 doi:10.20944/preprints202105.0125.v1

A New Iterative Scheme for Approximation of Fixed Points of Suzuki's Generalized Nonexpansive Mappings

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Abstract

In this paper, we introduce a new iteration scheme, named as the S**-iteration scheme, for approximation of fixed point of the nonexpansive mappings. This scheme is faster than Picard, Mann, Ishikawa, Noor, Agarwal, Abbas, Thakur, and Ullah iteration schemes. We show the stability of our instigated scheme and give a numerical example to vindicate our claim. We also put forward some weak and strong convergence theorems for Suzuki's generalized nonexpansive mappings in the setting of uniformly convex Banach spaces. Our results comprehend, improve, and consolidate many results in the existing literature.

Keywords: Uniformly convex Banach space; Iteration process; Suzuki's generalized nonexpansive mapping; weak convergence; Strong convergence
AMS Subject Classification: 47H09, 47H10.

1 Introduction

Fixed point theory provides very useful tools to solve most of the nonlinear problems, that have application in different fields, as they can be easily transformed into a fixed point problem. After establishing the existence of a fixed point we find its value using iterative processes. Till now many iterative processes have been developed, all of which can not be covered. Banach contraction principle [1], which is the most celebrated result in fixed point theory uses Picard iteration process for approximating the fixed point. The Picard iteration process is useful for the approximation of the fixed point of the contraction mappings but when one is dealing with nonexpansive mappings then it may fail to converge to the fixed point even if the fixed point is unique.

In 1953, Mann [2] introduced a new iterative scheme to approximate the fixed points of nonexpansive mappings. For a nonempty subset $\mathfrak{C}$ of a Banach space $\mathfrak{X}$, let $\mathfrak{T} : \mathfrak{C} \to \mathfrak{C}$ be a mapping. In this iterative scheme the sequence (t_n) is generated by $t_0 \in \mathfrak{C}$ as:

$$t_{n+1} = (1 - \alpha_n)t_n + \alpha_n \mathfrak{T} t_n \text{ for all } n \geq 0, \tag{1}$$

where $\alpha \in (0, 1)$. But the Mann iterative scheme fails to converge to fixed points of pseudo-contractive mappings.

In 1974, Ishikawa [3] introduced a two step Mann iterative scheme to approximate fixed points of pseudo-contractive mappings, where the sequence (t_n) is generated by $t_0 \in \mathfrak{C}$ as:

$$\left.\begin{aligned} t_{n+1} &= (1 - \alpha_n)t_n + \alpha_n \mathfrak{T} s_n \\ s_n &= (1 - \beta_n)t_n + \beta_n \mathfrak{T} t_n \end{aligned}\right\}, \tag{2}$$

for all $n \geq 0$, where $\alpha_n, \beta_n \in (0, 1)$.
Many authors studied Mann and Ishikawa iterative schemes for approximation of fixed point of nonexpansive mappings (for instance [4],[5] and [6]).

In 2000, Noor [7] established another iterative scheme, where the sequence (t_n) is generated by $t_0 \in \mathfrak{C}$ as:

$$\left.\begin{aligned} t_{n+1} &= (1 - \alpha_n)t_n + \alpha_n \mathfrak{T} s_n \\ s_n &= (1 - \beta_n)t_n + \beta_n \mathfrak{T} r_n \\ r_n &= (1 - \gamma_n)t_n + \gamma_n \mathfrak{T} t_n \end{aligned}\right\}, \tag{3}$$

 doi:10.20944/preprints202105.0125.v1

for all $n \geq 0$, where $\alpha_n, \beta_n, \gamma_n \in (0,1)$.

In 2007, Agarwal et al. [8] introduced a two-step iteration process for nearly asymptotically nonexpansive mappings, for arbitrary $t_0 \in \mathfrak{C}$, a sequence (t_n) is generated by

$$\left.\begin{array}{l} t_{n+1} = (1-\alpha_n)\mathfrak{T}t_n + \alpha_n \mathfrak{T}s_n \\ s_n = (1-\beta_n)t_n + \beta_n \mathfrak{T}t_n \end{array}\right\}, \tag{4}$$

for all $n \geq 0$, where $\alpha_n, \beta_n \in (0,1)$. This process converges faster than Mann iteration process for contraction mappings.

In 2014, Abbas and Nazir [9] developed an iterative scheme which is faster than Agarwal et al.'s [8] scheme, where a sequence (t_n) is formulated from arbitrary $t_0 \in \mathfrak{C}$ by

$$\left.\begin{array}{l} t_{n+1} = (1-\alpha_n)\mathfrak{T}s_n + \alpha_n \mathfrak{T}r_n \\ s_n = (1-\beta_n)\mathfrak{T}t_n + \beta_n \mathfrak{T}r_n \\ r_n = (1-\gamma_n)t_n + \gamma_n \mathfrak{T}t_n \end{array}\right\}, \tag{5}$$

for all $n \geq 0$, where $\alpha_n, \beta_n, \gamma_n \in (0,1)$.

In 2016, Thakur et al. [10] developed an iterative procedure, where a sequence (t_n) is generated iteratively by arbitrary $t_0 \in \mathfrak{C}$ and

$$\left.\begin{array}{l} t_{n+1} = (1-\alpha_n)\mathfrak{T}r_n + \alpha_n \mathfrak{T}s_n \\ s_n = (1-\beta_n)r_n + \beta_n \mathfrak{T}r_n \\ r_n = (1-\gamma_n)t_n + \gamma_n \mathfrak{T}t_n \end{array}\right\}, \tag{6}$$

for all $n \geq 0$, where $\alpha_n, \beta_n, \gamma_n \in (0,1)$.

In 2018, Ullah and Arshad [11] developed a new iteration process which converges faster than all the aforementioned process, where the sequence (t_n) is constructed by taking arbitrary $t_0 \in \mathfrak{C}$ and

$$\left.\begin{array}{l} t_{n+1} = \mathfrak{T}s_n \\ s_n = \mathfrak{T}((1-\alpha_n)r_n + \alpha_n \mathfrak{T}r_n) \\ r_n = (1-\beta_n)t_n + \beta_n \mathfrak{T}t_n \end{array}\right\}, \tag{7}$$

for all $n \geq 0$, where $\alpha_n, \beta_n \in (0,1)$.

