in C[0,1]$ and $x \in [0,1]$.

The Bernstein operators have been intensively studied and many generalizations were considered, one of which is the Kantorovich variant, due to L.V. Kantorovich, from 1930 (see [10]). The Kantorovich positive linear operators are $K_m : L_1[0,1] \rightarrow C[0,1]$, defined as

$$K_m(f,x) = (m+1) \sum_{k=0}^m p_{m,k}(x) \int_{\frac{k}{m+1}}^{\frac{k+1}{m+1}} f(s) ds,$$

where $p_{m,k}(x) = \binom{m}{k} x^k (1-x)^{m-k}$, for every $m \in \mathbb{N}$, $f \in L_1[0,1]$ and $x \in [0,1]$.

Among the numerous generalizations of the Kantorovich Bernstein type operators, we mention the one by Indrea et al. (see [9]), which introduces a new general class which preserves the test functions $e_1(x)$ and $e_2(x)$.

In [11] J.P. King introduced a new class of positive linear Bernstein-type operators which reproduce constant functions $(e_1(x))$ and $e_2(x)$. These operators are a generalization of the Bernstein operators, but they are not polynomial-type operators. With the results introduced by King, a new direction of research was initiated, which concerns the construction of new operators, with better approximation properties, obtained by modifying existing sequences of linear positive operators. This subject has been one of great interest. Gonska and Piţul (see [8]) studied estimates in terms of the first and second moduli of continuity for the operators introduced by King. Among the first generalizations of King's result we mention those of Agratini (see [1]), Cardena-Morales et al. (see [3]), Duman and



Citation: Vasian, B. I.; Garoiu, Ş. L.; Păcurar, C. M. On a New Class of Stancu-Kantorovich-Type Operators. *Preprints* 2021, 1, 0. https://doi.org/

Received: Accepted: Published:

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Özarslan (see [4], [5]) and Gonska et al. (see [7]). The subject is still of interest. Among the more recent studies we mention the one by Popa (see [14]) where Voronovskaja Type Theorems for King operators are studied.

Based on the results in [9], [15] and [17], we introduce a new class of King type approximation operators and establish some convergence properties based on a Korovkin type theorem. Our results are a generalization of previous results on the topic.

2. Preliminaries

In the following, we present the notions and results which will be used to prove the main results of the paper. We will denote by $\mathcal{F}(I)$ the set of all functions defined on $I \subset \mathbb{R}$.

Definition 1. Let *I* and *J* be two intervals of \mathbb{R} such that $I \cap J \neq \emptyset$. For $x \in I$, let $\mu_x : I \to \mathbb{R}$, $\mu_x(t) = t - x$, $t \in I$. For any $m \in \mathbb{N}$, we set the functions $\phi_{m,k} : J \to \mathbb{R}$, such that $\phi_{m,k}(x) \ge 0$ for every $x \in J$, $k \in \{0, 1, ..., m\}$ and the positive linear functionals $\Phi_{m,k} : U(I) \subset \mathcal{F}(I) \to \mathbb{R}$, $k \in \{0, 1, ..., m\}$. For $m \in \mathbb{N}$ we define the operator $L_m : U(I) \subset \mathcal{F}(I) \to V(J) \subset \mathcal{F}(J)$ as

$$L_m(f,x) = \sum_{k=0}^m \phi_{m,k}(x) \Phi_{m,k}(f), \ x \in J, \ f \in U(I).$$
(1)

Remark 1. The operators $(L_m)_{m \in \mathbb{N}}$ defined above are linear and positive on U(I).

Definition 2. [9] For any $f \in U(I)$, and $x \in I \cap J$, and for $i \in \mathbb{N}$, we have the following operator *defined as*

$$(\Gamma_{m,i}L_m)(x) = L_m\Big((e_1 - x)^i, x\Big) = \sum_{k=0}^m \phi_{m,k}(x)\Phi_{m,k}\Big((e_1 - x)^i\Big).$$
(2)

Definition 3. Let $I \subset \mathbb{R}$ be an interval and f be a bounded continuous function on I. The modulus of continuity is a function $\omega(f, \cdot) : [0, \infty) \to \mathbb{R}$ defined for any $h \ge 0$,

$$\omega(f,h) = \sup\{|f(x) - f(t)| : x, t \in I, |x - t| \le h\}.$$

Now, let us recall the well known result by Shisha and Mond, [16].

Theorem 1. Let *L* be a linear positive operator on *I*. If *f* is a continuous bounded function on *I*, then for every $x \in I$ and every h > 0, one has

$$|L(f,x) - f(x)| \le |f(x)| \cdot |L(e_0,x) - 1| + \left(L(e_0,x) + \frac{1}{h}\sqrt{L(e_0,x) \cdot L((e_0-x)^2,x)}\right)\omega(f,h).$$

3. A new Class of Stancu-Kantorovich Type Operators

In this section we introduce a modified Stancu-Kantorovich operator, namely, the modification introduced by King. For this class of new operators, we will study some properties taking into account their expressions on the test functions $e_i(t) = t^i$, i = 0, 1, 2, and imposing that the operator preserves the test functions e_i and e_j , $i \neq j$, $i, j \in \{0, 1, 2\}$.

