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Article

Lensing Effects in Retarded Gravity

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Abstract: Galaxies are huge physical systems having dimensions of many tens of thousands of light years. Thus any change at the galactic center will be noticed at the rim only tens of thousands of years later. Those retardation effects seems to be neglected in present day galactic modelling used to calculate rotational velocities of matter in the rims of the galaxy and surrounding gas. The significant differences between the predictions of Newtonian instantaneous action at a distance and observed velocities are usually explained by either assuming dark matter or by modifying the laws of gravity (MOND). In this paper we will show that taking general relativity seriously without neglecting retardation effects one can explain the apparent excess matter leading to gravitational lensing in both galaxies and galaxy clusters.

Keywords: spacetime symmetry; relativity of spacetime; retardation; lensing

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1. Introduction

The general theory of relativity (GR) is known to be symmetric under general coordinate modifications (diffeomorphism). This group of general coordinate transformations has a Lorentz subgroup, which is maintained even in the weak field approximation. This is manifested through the field equations containing the d'Alembert (wave) operator, which has a retarded potential solution.

From an observational point of view, it is well known that GR is verified by many observations. However, at the present time, the standard Newton–Einstein gravitational theory stands at something of a crossroads. It simultaneously has much in its favor observationally, while, at the same time, it has some very disquieting challenges. The observational successes that it has achieved in both astrophysics and cosmology have to be tempered by the fact that the theory needs to appeal to two as yet unconfirmed ingredients, dark matter and dark energy, in order to achieve these successes. The dark matter problem has not only been with us since the 1920s and 1930s (when it was initially known as the missing mass problem), but it has also become more and more severe as more and more dark matter has had to be introduced on larger and larger distance scales as new data have come online. In this paper we will be particularly concerned in particular with the excess dard matter needed to justify observed gravitational lensing. Moreover, extensive-now 40-year-underground and accelerator searches have failed to find any of it or establish its existence. The dark matter situation has become even more dire in the last few years as the Large Hadron Collider has failed to find any super symmetric particles, not only of the community's preferred form of dark matter, but also the form of it that is required in string theory, a theory that attempts to provide a quantized version of Newton–Einstein gravity.

While things may still eventually work out in favor of the standard dark matter paradigm, the situation is disturbing enough to warrant consideration of the possibility that the standard paradigm might at least need to be modified in some way if not outright replaced. The present proposal sets out to seek such a modification. Unlike other approaches such as Milgrom's MOND, Mannheim's Conformal Gravity or Moffat's MOG, the present approach is, in a sense, the minimalist one adhering strictly to the basic scientific principle dictated by Occam's razor. It seeks to replace dark matter by effects within standard General Relativity itself.

In 1933, Fritz Zwicky noticed that the velocities of a set of Galaxies within the Comma Cluster are much higher than those predicted by the virial calculation that assumes New-

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tonian theory [1]. He calculated that the amount of matter required to account for the velocities could be 400 times greater with respect to that of visible matter, which led to suggesting dark matter throughout the entire cluster. Volders, in 1959, indicated that stars in the outer rims of the nearby spiral galaxy M33 do not move "correctly" [2]. It is the result of the virial theorem coupled with Newtonian Gravity which implies that $MG/r \sim Mv^2$, that is to say, the expected rotation curve should at some point decrease as $1/\sqrt{r}$. This was well established during the seventies when Rubin and Ford [3,4] demonstrated that, for a large sample of spiral galaxies, this behavior can be considered a general feature: velocities at the our rim of the galaxies do not decrease—rather, in a general case, they attain a plateau at some velocity, which is different for each galaxy. In previous papers we have shown that such effects can be deduced from GR if retardation effects are not neglected. The derivation of the retardation force described in previous publications [5–11], see for example figure 1.