Recently, in 2020, Hassan et al. [12] introduced a new four-step iteration scheme for approximation of fixed point of the nonexpansive mappings named as S*-iteration scheme, where the sequence (t_n) is generated by taking arbitrary $t_0 \in \mathfrak{C}$ and

$$\left.\begin{array}{l} t_{n+1} = \mathfrak{T}((1-\alpha_n)s_n + \alpha_n \mathfrak{T}s_n) \\ s_n = \mathfrak{T}((1-\beta_n)r_n + \beta_n \mathfrak{T}r_n) \\ r_n = \mathfrak{T}((1-\gamma_n)q_n + \gamma_n \mathfrak{T}q_n) \\ q_n = \mathfrak{T}((1-\delta_n)t_n + \delta_n \mathfrak{T}t_n) \end{array}\right\}, \tag{8}$$

for all $n \geq 0$, where $\alpha_n, \beta_n, \gamma_n, \delta_n \in (0,1)$.

In this paper, we introduce a new three-step iteration process which is faster than Picard, Mann, Ishikawa, Noor, Agarwal, Abbas, Thakur, and Ullah iteration processes and prove the convergence results using our iterative scheme for Suzuki's generalized nonexpansive mappings in the context of uniformly convex Banach spaces. We also show that our process is analytically stable. With the help of an example, we compare the rate of convergence of our iteration process with the aforementioned iteration processes.

2 Preliminaries

Throughout this paper, $\mathfrak{C}$ is a non-empty closed convex subset of a uniformly convex Banach space $\mathfrak{X}$, $\mathbb{N}$ denotes the set of all positive integers, $\mathfrak{T} : \mathfrak{C} \to \mathfrak{C}$ be a mapping and $F(\mathfrak{T})$ denotes the set of all fixed points of $\mathfrak{T}$.

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Definition 2.1. *[13] A Banach space $\mathfrak{X}$ is said to be uniformly convex if for each $\epsilon \in (0,2]$, there exists a $\delta > 0$ such that for all $x, y \in \mathfrak{X}$,*

$$\left.\begin{array}{l} \|x\| \leq 1, \\ \|y\| \leq 1, \\ \|x-y\| > \epsilon \end{array}\right\} \textit{ implies } \left\|\frac{x+y}{2}\right\| \leq \delta. \tag{9}$$

Definition 2.2. *[14] A Banach space $\mathfrak{X}$ is said to satisfy Opial property if for each sequence (t_n) in $\mathfrak{X}$, converging weakly to $p \in \mathfrak{X}$, we have*

$$\limsup_{n\to\infty} \|t_n - p\| < \limsup_{n\to\infty} \|t_n - q\|, \tag{10}$$

for all $q \in \mathfrak{X}$ such that $p \neq q$.

Definition 2.3. *A mapping $\mathfrak{T} : \mathfrak{C} \to \mathfrak{C}$ is called a contraction if there exists $\alpha \in (0,1)$, such that*

$$\|\mathfrak{T}p - \mathfrak{T}q\| \leq \alpha\|p-q\|, \textit{ for all } p, q \in \mathfrak{C}. \tag{11}$$

Definition 2.4. *A mapping $\mathfrak{T} : \mathfrak{C} \to \mathfrak{C}$ is called quasi-nonexpansive if for all $p \in \mathfrak{C}$ and $q \in F(\mathfrak{T})$, we have*

$$\|\mathfrak{T}p - q\| \leq \|p - q\|. \tag{12}$$

In 2008, Suzuki introduced the concept of generalized nonexpansive mappings as follows.

Definition 2.5. *[15] A mapping $\mathfrak{T} : \mathfrak{C} \to \mathfrak{C}$ is called Suzuki's generalized nonexpansive mapping if for all $p, q \in \mathfrak{C}$, we have*

$$\frac{1}{2}\|p - \mathfrak{T}p\| \leq \alpha\|p-q\| \textit{ implies } \|\mathfrak{T}p - \mathfrak{T}q\| \leq \|p-q\|. \tag{13}$$

Suzuki [15] proved that the generalized nonexpansive mapping is weaker than nonexpansive mapping and stronger than quasi-nonexpansive mapping and obtained some fixed points and convergence theorems for Suzuki's generalized nonexpansive mappings. Recently, many authors have studied fixed-point theorems for Suzuki's generalized nonexpansive mapping (e.g.,[16]).

Senter and Dotson [5] introduced a class of mappings satisfying condition (I).

Definition 2.6. *A mapping $\mathfrak{T} : \mathfrak{C} \to \mathfrak{C}$ is said to satisfy condition (I), if there exists a nondecreasing function $\mathfrak{f} : [0,\infty) \to [0,\infty)$ with $\mathfrak{f}(0) = 0$ and $\mathfrak{f}(\delta) > 0$, for all $\delta > 0$ such that $\|q - \mathfrak{T}q\| \geq \mathfrak{f}(d(q, F(\mathfrak{T})))$, for all $q \in \mathfrak{C}$, where $d(q, F(\mathfrak{T})) = \inf_{q^* \in F(\mathfrak{T})} \|q - q^*\|$.*

Proposition 2.7. *[17] Let $\mathfrak{T} : \mathfrak{C} \to \mathfrak{C}$ be any mapping. Then*
(i) If $\mathfrak{T}$ is nonexpansive, then $\mathfrak{T}$ is a Suzuki's generalized nonexpansive mapping.
(ii) If $\mathfrak{T}$ is a Suzuki's generalized nonexpansive mapping and has a fixed point, then $\mathfrak{T}$ is a quasi-nonexpansive mapping.
(iii) If $\mathfrak{T}$ is a Suzuki's generalized nonexpansive mapping, then

$$\|p - \mathfrak{T}q\| \leq 3\|\mathfrak{T}p - p\| + \|p - q\|, \textit{ for all } p, q \in \mathfrak{C}. \tag{14}$$

Lemma 2.8. *[17] Suppose $\mathfrak{T} : \mathfrak{C} \to \mathfrak{C}$ is Suzuki's generalized nonexpansive mapping satisfying Opial property. If (t_n) converges weakly to p and $\lim_{n\to\infty} \|\mathfrak{T}t_n - t_n\| = 0$, then $\mathfrak{T}p = p$.*

Lemma 2.9. *[17] Let $\mathfrak{X}$ be a uniformly convex Banach space and $\mathfrak{C}$ a weakly convex compact subset of $\mathfrak{X}$. Assume that $\mathfrak{T} : \mathfrak{C} \to \mathfrak{C}$ is Suzuki's generalized nonexpansive mapping. Then $\mathfrak{T}$ has a fixed point.*

Lemma 2.10. *[18] Let $\mathfrak{X}$ be a uniformly convex Banach space and x_n be any real sequence such that $0 < a \leq x_n \leq b < 1$ for all $n \geq 1$. Suppose that (u_n) and (v_n) are any two sequences of $\mathfrak{X}$ such that $\limsup_{n\to\infty} \|u_n\| \leq r$, $\limsup_{n\to\infty} \|v_n\| \leq r$ and $\limsup_{n\to\infty} \|x_n u_n + (1-x_n)v_n\| = r$ hold for some $r \geq 0$. Then, $\limsup_{n\to\infty} \|u_n - v_n\| = 0$.*