Definition 4. Let I be an interval and $c_m, d_m : I \to \mathbb{R}$ be some functions that satisfy $c_m(x) \ge 0$, $d_m(x) \ge 0$ for all $x \in I$, $0 \le \alpha \le \beta$ and $m \in \mathbb{N}$. We define the following Stancu-Kantorovich type operators:

$$S_{m}^{(\alpha,\beta)*}(f,x) = (m+\beta+1)\sum_{k=0}^{m} \binom{m}{k} (c_{m}(x))^{k} (d_{m}(x))^{m-k} \int_{\frac{k+\alpha}{m+\beta+1}}^{\frac{k+\alpha+1}{m+\beta+1}} f(t)dt$$
(3)

for any $x \in I$, $m \in \mathbb{N}$ and $f \in L_1[0, 1]$.

Lemma 1. The operator proposed in relation (3) has the following properties

$$S_m^{(\alpha,\beta)*}(e_0,x) = (c_m(x) + d_m(x))^m,$$
(4)

$$S_{m}^{(\alpha,\beta)*}(e_{1},x) = \frac{m}{m+\beta+1}c_{m}(x)(c_{m}(x)+d_{m}(x))^{m-1} + \frac{2\alpha+1}{2(m+\beta+1)}(c_{m}(x)+d_{m}(x))^{m},$$
(5)

$$S_{m}^{(\alpha,\beta)*}(e_{2},x) = \frac{m(m-1)}{(m+\beta+1)^{2}}c_{m}^{2}(x)(c_{m}(x)+d_{m}(x))^{m-2}$$

$$+\frac{2m(1+\alpha)}{(m+\beta+1)^{2}}c_{m}(x)(c_{m}(x)+d_{m}(x))^{m-1}$$

$$+\frac{3\alpha(\alpha+1)+1}{3(m+\beta+1)^{2}}(c_{m}(x)+d_{m}(x))^{m}$$
(6)

for any $x \in I$, $m \in \mathbb{N}$, where $e_i(t) = t^i$, $i \in \{0, 1, 2\}$.

Proof. For $e_0(x) = 1$, we have:

$$S_m^{(\alpha,\beta)*}(e_0,x) = (m+\beta+1)\sum_{k=0}^m \binom{m}{k} (c_m(x))^k (d_m(x))^{m-k} \frac{1}{m+\beta+1},$$

which, from the binomial theorem, yields:

$$S_m^{(\alpha,\beta)*}(e_0,x) = (c_m(x) + d_m(x))^m.$$

Now, let us evaluate $S_m^{(\alpha,\beta)*}$ for $e_1(x) = x$:

$$S_m^{(\alpha,\beta)*}(e_1,x) = (m+\beta+1)\sum_{k=0}^m \binom{m}{k} (c_m(x))^k (d_m(x))^{m-k} \frac{1}{2} \left[\left(\frac{k+\alpha+1}{m+\beta+1}\right)^2 - \left(\frac{k+\alpha}{m+\beta+1}\right)^2 \right],$$

which is

$$S_m^{(\alpha,\beta)*}(e_1,x) = \frac{1}{m+\beta+1} \sum_{k=0}^m \binom{m}{k} k (c_m(x))^k (d_m(x))^{m-k} + \frac{2\alpha+1}{2(m+\beta+1)} (c_m(x)+d_m(x))^m.$$

Denoting k - 1 = l in the sum from the right hand side in the above relation, we get:

$$S_{m}^{(\alpha,\beta)*}(e_{1},x) = \frac{m}{m+\beta+1}c_{m}(x)\sum_{l=0}^{m-1} {m-1 \choose k}(c_{m}(x))^{l}(d_{m}(x))^{m-l-1} + \frac{2\alpha+1}{m+\beta+1}(c_{m}(x)+d_{m}(x))^{m}$$

and, again, by the binomial theorem, we get:

$$S_m^{(\alpha,\beta)*}(e_1,x) = \frac{m}{m+\beta+1}c_m(x)(c_m(x)+d_m(x))^{m-1} + \frac{2\alpha+1}{m+\beta+1}(c_m(x)+d_m(x))^m.$$

Lastly, we shall compute $S_m^{(\alpha,\beta)*}$ for $e_2(x) = x^2$:

$$S_m^{(\alpha,\beta)*}(e_2,x) = (m+\beta+1)\sum_{k=0}^m \binom{m}{k} (c_m(x))^k (d_m(x))^{m-k} \frac{1}{3} \left[\left(\frac{k+\alpha+1}{m+\beta+1}\right)^3 - \left(\frac{k+\alpha}{m+\beta+1}\right)^3 \right]$$