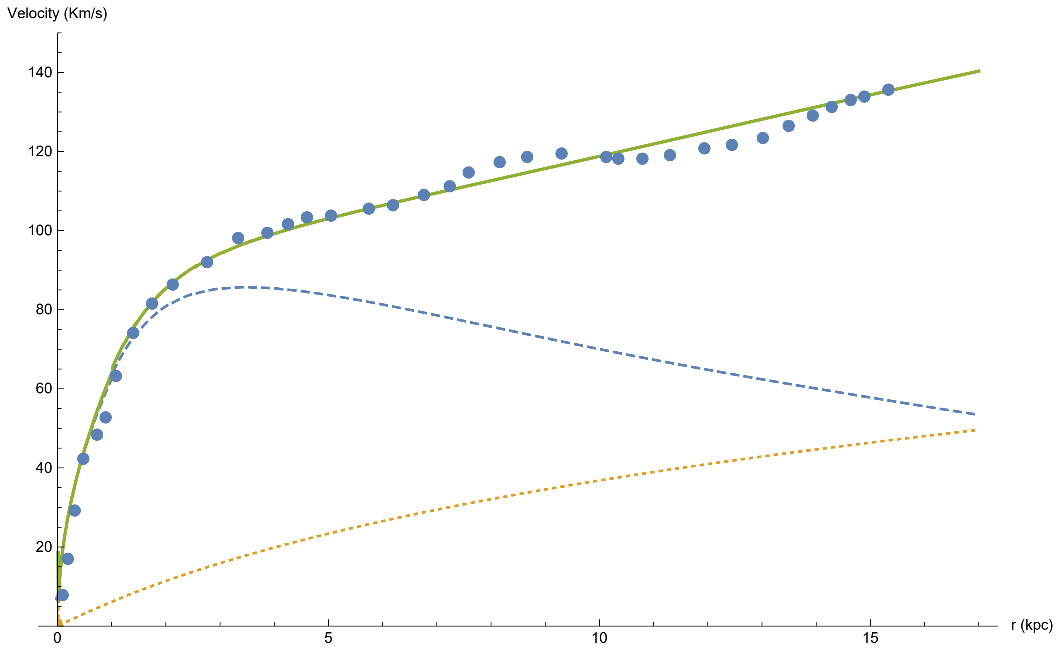

Figure 1. Rotation curve for M33. The observational points were supplied by Dr. Michal Wagman, a former PhD student at Ariel University, under my supervision, using [33]; the full line describes the complete rotation curve, which is the sum of the dotted line, describing the retardation contribution, and the dashed line, which is the Newtonian contribution.

Previous work assumed a test particle moving slowly with respect to the speed of light as is appropriated for the case of galactic rotation curves, this is not adequate when the test particle is a photon moving in the speed of light as in the case relevant to gravitational lensing. Here, a different mathematical approach is needed as described in the current paper.

A gravitational lens is a distribution of matter (such as a cluster of galaxies) between a distant light source and an observer, that is capable of bending the light from the source as the light travels towards the observer.

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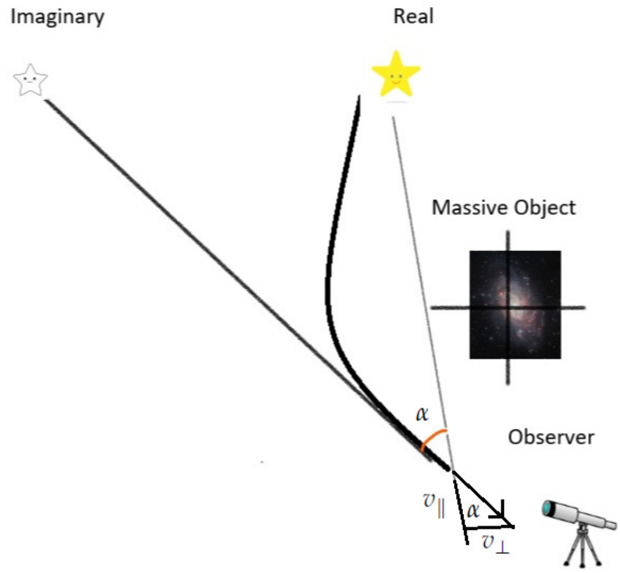

Figure 2. Light travelling toward the observer is bent due to the gravitational field of a massive object, thus a distant star appears to the observer at an angle α with resect to its true location. The tangent of alpha is the velocity of light perpendicular to the original direction $v_{\perp}$ divided by the velocity of light in the original direction $v_{\parallel}$.

This effect is known as gravitational lensing, and the amount of bending is one of the predictions of Albert Einstein's general theory of relativity [12,13]. It should be noted that Newtonian physics also predicts the bending of light, but only half of that predicted by general relativity [14].

Einstein made unpublished calculations on the subject of gravitational lensing as early as 1912 [15]. In 1915 Einstein showed how general relativity explained the anomalous perihelion advance of the planet Mercury without any arbitrary parameters ("fudge factors"),[16] and in 1919 an expedition led by Eddington confirmed general relativity's prediction for the deflection of starlight by the Sun during the total solar eclipse of May 29, 1919,[17,18] instantly making Einstein famous [16]. The reader should recall that there was a special significance to a British scientist confirming the prediction of a German scientist after the bloody battles of world war I between the British and the Germans.

The fact that distortion of space time is proportional to the amount of mass that caused the distortion led several researchers to use gravitational lensing as a tool for proving the existence of dark matter.