Definition 2.11. *[11] Let $\mathfrak{X}$ be a Banach space and $\mathfrak{C}$ a non-empty closed convex subset of $\mathfrak{X}$. Assume that (t_n) is a bounded sequence in $\mathfrak{X}$. For $p \in \mathfrak{X}$, we set $r(p,(t_n)) = \limsup_{n\to\infty} \|t_n - p\|$. The asymptotic radius of (t_n) relative to $\mathfrak{C}$ is the set $r(\mathfrak{C},(t_n)) = inf\{r(p,(t_n)) : p \in \mathfrak{C}\}$ and the asymptotic center of (t_n) relative to $\mathfrak{C}$ is given by the following set:*

$$\mathfrak{A}(\mathfrak{C},(t_n)) = \{p \in \mathfrak{C} : r(p,(t_n)) = r(\mathfrak{C},(t_n))\}. \tag{15}$$

It is known that, in a uniformly convex Banach space, $\mathfrak{A}(\mathfrak{C},(t_n))$ consists of exactly one point.

Definition 2.12. *[19] Let $\mathfrak{X}$ be a Banach space and $\mathfrak{T} : \mathfrak{X} \to \mathfrak{X}$. Suppose that $t_0 \in \mathfrak{X}$ and $t_{n+1} = f(\mathfrak{T}, t_n)$ defines an iteration procedure which gives a sequence of points $t_n \in \mathfrak{X}$. Assume that t_n converges to the fixed point p. Suppose (s_n) be a sequence in $\mathfrak{X}$ and ϵ_n be a sequence in $\mathbb{R}^+ = [0,\infty)$ given by $\epsilon_n = \|s_{n+1} - f(\mathfrak{T}, s_n)\|$. Then the iteration procedure defined by $t_{n+1} = f(\mathfrak{T}, t_n)$ is said to be $\mathfrak{T}$-stable or stable with respect to $\mathfrak{T}$ if $\lim_{n\to\infty} \epsilon_n = 0$ iff $\lim_{n\to\infty} s_n = p$.*

Definition 2.13. *[20] Let $\mathfrak{X}$ be a Banach space and $\mathfrak{T} : \mathfrak{X} \to \mathfrak{X}$. Then $\mathfrak{T}$ is called a contractive mapping on $\mathfrak{X}$ if there exist $L \geq 0$, $b \in [0,1)$ such that for each $p, q \in \mathfrak{X}$*

$$\|\mathfrak{T}p - \mathfrak{T}q\| \leq L\|p - \mathfrak{T}p\| + b\|p - q\|. \tag{16}$$

By using (7), Osilike [20] established several stability results most of which are generalizations of the results of Rhoades [21] and Harder and Hicks [22].

Definition 2.14. *[23] Let $\mathfrak{X}$ be a Banach space and $\mathfrak{T} : \mathfrak{X} \to \mathfrak{X}$. Then $\mathfrak{T}$ is called a contractive mapping on $\mathfrak{X}$ if there exist $a \in [0,1)$ and a monotone increasing function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ with $\psi(0) = 0$, such that for each $p, q \in \mathfrak{X}$,*

$$\|\mathfrak{T}p - \mathfrak{T}q\| \leq \psi(\|p - \mathfrak{T}p\|) + a\|p - q\|. \tag{17}$$

Lemma 2.15. *[24] If λ is a real number such that $0 \leq \lambda < 1$, and (ϵ_n) is the sequence of positive numbers such that*

$$\lim_{n\to\infty} \epsilon_n = 0, \tag{18}$$

then for any sequence of positive numbers t_n satisfying

$$t_{n+1} \leq \lambda t_n + \epsilon_n, \ \textit{for } n = 1, 2, ..., \tag{19}$$

we have

$$\lim_{n\to\infty} t_n = 0. \tag{20}$$

3 S**-Iteration Process

We introduce a new iteration scheme by generating the sequence (t_n) iteratively, taking arbitrary $t_0 \in \mathfrak{C}$, as

$$\left.\begin{array}{l} t_{n+1} = \mathfrak{T}((1-\mu_n)\mathfrak{T}r_n + \mu_n\mathfrak{T}s_n) \\ s_n = \mathfrak{T}((1-\nu_n)r_n + \nu_n\mathfrak{T}r_n) \\ r_n = \mathfrak{T}((1-\xi_n)t_n + \xi_n\mathfrak{T}t_n) \end{array}\right\}. \tag{21}$$

for all $n \geq 0$, where $(\mu_n), (\nu_n)$ and (ξ_n) are real sequences in the interval $(0,1)$.

Further, we show that S**-iteration process converges faster than all aforementioned iteration processes for contractive mappings due to Berinde [25] and is stable.

We will establish the convergence results for S**-iteration process foremost:

Theorem 3.1. *Let $\mathfrak{C}$ be a non-empty closed convex subset of a Banach space $\mathfrak{X}$ and $\mathfrak{T}$ a nonexpansive mapping on $\mathfrak{C}$. Let (t_n) be a sequence defined by (21) and $F(\mathfrak{T}) \neq \phi$. Then $\lim_{n\to\infty} \|t_n - q\|$ exists for all $q \in F(\mathfrak{T})$.*

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Proof. Let $q \in F(\mathfrak{T})$ for all $n \in \mathbb{N}$. From (13), we have

$$\begin{aligned} \|r_n - q\| &= \|\mathfrak{T}((1-\xi_n)t_n + \xi_n \mathfrak{T} t_n) - q\| \\ &\leq \|(1-\xi_n)t_n + \xi_n \mathfrak{T} t_n - q\| \\ &\leq (1-\xi_n)\|t_n - q\| + \xi_n \|\mathfrak{T} t_n - q\| \\ &\leq (1-\xi_n)\|t_n - q\| + \xi_n \|t_n - q\| \\ &= \|t_n - q\|, \end{aligned} \tag{22}$$

$$\begin{aligned} \|s_n - q\| &= \|\mathfrak{T}((1-\nu_n)r_n + \nu_n \mathfrak{T} r_n) - q\| \\ &\leq \|(1-\nu_n)r_n + \nu_n \mathfrak{T} r_n - q\| \\ &\leq (1-\nu_n)\|r_n - q\| + \nu_n \|\mathfrak{T} r_n - q\| \\ &\leq (1-\nu_n)\|r_n - q\| + \nu_n \|r_n - q\| \\ &= \|r_n - q\| \\ &\leq \|t_n - q\|, \end{aligned} \tag{23}$$

$$\begin{aligned} \|t_{n+1} - q\| &= \|\mathfrak{T}((1-\mu_n)\mathfrak{T} r_n + \mu_n \mathfrak{T} s_n) - q\| \\ &\leq \|(1-\mu_n)\mathfrak{T} r_n + \mu_n \mathfrak{T} s_n - q\| \\ &\leq (1-\mu_n)\|\mathfrak{T} r_n - q\| + \mu_n \|\mathfrak{T} s_n - q\| \\ &\leq (1-\mu_n)\|r_n - q\| + \mu_n \|s_n - q\| \\ &\leq (1-\mu_n)\|t_n - q\| + \mu_n \|t_n - q\| \\ &= \|t_n - q\|. \end{aligned} \tag{24}$$