Now, by doing the calculations in the square brackets we get:

$$S_{m}^{(\alpha,\beta)*}(e_{2},x) = \frac{1}{(m+\beta+1)^{2}} \sum_{k=0}^{m} {m \choose k} k^{2} (c_{m}(x))^{k} (d_{m}(x))^{m-k} + \frac{2\alpha+1}{(m+\beta+1)^{2}} \sum_{k=0}^{m} {m \choose k} k (c_{m}(x))^{k} (d_{m}(x))^{m-k} + \frac{3\alpha(\alpha+1)+1}{3(m+\beta+1)^{2}} (c_{m}(x) + d_{m}(x))^{m}.$$

By denoting k - 2 = l in the first sum from the right hand side in the relation from above and by the binomial theorem we will have:

$$S_m^{(\alpha,\beta)*}(e_2,x) = \frac{m(m-1)}{(m+\beta+1)^2} c_m^2(x) (c_m(x) + d_m(x))^{m-2} + \frac{2m(\alpha+1)}{(m+\beta+1)^2} c_m(x) (c_m(x) + d_m(x))^{m-1} + \frac{3\alpha(\alpha+1) + 1}{3(m+\beta+1)^2} (c_m(x) + d_m(x))^m,$$

which completes the proof. \Box

3.1. Stancu-Kantorovich type operators which preserve the functions e_0 and e_1

In this section we shall construct an operator of Stancu-Kantorovich type as in (3), that preserves the test functions e_0 and e_1 , i.e. an operator that satisfies

$$S_{m}^{(\alpha,\beta)*}(e_{0},x) = 1$$

$$S_{m}^{(\alpha,\beta)*}(e_{1},x) = x$$

$$S_{m}^{(\alpha,\beta)*}(e_{2},x) \xrightarrow{u} x^{2}.$$
(7)

Now, from the conditions in (7) and relations (4), (5) we get

$$c_m(x) = \frac{m+\beta+1}{m}x - \frac{2\alpha+1}{m},$$
 (8)

and

$$d_m(x) = \frac{-(m+\beta+1)x}{m} + \frac{m+2\alpha+1}{m},$$
(9)

for any $m \in \mathbb{N}$ and $x \in I$.

In order to have a positive operator we shall assume that the functions c_m and d_m are positive. This condition yields the following inequality

$$\frac{2\alpha+1}{m+\beta+1} \le x \le \frac{m+2\alpha+1}{m+\beta+1}, \, \forall x \in I, \, \forall m \in \mathbb{N}.$$

Lemma 2. For $0 \le 2\alpha \le \beta$ and any integers $m_0 < m$ we have

$$\left[\frac{2\alpha+1}{m_0+\beta+1};\frac{m_0+2\alpha+1}{m_0+\beta+1}\right] \subset \left[\frac{2\alpha+1}{m+\beta+1};\frac{m+2\alpha+1}{m+\beta+1}\right].$$

Proof. Let us consider the sequences $(x_m)_{m \ge 1}$,

$$x_m = \frac{2\alpha + 1}{m + \beta + 1}$$

and $(y_m)_{m>1}$,

$$y_m = \frac{m+2\alpha+1}{m+\beta+1}.$$

Imposing the condition $0 \le 2\alpha \le \beta$ we have that $(x_m)_{m\ge 1}$ is a decreasing sequence, and $(y_m)_{m\ge 1}$ is an increasing sequence, thus implying that our inclusion holds for any $m \ge m_0$. \Box

Remark 2. *From now on, we will consider* $0 \le 2\alpha \le \beta$ *.*

Remark 3. Since on the interval $\left[\frac{2\alpha+1}{m_0+\beta+1}; \frac{m_0+2\alpha+1}{m_0+\beta+1}\right]$ we have that c_m , $d_m \ge 0$, for every $m \in \mathbb{N}$, we will consider $I = \left[\frac{2\alpha+1}{m_0+\beta+1}; \frac{m_0+2\alpha+1}{m_0+\beta+1}\right]$, where m_0 is a positive integer which is arbitrarily fixed. Note that for any $\varepsilon > 0$, if we take m_0 sufficiently large, then $[\varepsilon, 1 - \varepsilon] \subset I$.

Now, taking into account the sequences c_m and d_m obtained in (8) and (9) the operator in (3) will be

$$S_{1,m}^{(\alpha,\beta)*}(f,x) = (m+\beta+1)\sum_{k=0}^{m} \binom{m}{k} \left(\frac{m+\beta+1}{m}x - \frac{2\alpha+1}{m}\right)^{k}$$
(10)

$$\times \left(\frac{-(m+\beta+1)x}{m} + \frac{m+2\alpha+1}{m}\right)^{m-k} \int_{\frac{k+\alpha}{m+\beta+1}}^{\frac{k+\alpha+1}{m+\beta+1}} f(t)dt,$$

for any $x \in I$.

Lemma 3. The operator $S_{1,m}^{(\alpha,\beta)*}$ from (10) satisfies

$$S_{1,m}^{(\alpha,\beta)*}(e_0, x) = 1;$$

$$S_{1,m}^{(\alpha,\beta)*}(e_1, x) = x;$$

$$S_{1,m}^{(\alpha,\beta)*}(e_2, x) = \frac{m-1}{m}x^2 + \frac{m+2\alpha-1}{m(m+\beta+1)}x + \frac{12\alpha(\alpha+1) - m(5-36\alpha) - 3}{12m(m+\beta+1)^2}$$
(11)

for $x \in I$.