Strong lensing is the observed distortion of background galaxies into arcs when their light passes through such a gravitational lens. It has been observed around many distant clusters including Abell 1689 [19]. By measuring the distortion geometry, the mass of the intervening cluster can be obtained. In the dozens of cases where this has been done, the mass-to-light ratios obtained correspond to the dynamical dark matter measurements of clusters [20]. Here we will show that is not a coincidence, and retardation dictates that this should be so. Lensing can lead to multiple copies of an image. By analyzing the distribution of multiple image copies, scientists have been able to deduce and map the distribution of dark matter around the MACS J0416.1-2403 galaxy cluster [21,22].

Weak gravitational lensing investigates minute distortions of galaxies, using statistical analyses from vast galaxy surveys. By examining the apparent shear deformation of the adjacent background galaxies, the mean distribution of dark matter can be characterized. The mass-to-light ratios correspond to dark matter densities predicted by other large-scale structure measurements [23].

We underline that dark matter does not bend light itself; mass (in this case the alleged mass of the dark matter) bends spacetime. Light follows the curvature of spacetime,

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resulting in the lensing effect [24,25]. Here we will show that the above described effects may be attributed to retardation effects and baryonic matter, and no additional form of matter needs to be speculated.

The structure of the paper is as follows: First we describe the main results of general relativity, this is followed by the weak field approximation. Next we derive the geodesic equations for a particle moving at a speed of light ("a photon"). This is followed in a comparison of the current approach to that of Weinberg [28]. The classical result of Einstein and Eddington are re-derived. Finally we discuss the effect of retardation on the lensing phenomena and show in what way can retardation cause a "dark matter" phenomena.

2. General Relativity

The general theory of relativity is based on two fundamental equations, the first being Einstein equations [26–29]:

$$G_{\mu\nu} = -\frac{8\pi G}{c^4} T_{\mu\nu} \tag{1}$$

$G_{\mu\nu}$ stands for the Einstein tensor, $T_{\mu\nu}$ indicates the stress–energy tensor, $G \simeq 6.67\, 10^{-11}$ m^{3}kg^{-1}s^{-2} is the universal gravitational constant and $c \simeq 3\, 10^8$ m s^{-1} indicates the velocity of light in the absence of matter (Greek letters are indices in the range $0-3$). The second fundamental equation that GR is based on is the geodesic equation:

$$\frac{d^2 x^\alpha}{dp^2} + \Gamma^\alpha_{\mu\nu}\frac{dx^\mu}{dp}\frac{dx^\nu}{dp} = \frac{du^\alpha}{dp} + \Gamma^\alpha_{\mu\nu} u^\mu u^\nu = 0 \tag{2}$$

$x^\alpha(p)$ are the coordinates of the particle in spacetime, p is a typical parameter along the trajectory that for massive particles is chosen to be the length of the trajectory ($p = s$), $u^\mu = \frac{dx^\mu}{dp}$ is the μ-th component of the 4-velocity of a massive particle moving along the geodesic trajectory (increment of x per p) and $\Gamma^\alpha_{\mu\nu}$ is the affine connection (Einstein summation convention is assumed). The stress–energy tensor of matter is usually taken in the form:

$$T_{\mu\nu} = (pr + \rho c^2) u_\mu u_\nu - pr\, g_{\mu\nu} \tag{3}$$

In the above, pr is the pressure and ρ is the **mass** density. We remind the reader that lowering and raising indices is done through the metric $g_{\mu\nu}$ and inverse metric $g^{\mu\nu}$, such that $u_\mu = g_{\mu\nu} u^\nu$. The same metric serves to calculate s:

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu, \tag{4}$$

for a photon $ds = 0$. The affine connection is connected to the metric as follows:

$$\Gamma^\alpha_{\mu\nu} = \frac{1}{2} g^{\alpha\beta}(g_{\beta\mu,\nu} + g_{\beta\nu,\mu} - g_{\mu\nu,\beta}), \qquad g_{\beta\mu,\nu} \equiv \frac{\partial g_{\beta\mu}}{\partial x^\nu} \tag{5}$$

Using the affine connection we calculate the Riemann and Ricci tensors and the curvature scalar:

$$R^\mu_{\nu\alpha\beta} = \Gamma^\mu_{\nu\alpha,\beta} - \Gamma^\mu_{\nu\beta,\alpha} + \Gamma^\sigma_{\nu\alpha}\Gamma^\mu_{\sigma\beta} - \Gamma^\sigma_{\nu\beta}\Gamma^\mu_{\sigma\alpha}, \quad R_{\alpha\beta} = R^\mu_{\alpha\beta\mu}, \quad R = g^{\alpha\beta} R_{\alpha\beta} \tag{6}$$

which, in turn, serves to calculate the Einstein tensor:

$$G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R. \tag{7}$$

Hence, the given matter distribution determines the metric through Equation (1) and the metric determines the geodesic trajectories through Equation (2). Those equations are well known to be symmetric under smooth coordinate transformations (diffeomorphism).