Hence $\lim_{n\to\infty} \|t_n - q\|$ exists for all $q \in F(\mathfrak{T})$. □

Theorem 3.2. *Let $\mathfrak{C}$ be a non-empty closed convex subset of a uniformly convex Banach space $\mathfrak{X}$ and $\mathfrak{T} : \mathfrak{C} \to \mathfrak{C}$ a nonexpansive mapping. Let (t_n) be defined by the iteration process (21) and $F(\mathfrak{T}) \neq \phi$. Then the sequence (t_n) converges to a point of $F(\mathfrak{T})$ iff $\liminf_{n\to\infty} d(t_n, F(\mathfrak{T})) = 0$, where $d(t_n, F(\mathfrak{T})) = inf\{\|t_n - q\| : q \in F(\mathfrak{T})\}$.*

Proof. It is obvious that if the sequence (t_n) converges to a point of $F(\mathfrak{T})$ then $\liminf_{n\to\infty} d(t_n, F(\mathfrak{T})) = 0$. Now, suppose that $\liminf_{n\to\infty} d(t_n, F(\mathfrak{T})) = 0$. From Theorem 3.1, we have $\lim_{n\to\infty} \|t_n - q\|$ exists for all $q \in F(\mathfrak{T})$, so $\lim_{n\to\infty} d(t_n, F(\mathfrak{T}))$ exists and $\liminf_{n\to\infty} d(t_n, F(\mathfrak{T})) = 0$ by assumption. Now, we will prove that (t_n) is a cauchy sequence in $\mathfrak{C}$. For a given $\epsilon > 0$, $\exists\ N \in \mathbb{N}$ s.t., for all $n \in \mathbb{N}$,

$$d(t_n, F(\mathfrak{T})) < \epsilon/2. \tag{25}$$

In particular, $inf\{\|t_n - q\| : q \in F(\mathfrak{T})\} < \epsilon/2$. Hence, there exists $q^* \in F(\mathfrak{T})$ s.t. $\|t_n - q^*\| < \epsilon/2$. Now, for all $m, n \in \mathbb{N}$,

$$\|t_{m+n} - t_n\| \leq \|t_{m+n} - q^*\| + \|t_n - q^*\| \leq 2\|t_n - q^*\| < \epsilon, \tag{26}$$

which shows that (t_n) is a cauchy sequence in $\mathfrak{C}$. Also $\mathfrak{C}$ is given to be a closed subset of $\mathfrak{X}$, therefore there exists $q \in \mathfrak{C}$ s.t. $\lim_{n\to\infty} t_n = q$. Now, $\lim_{n\to\infty} d(t_n, F(\mathfrak{T})) = 0$ gives $d(q, F(\mathfrak{T})) = 0$ which implies that $q \in F(\mathfrak{T})$. □

We prove that our iteration process is $\mathfrak{T}$-stable.

Theorem 3.3. *Let $\mathfrak{X}$ be a Banach Space and $\mathfrak{T} : \mathfrak{X} \to \mathfrak{X}$ a mapping satisfying (21). Suppose $\mathfrak{T}$ has a fixed point q and (t_n) be a sequence in $\mathfrak{X}$ satisfying (21). Then (21) is $\mathfrak{T}$-stable.*

Proof. Let (w_n) be an arbitrary sequence in $\mathfrak{X}$ and the sequence which is generated by (21) is $t_{n+1} = f(\mathfrak{T}, t_n)$ converging to a unique fixed point q and $\epsilon_n = \|w_{n+1} - f(\mathfrak{T}, w_n)\|$. We show that $\lim_{n\to\infty} \epsilon_n = 0$ iff

$\lim_{n\to\infty} w_n = q$. First, assume that $\lim_{n\to\infty} \epsilon_n = 0$ and

$$\begin{aligned}
\|w_{n+1} - q\| &= \|w_{n+1} - f(\mathfrak{T}, w_n) + f(\mathfrak{T}, w_n) - q\| \\
&\leq \|w_{n+1} - f(\mathfrak{T}, w_n)\| + \|f(\mathfrak{T}, w_n) - q\| \\
&\leq \|w_{n+1} - \mathfrak{T}((1-\mu_n)\mathfrak{T}u_n + \mu_n\mathfrak{T}v_n)\| + \|\mathfrak{T}((1-\mu_n)\mathfrak{T}u_n + \mu_n\mathfrak{T}v_n) - q\| \\
&\leq \epsilon_n + b[(1-\mu_n)\|\mathfrak{T}u_n - q\| + \mu_n\|\mathfrak{T}v_n - q\|] \\
&\leq \epsilon_n + b^2[(1-\mu_n)\|u_n - q\| + \mu_n\|v_n - q\|] \\
&\leq \epsilon_n + b^2[(1-\mu_n)\|\mathfrak{T}((1-\xi_n)w_n + \xi_n\mathfrak{T}w_n) - q\| + \mu_n\|\mathfrak{T}((1-\nu_n)u_n + \nu_n\mathfrak{T}u_n) - q\|] \\
&\leq \epsilon_n + b^3[(1-\mu_n)\|(1-\xi_n)w_n + \xi_n\mathfrak{T}w_n - q\| + \mu_n\|(1-\nu_n)u_n + \nu_n\mathfrak{T}u_n - q\|] \\
&\leq \epsilon_n + b^3[(1-\mu_n)(1-\xi_n(1-b))\|w_n - q\| + \mu_n(1-\nu_n(1-b))\|u_n - q\|] \\
&\leq \epsilon_n + b^3[(1-\mu_n)(1-\xi_n(1-b))\|w_n - q\| + \mu_n(1-\nu_n(1-b))\|\mathfrak{T}((1-\xi_n)w_n + \xi_n\mathfrak{T}w_n) - q\|] \\
&\leq \epsilon_n + b^3[(1-\mu_n)(1-\xi_n(1-b))\|w_n - q\| + \mu_n(1-\nu_n(1-b))b\|(1-\xi_n)w_n + \xi_n\mathfrak{T}w_n - q\|] \\
&\leq \epsilon_n + b^3[(1-\mu_n)(1-\xi_n(1-b))\|w_n - q\| + b\mu_n(1-\nu_n(1-b))(1-\xi_n(1-b))\|w_n - q\|] \\
&= \epsilon_n + b^3[(1-\mu_n)(1-\xi_n(1-b)) + b\mu_n(1-\nu_n(1-b))(1-\xi_n(1-b))]\|w_n - q\|.
\end{aligned} \tag{27}$$