Proof. The first two relations from (11) are obvious and the third follows by applying relations (5), (8) and (9) and after some computations. \Box

Lemma 4. The following relations hold

$$\left(\Gamma_{m,0}S_{1,m}^{(\alpha,\beta)*}\right)(x) = 1,$$
(12)

$$\left(\Gamma_{m,1}S_{1,m}^{(\alpha,\beta)*}\right)(x) = 0, \tag{13}$$

$$\left(\Gamma_{m,2}S_{1,m}^{(\alpha,\beta)*}\right)(x) = \frac{12m(x-x^2) + \left(-24x^2(\beta+1) + 12x(2\alpha+\beta) + 36\alpha-5\right)}{12(m+\beta+1)^2} + \frac{-12x^2(\beta+1)^2 + 12x(2\alpha\beta+2\alpha-\beta-1) + 3(4\alpha^2+4\alpha-1)}{12m(m+\beta+1)^2}.$$

$$(14)$$

Proof. Using the previous lemma and relation (2) we get

$$\left(\Gamma_{m,0} S_{1,m}^{(\alpha,\beta)*} \right)(x) = 1, \left(\Gamma_{m,1} S_{1,m}^{(\alpha,\beta)*} \right)(x) = \left[S_{1,m}^{(\alpha,\beta)*}(e_1,x) - x S_{1,m}^{(\alpha,\beta)*}(e_0,x) \right] = 0$$

and

$$\begin{split} & \left(\Gamma_{m,2}S_{1,m}^{(\alpha,\beta)*}\right)(x) = S_{1,m}^{(\alpha,\beta)*}(e_2,x) - 2xS_{1,m}^{(\alpha,\beta)*}(e_1,x) + x^2S_{1,m}^{(\alpha,\beta)*}(e_0,x) \\ & = \frac{m-1}{m}x^2 + \frac{m+2\alpha-1}{m(m+\beta+1)}x + \frac{12\alpha(\alpha+1) - m(5-36\alpha) - 3}{12m(m+\beta+1)^2} - x^2. \end{split}$$

which, after some calculations, yields (14). \Box

Lemma 5. We have

$$\lim_{m \to \infty} m \left(\Gamma_{m,2} S_{1,m}^{(\alpha,\beta)*} \right)(x) = 12x(1-x)$$
(15)

uniformly with regard to $x \in I$. Consequently, for any $\varepsilon > 0$ there exists an integer $m_{\varepsilon} \ge m_0$, sufficiently large, such that

$$\left(\Gamma_{m,2}S_{1,m}^{(\alpha,\beta)*}\right)(x) \leq \frac{3+\varepsilon}{m},$$
(16)

for any $x \in I$ and $m \in \mathbb{N}$ such that $m \geq m_{\varepsilon}$.

Proof. The relation (15) follows from (12) and (14). The existence of m_{ε} follows from the definition of the limit of a function and the inequality (16) follows from (15) by applying the inequality $x(1-x) \le \frac{1}{4}$, which is true for every $x \in [0,1]$. \Box

Theorem 2. Let $f : [0,1] \to \mathbb{R}$ be a continuous function on [0,1]. Then, we have

$$\lim_{m\to\infty}S_{1,m}^{(\alpha,\beta)*}f=f$$

uniformly on I and for every $\varepsilon > 0$ there exists $m_{\varepsilon} \in \mathbb{N}$ such that

$$\left|S_{1,m}^{(\alpha,\beta)*}(f,x) - f(x)\right| \le (1 + \sqrt{3+\varepsilon})\omega\left(f;\frac{1}{\sqrt{m}}\right),$$

for any $x \in I$ *and* $m \in \mathbb{N}$ *such that* $m \geq m_{\varepsilon}$ *.*

Proof. The Theorem from above follows from Theorem 1 by taking $h = \frac{1}{\sqrt{m}}$.

3.1.1. Graphic properties of approximation

Figure 1. $\alpha = 0.1$, $\beta = 0.2$, m = 25 iterations



Figure 2. $\alpha = 0.1$, $\beta = 0.5$, m = 500 iterations

Remark 4. It can be seen in both Figure 1 and Figure 2 that our operators approximate the given functions for α and β chosen such that $\alpha \leq 2\beta$.