Since $b \in [0, 1)$ and μ_n, ν_n and ξ_n are in $[0, 1]$, we have

$$b^3[(1-\mu_n)(1-\xi_n(1-b)) + b\mu_n(1-\nu_n(1-b))(1-\xi_n(1-b))] < 1. \tag{28}$$

Hence by Lemma 2.15, we have $\lim_{n\to\infty} \|w_n - q\| = 0$, which gives $\lim_{n\to\infty} w_n = q$. Conversely, suppose that $\lim_{n\to\infty} w_n = q$. Then

$$\begin{aligned}
\epsilon_n &= \|w_{n+1} - f(\mathfrak{T}, w_n)\| \\
&= \|w_{n+1} - q + q - f(\mathfrak{T}, w_n)\| \\
&\leq \|w_{n+1} - q\| + \|\mathfrak{T}((1-\mu_n)\mathfrak{T}u_n + \mu_n\mathfrak{T}v_n) - q\| \\
&\leq \|w_{n+1} - q\| + b[(1-\mu_n)\|\mathfrak{T}u_n - q\| + \mu_n\|\mathfrak{T}v_n - q\|] \\
&\leq \|w_{n+1} - q\| + b^2[(1-\mu_n)\|u_n - q\| + \mu_n\|v_n - q\|] \\
&\leq \|w_{n+1} - q\| + b^2[(1-\mu_n)\|\mathfrak{T}((1-\xi_n)w_n + \xi_n\mathfrak{T}w_n) - q\| + \mu_n\|\mathfrak{T}((1-\nu_n)u_n + \nu_n\mathfrak{T}u_n) - q\|] \\
&\leq \|w_{n+1} - q\| + b^3[(1-\mu_n)\|(1-\xi_n)w_n + \xi_n\mathfrak{T}w_n - q\| + \mu_n\|(1-\nu_n)u_n + \nu_n\mathfrak{T}u_n - q\|] \\
&\leq \|w_{n+1} - q\| + b^3[(1-\mu_n)(1-\xi_n(1-b))\|w_n - q\| + \mu_n(1-\nu_n(1-b))\|u_n - q\|] \\
&\leq \|w_{n+1} - q\| + b^3[(1-\mu_n)(1-\xi_n(1-b))\|w_n - q\| + \mu_n(1-\nu_n(1-b))\|\mathfrak{T}((1-\xi_n)w_n + \xi_n\mathfrak{T}w_n) - q\|] \\
&\leq \|w_{n+1} - q\| + b^3[(1-\mu_n)(1-\xi_n(1-b))\|w_n - q\| + \mu_n(1-\nu_n(1-b))b\|(1-\xi_n)w_n + \xi_n\mathfrak{T}w_n - q\|] \\
&\leq \|w_{n+1} - q\| + b^3[(1-\mu_n)(1-\xi_n(1-b))\|w_n - q\| + b\mu_n(1-\nu_n(1-b))(1-\xi_n(1-b))\|w_n - q\|] \\
&= \|w_{n+1} - q\| + b^3[(1-\mu_n)(1-\xi_n(1-b)) + b\mu_n(1-\nu_n(1-b))(1-\xi_n(1-b))]\|w_n - q\|.
\end{aligned} \tag{29}$$

Taking limit as $n \to \infty$ in (29) gives $\lim_{n\to\infty} \epsilon_n = 0$. □

Now, we give an example to compare the rate of convergence of our iteration scheme with others.

Example 3.4. *Let $\mathfrak{X} = \mathbb{R}$, $\mathfrak{C} = [1, 30]$ and $\mathfrak{T} : \mathfrak{C} \to \mathfrak{C}$ be a mapping defined by $\mathfrak{T}t = \sqrt{t^2 - 7t + 42}$ for all $t \in \mathfrak{C}$. For $t_1 = 10$ and $\mu_n = \nu_n = \xi_n = 3/4, n = 1, 2, 3, \ldots$. From Table 1 we can see that all the iteration procedures are converging to $q^* = 6$.*

In Figure 1, black curve represents our iteration process. The graphical view shows that our iteration process requires less number of iterations as compared to the other iteration processes. The number of iterations in which these processes attain the fixed point is given in Table 2.

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Table 1: Comparison of the rate of convergence with different iteration schemes

Step	Picard	Ishikawa	Noor	Agarwal	Abbas	Thakur	K. Ullah	S**-iteration
1	10	10	10	10	10	10	10	10
2	8.48528137423857	8.23747931068045	7.91839108945248	7.85879965424009	7.48299381981887	7.19365372863816	6.50327292180584	6.17058820431927
3	7.38938633313552	7.12242990301049	6.81657619384279	6.68921521738991	6.39390416650416	6.23393093417649	6.03083955536974	6.00307677148481
4	6.69905411594662	6.52015557529087	6.32662170051318	6.21650474463483	6.08687765191737	6.03851294451616	6.00170547093161	6.00005334404782
5	6.32328611061962	6.23041600151899	6.12724957655622	6.06293951132202	6.01810044948545	6.00610953889799	6.00009371908266	6.00000092420172
6	6.14173790245218	6.09984579840025	6.04906294167737	6.01781666894804	6.00372234157488	6.00096320048653	6.00000514824794	6.00000001601188
7	6.06042730715025	6.04283750316118	6.01884104150881	6.00500351572644	6.00076340654292	6.00015170386309	6.00000028280202	6.00000000027741
8	6.02542844909809	6.01829910956501	6.00722412363746	6.00140196803230	6.00015647717348	6.00002388960866	6.00000001553478	6.00000000000481
9	6.01063963746905	6.00780230235401	6.00276826752347	6.00039257571411	6.00003206977691	6.00000376193069	6.00000000085335	6.00000000000008
10	6.00444097224968	6.00332405204450	6.00106055283451	6.00010990841270	6.00000657250009	6.00000059239429	6.00000000004688	6.00000000000000
11	6.00185176287143	6.00141567799054	6.00040627375696	6.00003076923152	6.00000134698623	6.00000009328476	6.00000000000258	6.00000000000000
12	6.00077180397513	6.00060283424769	6.00015562908162	6.00000861382931	6.00000027605478	6.00000001468962	6.00000000000014	6.00000000000000
13	6.00032162600948	6.00025668732426	6.00005961522584	6.00000241142726	6.00000005657536	6.00000000231318	6.00000000000001	6.00000000000000
14	6.00013401796082	6.00010929479003	6.00002283607749	6.00000067507432	6.00000001159469	6.00000000036426	6.00000000000000	6.00000000000000
15	6.00005584205388	6.00004653606033	6.00000874752131	6.00000018898567	6.00000000237625	6.00000000005736	6.00000000000000	6.00000000000000
16	6.00002326773719	6.00001981425228	6.00000335079762	6.00000005290615	6.00000000048699	6.00000000000903	6.00000000000000	6.00000000000000
17	6.00000969492778	6.00000843654986	6.00000128354549	6.00000001481097	6.00000000009981	6.00000000000142	6.00000000000000	6.00000000000000
18	6.00000403955972	6.00000359212706	6.00000049167066	6.00000000414629	6.00000000002046	6.00000000000022	6.00000000000000	6.00000000000000
19	6.00000168315101	6.00000152946077	6.00000018833772	6.00000000116075	6.00000000000419	6.00000000000004	6.00000000000000	6.00000000000000
20	6.00000070131311	6.00000065121579	6.00000007214401	6.00000000032495	6.00000000000086	6.00000000000001	6.00000000000000	6.00000000000000
21	6.00000029221383	6.00000027727549	6.00000002763524	6.00000000009097	6.00000000000018	6.00000000000000	6.00000000000000	6.00000000000000
22	6.00000012175577	6.00000011805871	6.00000001058586	6.00000000002547	6.00000000000004	6.00000000000000	6.00000000000000	6.00000000000000
23	6.00000005073157	6.00000005026718	6.00000000405499	6.00000000000713	6.00000000000001	6.00000000000000	6.00000000000000	6.00000000000000
24	6.00000002113816	6.00000002140282	6.00000000155329	6.00000000000199	6.00000000000000	6.00000000000000	6.00000000000000	6.00000000000000
25	6.00000000880757	6.00000000911292	6.00000000059499	6.00000000000056	6.00000000000000	6.00000000000000	6.00000000000000	6.00000000000000
26	6.00000000366982	6.00000000388011	6.00000000022792	6.00000000000016	6.00000000000000	6.00000000000000	6.00000000000000	6.00000000000000
27	6.00000000152909	6.00000000165208	6.00000000008731	6.00000000000004	6.00000000000000	6.00000000000000	6.00000000000000	6.00000000000000
28	6.00000000063712	6.00000000070342	6.00000000003344	6.00000000000001	6.00000000000000	6.00000000000000	6.00000000000000	6.00000000000000
29	6.00000000026547	6.00000000029950	6.00000000001281	6.00000000000000	6.00000000000000	6.00000000000000	6.00000000000000	6.00000000000000
30	6.00000000011061	6.00000000012752	6.00000000000491	6.00000000000000	6.00000000000000	6.00000000000000	6.00000000000000	6.00000000000000
31	6.00000000004609	6.00000000005429	6.00000000000188	6.00000000000000	6.00000000000000	6.00000000000000	6.00000000000000	6.00000000000000
32	6.00000000001920	6.00000000002312	6.00000000000072	6.00000000000000	6.00000000000000	6.00000000000000	6.00000000000000	6.00000000000000
33	6.00000000000800	6.00000000000985	6.00000000000028	6.00000000000000	6.00000000000000	6.00000000000000	6.00000000000000	6.00000000000000
34	6.00000000000333	6.00000000000419	6.00000000000011	6.00000000000000	6.00000000000000	6.00000000000000	6.00000000000000	6.00000000000000
35	6.00000000000139	6.00000000000179	6.00000000000004	6.00000000000000	6.00000000000000	6.00000000000000	6.00000000000000	6.00000000000000
36	6.00000000000058	6.00000000000076	6.00000000000002	6.00000000000000	6.00000000000000	6.00000000000000	6.00000000000000	6.00000000000000
37	6.00000000000024	6.00000000000032	6.00000000000001	6.00000000000000	6.00000000000000	6.00000000000000	6.00000000000000	6.00000000000000
38	6.00000000000010	6.00000000000014	6.00000000000000	6.00000000000000	6.00000000000000	6.00000000000000	6.00000000000000	6.00000000000000
39	6.00000000000004	6.00000000000006	6.00000000000000	6.00000000000000	6.00000000000000	6.00000000000000	6.00000000000000	6.00000000000000
40	6.00000000000002	6.00000000000003	6.00000000000000	6.00000000000000	6.00000000000000	6.00000000000000	6.00000000000000	6.00000000000000
41	6.00000000000001	6.00000000000001	6.00000000000000	6.00000000000000	6.00000000000000	6.00000000000000	6.00000000000000	6.00000000000000
42	6.00000000000000	6.00000000000000	6.00000000000000	6.00000000000000	6.00000000000000	6.00000000000000	6.00000000000000	6.00000000000000
43	6.00000000000000	6.00000000000000	6.00000000000000	6.00000000000000	6.00000000000000	6.00000000000000	6.00000000000000	6.00000000000000

 doi:10.20944/preprints202105.0125.v1

Table 2: Number of iterations in which the fixed point is attained.

Iterative method	Number of iterations
Picard	42
Ishikawa	42
Noor	38
Agarwal	29
Abbas	24
Thakur	21
K.Ullah	14
S**-iteration	10

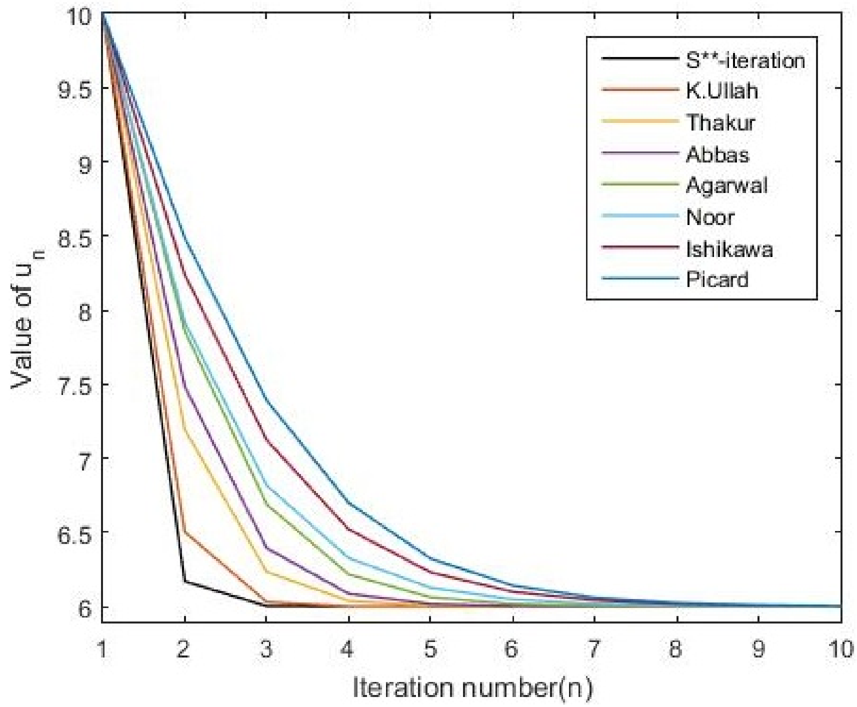

Figure 1: Graphical representation of convergence of iterative schemes.