3.2. Stancu-Kantorovich type operators which preserve the functions e_0 and e_2

In this section we shall construct an operator of Stancu-Kantorovich type as in (3), that preserves the test functions e_0 and e_2 i.e. an operator that satisfies

$$S_{m}^{(\alpha,\beta)*}(e_{0},x) = 1$$

$$S_{m}^{(\alpha,\beta)*}(e_{2},x) = x^{2}$$

$$S_{m}^{(\alpha,\beta)*}(e_{1},x) \xrightarrow{u} x.$$
(17)

Now, imposing the condition (17) and the equations (4) and (6) we get:

$$c_m(x) + d_m(x) = 1, \ \forall x \in I, \ m \in \mathbb{N},$$
(18)

and the following quadratic equation, in $c_m(x)$:

$$m(m-1)c_m^2(x) + 2m(1+\alpha)c_m(x) + \alpha(\alpha+1) + \frac{1}{3} = x^2(m+\beta+1)^2, \ \forall x \in I, \ m \in \mathbb{N}.$$
(19)

Note that for $\alpha \ge 0$, $\beta \ge 0$, the discriminant

$$\delta_m(x) = 4m \left[m \left(\frac{2}{3} + \alpha \right) + \alpha^2 + \alpha + \frac{1}{3} + x^2 (m-1)(m+\beta+1)^2 \right]$$
(20)

of the quadratic equation (19) is positive.

We make the following notation:

$$\Delta_m(x) = rac{\delta_m(x)}{4}.$$

Now, solving the equation (19) we obtain, for $m \ge 2$:

$$c_m(x) = \frac{-m(1+\alpha) + \sqrt{\Delta_m(x)}}{m(m-1)}$$
(21)

and, from relation (18) we get

$$d_m(x) = \frac{m(m+\alpha) - \sqrt{\Delta_m(x)}}{m(m-1)}.$$
(22)

In order to have positive linear operators we shall impose that the functions c_m and d_m from (21) and (22), are positive. In this case we obtain the following inequalities

$$\frac{\sqrt{\alpha^2 + \alpha + \frac{1}{3}}}{m + \beta + 1} \le x \le \frac{\sqrt{m(m + 2\alpha + 1) + \alpha^2 + \alpha + \frac{1}{3}}}{m + \beta + 1}$$

Lemma 6. Let $0 < \varepsilon' < \frac{1}{2}$ be fixed. There is an integer $m_0 \in \mathbb{N}$ such that

$$[\varepsilon', 1-\varepsilon'] \subset \left[\frac{\sqrt{\alpha^2 + \alpha + \frac{1}{3}}}{m+\beta+1}; \frac{\sqrt{m(m+2\alpha+1) + \alpha^2 + \alpha + \frac{1}{3}}}{m+\beta+1}\right],$$
(23)

for every $m \in \mathbb{N}$ such that $m \ge m_{\varepsilon}$ and α , β satisfying $\alpha \le \beta$.

Proof. We have that

$$\frac{\sqrt{\alpha^2 + \alpha + \frac{1}{3}}}{m + \beta + 1} \to 0$$

and

$$\frac{\sqrt{m(m+2\alpha+1)+\alpha^2+\alpha+\frac{1}{3}}}{m+\beta+1} \to 1,$$

therefore, for all $\varepsilon > 0$, the inclusion (23) holds. \Box

Remark 5. Since the functions c_m and d_m are positive on the interval considered in (23), from now on, we will consider $I = [\varepsilon', 1 - \varepsilon']$, for all $\varepsilon' > 0$ and $m \ge m_0$.

Now, taking into account the sequences c_m and d_m obtained in (21) and (22) the operator in (3) will be

$$S_{2,m}^{(\alpha,\beta)*}(f,x) = \frac{m+\beta+1}{(m(m-1))^m} \sum_{k=0}^m \binom{m}{k} \left(-m(1+\alpha) + \sqrt{\Delta_m(x)}\right)^k$$
(24)

$$\times \left(m(m+\alpha) - \sqrt{\Delta_m(x)}\right)^{m-k} \int_{\frac{k+\alpha}{m+\beta+1}}^{\frac{k+\alpha+1}{m+\beta+1}} f(t)dt,$$

for any $x \in I$ and $m \ge m_0$.

Lemma 7. The operator $S_{2,m}^{(\alpha,\beta)*}$ from (24) satisfies

$$S_{2,m}^{(\alpha,\beta)*}(e_0, x) = 1;$$

$$S_{2,m}^{(\alpha,\beta)*}(e_1, x) = \frac{-(m+2\alpha+1)+2\sqrt{\Delta_m(x)}}{2(m+\beta+1)(m-1)};$$

$$S_{2,m}^{(\alpha,\beta)*}(e_2, x) = x^2.$$
(25)

for $x \in I$ and $m \geq m_0$.

Proof. The first and last relation from (25) are obvious and the seconds follows by applying relations (5) and (21). \Box

Now, we can obtain the following result.

Lemma 8. The following relations hold

$$\left(\Gamma_{m,0}S_{2,m}^{(\alpha,\beta)*}\right)(x) = 1,$$
(26)

$$\left(\Gamma_{m,1}S_{2,m}^{(\alpha,\beta)*}\right)(x) = x - \frac{-(m+2\alpha+1) + 2\sqrt{\Delta_m(x)}}{2(m+\beta+1)(m-1)},\tag{27}$$

$$\left(\Gamma_{m,2}S_{2,m}^{(\alpha,\beta)*}\right)(x) = 2x\left(x - \frac{-(m+2\alpha+1) + 2\sqrt{\Delta_m(x)}}{2(m+\beta+1)(m-1)}\right),\tag{28}$$

for any $x \in I$ and $m \in \mathbb{N}$.