4 Some Convergence Results for Suzukis Generalized Nonexpansive Mappings

Now, we will prove some weak and strong convergence results for the sequence generated by the S**-iteration process for Suzuki's generalized nonexpansive mappings in the setting of uniformly convex Banach spaces.

Lemma 4.1. *Let $\mathfrak{C}$ be a non-empty closed convex subset of a Banach space $\mathfrak{X}$ and $\mathfrak{T} : \mathfrak{C} \to \mathfrak{C}$ a Suzuki's generalised nonexpansive mapping with $F(\mathfrak{T}) \neq \phi$. For $t_0 \in \mathfrak{C}$, the sequence (t_n) is generated by the S**-iteration process. Then $\lim_{n\to\infty} \|t_n - q\|$ exists for all $q \in F(\mathfrak{T})$.*

Proof. From Proposition 2.7(i) and Theorem 3.1, we get our result. □

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Lemma 4.2. *Let $\mathfrak{C}$ be a non-empty closed convex subset of a uniformly convex Banach space $\mathfrak{X}$ and $\mathfrak{T}:\mathfrak{C}\to\mathfrak{C}$ a Suzuki's generalized nonexpansive mapping with $F(\mathfrak{T})\neq\phi$. For arbitrary $t_0\in\mathfrak{C}$ the sequence (t_n) is generated by the S^{**}-iteration process. Then $F(\mathfrak{T})\neq\phi$ if and only if (t_n) is bounded and $\lim_{n\to\infty}\|\mathfrak{T}t_n-t_n\|=0$.*

Proof. Suppose that $F(\mathfrak{T})\neq\phi$ and let $q\in\mathfrak{C}$. Then by previous Lemma, $\lim_{n\to\infty}\|t_n-q\|$ exists and t_n is bounded. Let

$$\lim_{n\to\infty}\|t_n-q\|=c. \tag{30}$$

From (22) and (30), we have

$$\limsup_{n\to\infty}\|r_n-q\|\leq\limsup_{n\to\infty}\|t_n-q\|=c. \tag{31}$$

By Proposition 2.7(iii), we have

$$\limsup_{n\to\infty}\|\mathfrak{T}t_n-q\|\leq\limsup_{n\to\infty}\|t_n-q\|=c. \tag{32}$$

On the other hand

$$\begin{aligned}\|t_{n+1}-q\|&=\|\mathfrak{T}((1-\mu_n)\mathfrak{T}r_n+\mu_n\mathfrak{T}s_n)-q\|\\&\leq\|(1-\mu_n)\mathfrak{T}r_n+\mu_n\mathfrak{T}s_n-q\|\\&\leq(1-\mu_n)\|r_n-q\|+\mu_n\|s_n-q\|\\&\leq(1-\mu_n)\|r_n-q\|+\mu_n\|r_n-q\|\\&\leq(1-\mu_n)\|t_n-q\|+\mu_n\|r_n-q\|.\end{aligned}$$

This implies that

$$\begin{aligned}\frac{\|t_{n+1}-q\|-\|t_n-q\|}{\mu_n}&\leq[\|r_n-q\|-\|t_n-q\|]\\\|t_{n+1}-q\|-\|t_n-q\|&\leq\frac{\|t_{n+1}-q\|-\|t_n-q\|}{\mu_n}\\&\leq[\|r_n-q\|-\|t_n-q\|]\\\|t_{n+1}-q\|&\leq\|r_n-q\|.\end{aligned}$$

Therefore,

$$c\leq\liminf_{n\to\infty}\|r_n-q\|. \tag{33}$$

From (31) and (33), we get

$$\begin{aligned}c&=\lim_{n\to\infty}\|r_n-q\|\\&=\lim_{n\to\infty}\|(1-\xi_n)t_n+\xi_n\mathfrak{T}t_n-q\|\\&=\lim_{n\to\infty}\|(1-\xi_n)(t_n-q)+\xi_n(\mathfrak{T}t_n-q)\|.\end{aligned} \tag{34}$$

From (30), (32), (34) and Lemma 2.10, we have $\lim_{n\to\infty}\|\mathfrak{T}t_n-t_n\|=0$.

Conversely, suppose that (t_n) is bounded and $\lim_{n\to\infty}\|\mathfrak{T}t_n-t_n\|=0$. Let $q\in\mathfrak{A}(\mathfrak{C},(t_n))$. From Proposition 2.7(iii), we get

$$\begin{aligned}r(\mathfrak{T}q,(t_n))&=\limsup_{n\to\infty}\|t_n-\mathfrak{T}q\|\\&\leq\limsup_{n\to\infty}[3\|\mathfrak{T}t_n-t_n\|+\|t_n-q\|]\\&\leq\limsup_{n\to\infty}\|t_n-q\|\\&=r(q,(t_n)).\end{aligned}$$

This shows that $\mathfrak{T}q\in\mathfrak{A}(\mathfrak{C}(t_n))$. Since $\mathfrak{X}$ is uniformly convex, $\mathfrak{A}(\mathfrak{C}(t_n))$ is singleton. Thus, $\mathfrak{T}q=q$ i.e., $F(\mathfrak{T})\neq\phi$. □

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Theorem 4.3. *Let $\mathfrak{C}$ be a non-empty closed convex subset of a uniformly convex Banach space $\mathfrak{X}$ with the Opial property and $\mathfrak{T} : \mathfrak{C} \to \mathfrak{C}$ a Suzuki's generalised nonexpansive mapping. For arbitrary $t_0 \in \mathfrak{C}$, let the sequence (t_n) be generated by the S^{**}-iteration process with $F(\mathfrak{T}) \neq \phi$. Then (t_n) converges weakly to a fixed point of $\mathfrak{T}$.*