Proof. Using the previous lemma and the definition of the operator $\Gamma_{m,i}$ from (2) we get the results after some calculations. \Box

Lemma 9. We have:

$$\lim_{m \to \infty} m \left(\Gamma_{m,1} S_{2,m}^{(\alpha,\beta)*} \right)(x) = \frac{1}{2} (1-x),$$
(29)

$$\lim_{m \to \infty} m \left(\Gamma_{m,2} S_{2,m}^{(\alpha,\beta)*} \right)(x) = x(1-x), \tag{30}$$

uniformly with regard to $x \in I$. For any $\varepsilon > 0$ there exists $m_{\varepsilon} > m_0$ such that

$$\left(\Gamma_{m,2}S_{2,m}^{(\alpha,\beta)*}\right)(x) \le \frac{1}{m}\left(\frac{1}{4}+\varepsilon\right),$$
(31)

for any $x \in I$ and $m \in \mathbb{N}$ such that $m \geq m_{\varepsilon}$.

Proof. We have:

$$\lim_{m \to \infty} m \left(x - \frac{-(m+2\alpha+1)+2\sqrt{\Delta_m(x)}}{2(m+\beta+1)(m-1)} \right) = \\ \lim_{m \to \infty} \left(-\frac{2\sqrt{\Delta_m(x)} - 2(m+\beta+1)(m-1)x}{m} \frac{m^2}{2(m+\beta+1)(m-1)} + \frac{m(m+2\alpha+1)}{2(m+\beta+1)(m-1)} \right) = \\ \frac{1}{2} - \lim_{m \to \infty} \frac{\sqrt{\Delta_m(x)} - (m+\beta+1)(m-1)x}{m},$$

and after some calculations we get:

$$\lim_{m \to \infty} m \left(x - \frac{-(m+2\alpha+1) + 2\sqrt{\Delta_m(x)}}{2(m+\beta+1)(m-1)} \right) = \frac{1}{2}(1-x).$$
(32)

Now, replacing the right hand side term in (27) and (28) with (32) we will get the convergences in (29) and (30). Using the definition of the limit of a function and the inequality $x(1-x) \leq \frac{1}{4}, \forall x \in [0,1]$ we have that for every $\varepsilon > 0$ there exists $m_{\varepsilon} \in \mathbb{N}$ such that the inequality (31) holds, for every $m \ge m_{\varepsilon}$. \Box

Now, using the above results we obtain the following theorem.

Theorem 3. Let $f : [0,1] \to \mathbb{R}$ be a continuous function on [0,1]. Then, we have

$$\lim_{m \to \infty} \left(S_{2,m}^{(\alpha,\beta)*} f \right)(x) = f(x)$$

uniformly on I and for every $\varepsilon > 0$ there exists $m_{\varepsilon} \in \mathbb{N}$ such that

1

$$\left| \left(S_{2,m}^{(\alpha,\beta)*} f \right)(x) - f(x) \right| \le \left(1 + \sqrt{\frac{1}{4} + \varepsilon} \right) \omega \left(f; \frac{1}{\sqrt{m}} \right),$$

for any $x \in I$ and $m \in \mathbb{N}$ such that $m \geq m_{\varepsilon}$.

Proof. The Theorem follows from relation (31) and from Theorem 1 by taking $h = \frac{1}{\sqrt{m}}$.

3.2.1. Graphic properties of approximation



Figure 3. $\alpha = 0.1$, $\beta = 0.65$, m = 50 iterations



Figure 4. $\alpha = 0.1$, $\beta = 0.65$, m = 50 iterations

Remark 6. Also, in this case, it can be seen in Figure 3 and Figure 4 that our operators approximate the given functions.

3.3. Stancu-Kantorovich type operators which preserve the functions e_1 and e_2

In this section we shall construct an operator of Stancu-Kantorovich type as in (3), that preserves the test functions e_1 and e_2 , i.e. an operator that satisfies

$$S_m^{(\alpha,\beta)*}(e_0, x) \xrightarrow{u} 1$$

$$S_m^{(\alpha,\beta)*}(e_1, x) = x$$

$$S_m^{(\alpha,\beta)*}(e_2, x) = x^2.$$
(33)

In order to obtain the main results of this section, we shall consider the following notation

$$S_m^{(\alpha,\beta)*}(e_0,x) = 1 + w_m(x),$$
(34)

where $x \in I$, $m \in \mathbb{N}$ and $w_m : I \to \mathbb{R}$.

With the previous notation we have the following remark.

Remark 7. In order to have positive operators $S_m^{(\alpha,\beta)*}$, $m \in \mathbb{N}$, $0 \le \alpha \le \beta$ and for the relation (34) to hold, we shall impose that $S_m^{(\alpha,\beta)*}(e_0, x) \ge 0$, which implies

$$1 + w_m(x) \ge 0, \ \forall x \in I, \ m \in \mathbb{N}$$
(35)

From (34), we get

$$(c_m(x) + d_m(x))^m = 1 + w_m(x), \forall x \in I, \ m \in \mathbb{N}$$
(36)

which implies

$$c_m(x) + d_m(x) = (1 + w_m(x))^{\frac{1}{m}}, \forall x \in I, \ m \in \mathbb{N}.$$
 (37)

Now, from the above considerations and imposing the conditions

$$S_m^{(\alpha,\beta)*}(e_1,x) = x, \ \forall x \in I, \ m \in \mathbb{N}$$
(38)

and

$$S_m^{(\alpha,\beta)*}(e_2,x) = x^2, \forall x \in I, \ m \in \mathbb{N},$$
(39)

we will obtain the following lemma.

Lemma 10. We have

$$c_m(x) = \frac{m+\beta+1}{m} \left[x - \frac{2\alpha+1}{2(m+\beta+1)} (1+w_m(x)) \right] (1+w_m(x))^{\frac{1-m}{m}}$$
(40)

and

$$d_m(x) = (1 + w_m(x))^{\frac{1}{m}} \times$$

$$\left[1 - \frac{m + \beta + 1}{m} \cdot \frac{1}{1 + w_m(x)} \left(x - \frac{2\alpha + 1}{2(m + \beta + 1)} (1 + w_m(x)) \right) \right].$$
(41)

Proof. Equallying relations (5) and (38), we get

$$\frac{m}{m+\beta+1}c_m(x)(c_m(x)+d_m(x))^{m-1}+\frac{2\alpha+1}{2(m+\beta+1)}(c_m(x)+d_m(x))^m=x.$$

Now, using the formula (37), we have

$$\frac{m}{m+\beta+1}c_m(x)(1+w_m(x))^{\frac{m-1}{m}}+\frac{2\alpha+1}{2(m+\beta+1)}(1+w_m(x))=x,$$

which gives relation (40).

Now, using relations (37) and (40), by direct computation, we will get formula (41). \Box Imposing the condition (39) we will obtain the following quadratic equation in $w_m(x)$:

$$\begin{split} w_m^2(x)[-5m-3-\alpha(1+m+\alpha)]+\\ w_m(x)\{-12m\Big[(m+1)^2+\beta(\beta+2m+2)\Big]x^2+\\ 12\Big[(m+1)^2+2\alpha(1+m)+\beta(1+m+2\alpha)\Big]x-2[5m+3+12\alpha(1+m+\alpha)]\}+\\ +\{-12\Big[(m+1)^2+\beta(\beta+2m+2)\Big]x^2+\\ +12\Big[(m+1)^2+2\alpha(1+m)+\beta(1+m+2\alpha)\Big]x-[5m+3+12\alpha(1+m+\alpha)]\}=0, \end{split}$$

which has the following solutions:

$$w_{m,1}(x) = \frac{1}{2(-3-5m-\alpha-m\alpha-\alpha^2)} \left(x^2 \left(12m+24m^2+12m^3+24\beta+24m\beta+12\beta^2 \right) + x(-144-144m-24\alpha-24m\alpha-12\beta-12m\beta-24\alpha\beta) + \left(6+10m+24\alpha+24m\alpha+24\alpha^2 - \sqrt{t_1^2(x)+t_2(x)} \right) \right) \right)$$

$$w_{m,2}(x) = \frac{1}{2(-3-5m-\alpha-m\alpha-\alpha^2)} \left(x^2 \left(12m+24m^2+12m^3+24\beta+24m\beta+12\beta^2 \right) + x(-144-144m-24\alpha-24m\alpha-12\beta-12m\beta-24\alpha\beta) + \left(6+10m+24\alpha+24m\alpha+24\alpha^2 + \sqrt{t_1^2(x)+t_2(x)} \right) \right) \right)$$

where

$$t_{1}(x) = -6 - 10m - 24\alpha - 24m\alpha - 24\alpha^{2} + x(144 + 144m + 24\alpha + 24m\alpha + 12\beta + 12m\beta + 24\alpha\beta) + x^{2}(-12m - 24m^{2} - 12m^{3} - 24\beta - 24m\beta - 12\beta^{2});$$

$$t_{2}(x) = 4(-3 - 5m - 12\alpha - 12m\alpha - 12\alpha^{2})(-3 - 5m - \alpha - m\alpha - \alpha^{2}) + 4x^{2}(-3 - 5m - \alpha - m\alpha - \alpha^{2})(-144 - 144m - 36\beta - 24m\beta) + 4x(-3 - 5m - \alpha - m\alpha - \alpha^{2})(144 + 144m + 24\alpha + 24m\alpha + 12\beta + 12m\beta + 24\alpha\beta).$$

Remark 8. It is easy to verify that $\lim_{m\to\infty} w_{m,2}(x) = -\infty$ and $\lim_{m\to\infty} w_{m,1}(x) = 0$, uniformly for $x \in (0,1)$.