Proof. Since $F(\mathfrak{T}) \neq \phi$, so by Lemma 4.2, we have that (t_n) is bounded and $\lim_{n\to\infty} \|\mathfrak{T}t_n - t_n\| = 0$. As $\mathfrak{X}$ is uniformly convex so it is reflexive, thus by Eberlin's theorem, there exists a subsequence of (t_n), say (t_{n_i}) which converges weakly to some $q_1 \in \mathfrak{X}$. Now, $\mathfrak{C}$ is a closed and convex subset of $\mathfrak{X}$ so by Mazur's theorem $q_1 \in \mathfrak{C}$. By Lemma 2.8, $q_1 \in F(\mathfrak{T})$. Next we show that (t_n) converges weakly to q_1. Let us assume that it is not true. So there must exists a subsequence of (t_n), say (t_{n_j}), such that (t_{n_j}) converges weakly to $q_2 \in \mathfrak{C}$, with $q_1 \neq q_2$. Using Lemma 2.8, we have $q_2 \in F(\mathfrak{T})$. Now, since $\lim_{n\to\infty} \|t_n - q\|$ exists for all $q \in F(\mathfrak{T})$. Using Lemma 4.2 and Opial property, we have

$$\begin{aligned}
\lim_{n\to\infty} \|t_n - q_1\| &= \lim_{i\to\infty} \|t_{n_i} - q_1\| && (35)\\
&< \lim_{i\to\infty} \|t_{n_i} - q_2\| && (36)\\
&= \lim_{n\to\infty} \|t_n - q_2\| && (37)\\
&= \lim_{j\to\infty} \|t_{n_j} - q_2\| && (38)\\
&< \lim_{j\to\infty} \|t_{n_j} - q_1\| && (39)\\
&= \lim_{n\to\infty} \|t_n - q_1\|, && (40)
\end{aligned}$$

which is a contradiction. Hence $q_1 = q_2$. This shows that (t_n) converges weakly to a fixed point of $\mathfrak{T}$. □

Theorem 4.4. *Let $\mathfrak{C}$ be a non-empty closed convex subset of a uniformly convex Banach space $\mathfrak{X}$ and $\mathfrak{T} : \mathfrak{C} \to \mathfrak{C}$ a Suzuki's generalised nonexpansive mapping. For arbitrary $t_0 \in \mathfrak{C}$, let the sequence (t_n) be generated by the S^{**}-iteration process with $F(\mathfrak{T}) \neq \phi$. Then (t_n) converges strongly to a fixed point of $\mathfrak{T}$.*

Proof. From Lemma 2.9, we get $F(\mathfrak{T}) \neq \phi$ and so by Lemma 4.2, we get $\lim_{n\to\infty} \|\mathfrak{T}t_n - t_n\| = 0$. By the compactness of $\mathfrak{C}$, there exists a subsequence of (t_n), say (t_{n_i}), converging strongly to q for some $q \in \mathfrak{C}$. Now, by using Proposition 2.7(iii), we get

$$\|t_{n_i} - \mathfrak{T}q\| \leq 3\|\mathfrak{T}t_{n_i} - t_{n_i}\| + \|t_{n_i} - q\|. \tag{41}$$

Taking limit $i \to \infty$, we get $\mathfrak{T}q = q$ i.e., $q \in F(\mathfrak{T})$. By using Lemma 4.1, $\lim_{n\to\infty} \|t_n - q\|$ exists for all $q \in F(\mathfrak{T})$. Thus, (t_n) converges strongly to q.

□

Theorem 4.5. *Let $\mathfrak{C}$ be a non-empty closed convex subset of a uniformly convex Banach space $\mathfrak{X}$ and $\mathfrak{T} : \mathfrak{C} \to \mathfrak{C}$ a Suzuki's generalised nonexpansive mapping. For arbitrary $t_0 \in \mathfrak{C}$, let the sequence (t_n) be generated by the S^{**}-iteration process with $F(\mathfrak{T}) \neq \phi$. If $\mathfrak{T}$ satisfies condition (I), then (t_n) converges strongly to a fixed point of $\mathfrak{T}$.*

Proof. By Lemma 4.2, $\lim_{n\to\infty} \|t_n - q\|$ exists for all $q \in F(\mathfrak{T})$ and so $\lim_{n\to\infty} d(t_n, F(\mathfrak{T}))$ exists. Let $\lim_{n\to\infty} \|t_n - q\| = \alpha$, for some $\alpha \geq 0$. If $\alpha = 0$, then we are done. Suppose $\alpha > 0$, from condition (I) and the hypothesis, we have

$$f(d(t_n, F(\mathfrak{T}))) \leq \|\mathfrak{T}t_n - t_n\|. \tag{42}$$

As $F(\mathfrak{T}) \neq \phi$, by Lemma 4.1, we have $\lim_{n\to\infty} \|\mathfrak{T}t_n - t_n\| = 0$. Hence (42) implies that

$$\lim_{n\to\infty} f(d(t_n, F(\mathfrak{T}))) = 0. \tag{43}$$

Since f is a nondecreasing function, by equation (43), we get $\lim_{n\to\infty} d(t_n, F(\mathfrak{T})) = 0$. Thus, we have a subsequence (t_{n_k}) of (t_n) and a sequence (z_k) of $F(\mathfrak{T})$ such that

$$\|t_{n_k} - z_k\| < \frac{1}{2^k}, \text{ for all } k \in \mathbb{N}. \tag{44}$$

From equation (44),

$$\|t_{n_{k+1}} - z_k\| \leq \|t_{n_k} - z_k\| < \frac{1}{2^k}, \tag{45}$$

$$\|z_{k+1} - z_k\| \leq \|z_{k+1} - t_{k+1}\| + \|t_{k+1} - z_k\| \tag{46}$$

$$\leq \frac{1}{2^{k+1}} + \frac{1}{2^k} \tag{47}$$

$$< \frac{1}{2^{k-1}}. \tag{48}$$

Letting $i \to \infty$, we get $\frac{1}{2^{k-1}} \to 0$. Hence z_k is a cauchy sequence in $F(\mathfrak{T})$, so it converges to q. As $F(\mathfrak{T})$ is closed, $q \in F(\mathfrak{T})$ and therefore t_{n_k} converges strongly to q. Since $\lim_{n\to\infty} \|t_n - q\|$ exists, we have $t_n \to q \in F(\mathfrak{T})$. Hence proved. □

Data Availability

No data were used to support this study.

Compliance With Ethical Standards

Conflict of Interest: Authors declare that they have no conflict of interest.

Acknowledgments

The first and third author would like to thank CSIR-HRDG Fund, under grant EMR-09/386(0064)/2019-EMR-1 and EMR-09/386(0059)/2017-EMR-1 respectively, for financial support.

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