From now on, in this section we will consider $w_m(x) = w_{m,1}(x)$.

In order to have a positive operator, the quantities $c_m(x)$ and $d_m(x)$ from relations (40) and (41) shall be positive. With that condition we get the following inequalities

$$x - rac{2lpha + 1}{2(m + eta + 1)}(1 + w_m(x)) \ge 0,$$

and

$$1 - \frac{m+\beta+1}{m} \cdot \frac{1}{1+w_m(x)} \left(x - \frac{2\alpha+1}{2(m+\beta+1)} (1+w_m(x)) \right) \ge 0$$

for all $x \in I$, $m \in \mathbb{N}$ and $0 \le \alpha \le \beta$ which lead to

$$\frac{2(m+\beta+1)}{2m+2\alpha+1}x - 1 \le w_m(x) \le \frac{2(m+\beta+1)}{2\alpha+1}x - 1,$$
(42)

for all $x \in I$, $m \in \mathbb{N}$ and $0 \le \alpha \le \beta$.

Lemma 11. Let $0 < \varepsilon' < \frac{1}{2}$. Then there exists $m_0 \in \mathbb{N}$ such that relation (42) holds for any $x \in [\varepsilon', 1 - \varepsilon']$ and $m \in \mathbb{N}$, $m \ge m_0$.

Proof. We have $w_m \rightarrow 0$, uniformly on (0, 1), and

$$\lim_{m\to\infty}\left(\frac{2(m+\beta+1)}{2m+2\alpha+1}x-1\right)=x-1\leq\varepsilon',$$

uniformly for $x \in I$. \Box

From now on, we will consider $I = [\varepsilon', 1 - \varepsilon']$, with fixed $0 < \varepsilon' < \frac{1}{2}$. We can write the operators in (3) as

$$S_{3,m}^{(\alpha,\beta)*}(f,x) = (m+\beta+1) \sum_{k=0}^{m} {m \choose k} (1+w_m(x))^{1-k} \\ \times \left(\frac{m+\beta+1}{m} \left(x - \frac{2\alpha+1}{2(m+\beta+1)}(1+w_m(x))\right)\right)^k \\ \times \left(1 - \frac{m+\beta+1}{m(1+w_m(x))} \left(x - \frac{2\alpha+1}{2(m+\beta+1)}(1+w_m(x))\right)\right)^{m-k} \int_{\frac{k+\alpha}{m+\beta+1}}^{\frac{k+\alpha}{m+\beta+1}} f(t)dt.$$

Lemma 12. *For* $x \in I$ *and* $m \in \mathbb{N}$ *, we have*

$$egin{aligned} & \left(\Gamma_{m,0} S^{(lpha,eta)*}_{3,m}
ight)(x) = 1 + w_m(x), \ & \left(\Gamma_{m,1} S^{(lpha,eta)*}_{3,m}
ight)(x) = -x w_m(x), \ & \left(\Gamma_{m,2} S^{(lpha,eta)*}_{3,m}
ight)(x) = x^2 w_m(x). \end{aligned}$$

Proof. The proof follows immediately from relation (2) and from the conditions (38), (39). \Box

Theorem 4. We have

$$\lim_{m \to \infty} S_{3,m}^{(\alpha,\beta)*}(f,x) = f(x)$$

uniformly on I for every $f \in C([0,1])$.

Proof. By applying Theorem 1. \Box

Remark 9. It was proved in [6] that there is no sequence of positive, linear and analytic operators $L : C[0,1] \rightarrow C[0,1]$ that preserve the test functions e_1 and e_2 . Therefore we have operators $S_{3,m}^{(\alpha,\beta)*} : C[0,1] \rightarrow C[\varepsilon', 1-\varepsilon']$.

3.3.1. Graphic properties of approximation

As a first comparison, we considered the function $f(x) = \sin(20x)$ and we obtained the following graphics where PolKS(x) represent our operators that preserve e_1 and e_2 , and P(x) is the operator obtained by Indrea et al. in [9], which is also a particular case of our operators considered in the third section for $\alpha = \beta = 0$.

sin20x.png sin20x.pdf sin20x.jpg sin20x.mps sin20x.jpeg sin20x.jbig2 sin20x.jb2 sin20x.PNG sin20x.PDF sin20x.JPG sin20x.JPEG sin20x.JBIG2 sin20x.JB2 sin20x.eps

Figure 5. $f(x) = \sin(20x)$, $\alpha = 10$, $\beta = 20$, m = 50 iterations

Now, we considered the function f(x) = |x - 0.5| and we obtained the following graphic:

modul2.png modul2.pdf modul2.jpg modul2.mps modul2.jpeg modul2.jbig2 modul2.jb2 modul2.PNG modul2.PDF modul2.JPG modul2.JPEG modul2.JBIG2 modul2.JB2 modul2.eps

Figure 6. $f(x) = |x - 0.5|, \alpha = 10, \beta = 20, m = 50$ iterations

Author Contributions: All authors contributed equally to this work.

Funding: The APC was funded by Transilvania University of Brașov.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

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