$$x'_\alpha = x'_\alpha(x_\mu). \tag{8}$$

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3. Linear Approximation of GR - The Metric

Only in extreme cases of compact objects (black holes and neutron stars) and the primordial reality or the very early universe does one need not consider the solution of the full non-linear Einstein Equation [5]. In typical cases of astronomical interest (the galactic case included) one can use a linear approximation to those equations around the flat Lorentz metric $\eta_{\mu\nu}$ such that (Private communication with the late Professor Donald Lynden-Bell):

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \quad \eta_{\mu\nu} \equiv \text{ diag } (1,-1,-1,-1), \qquad |h_{\mu\nu}| \ll 1 \tag{9}$$

One then defines the quantity:

$$\bar{h}_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h, \quad h = \eta^{\mu\nu}h_{\mu\nu}, \tag{10}$$

$\bar{h}_{\mu\nu} = h_{\mu\nu}$ for non diagonal terms. For diagonal terms:

$$\bar{h} = -h \Rightarrow h_{\mu\nu} = \bar{h}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\bar{h}. \tag{11}$$

The general coordinate transformation symmetry of Equation (8) has a subgroup of infinitesimal transformations which are manifested in the gauge freedom of $h_{\mu\nu}$ in the weak field approximation. It can be shown ([26] page 75, exercise 37, see also [27–29]) that one can choose a gauge such that the Einstein equations are:

$$\Box\bar{h}_{\mu\nu} \equiv \bar{h}_{\mu\nu,\alpha}{}^{\alpha} = -\frac{16\pi G}{c^4}T_{\mu\nu}, \qquad \bar{h}_{\mu\alpha,}{}^{\alpha} = 0. \tag{12}$$

The d'Alembert operator $\Box$ is clearly invariant under the Lorentz symmetry group (another subgroup of the general coordinate transformation symmetry described by Equation (8)), of which the Newtonian Laplace operator $\vec{\nabla}^2$ is not, but this comes with the price that "action at a distance" solutions are forbidden and only retarded solutions are allowed. The $T_{\mu\nu}$ stress energy tensor should be calculated at the appropriate frame and thus for the massive body causing the lensing, matter is approximately at rest.

Equation (12) can always be integrated to take the form [30]:

$$\begin{aligned} \bar{h}_{\mu\nu}(\vec{x},t) &= -\frac{4G}{c^4}\int\frac{T_{\mu\nu}(\vec{x}',t-\frac{R}{c})}{R}d^3x', \\ t &\equiv \frac{x^0}{c}, \quad \vec{x} \equiv x^a \quad a,b \in [1,2,3], \\ \vec{R} &\equiv \vec{x}-\vec{x}', \quad R = |\vec{R}|. \end{aligned} \tag{13}$$

For reasons why the symmetry between space and time is broken, see [31,32]. The factor before the integral is small: $\frac{4G}{c^4} \simeq 3.3 \times 10^{-44}$ in MKS units; hence, in the above calculation one can take $T_{\mu\nu}$, which is zero order in $h_{\alpha\beta}$.

In the zeroth order:

$$u^0 = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}, \quad u^a = \vec{u} = \frac{\frac{\vec{v}}{c}}{\sqrt{1-\frac{v^2}{c^2}}}, \quad v^{\mu} \equiv \frac{dx^{\mu}}{dt}, \quad \vec{v} \equiv \frac{d\vec{x}}{dt}, \quad v = |\vec{v}|, \tag{14}$$

in which we assume that massive body causing the lensing effect is composed of massive particles (those equations will not be correct for the photon which is affected by the lensing potential).

Assuming the reasonable assumption that the said massive body is composed of particles of non relativistic velocities:

$$u^0 \simeq 1, \qquad \vec{u} \simeq \frac{\vec{v}}{c}, \qquad u^a \ll u^0 \qquad \text{for} \quad v \ll c. \tag{15}$$

Let us now look at equation (3). We assume $\rho c^2 \gg pr$ and, taking into account equation (15), we arrive at $T_{00} = \rho c^2$, while other tensor components are significantly smaller. Thus, $\bar{h}_{00}$ is significantly larger than other components of $\bar{h}_{\mu\nu}$ which are ignored from now on. One should notice that it is possible to deduce from the gauge condition in equation (12) the relative order of magnitude of the relative components of $h_{\mu\nu}$:

$$\bar{h}_{\alpha 0,}{}^{0} = -\bar{h}_{\alpha a,}{}^{a} \qquad \Rightarrow \bar{h}_{00,}{}^{0} = -\bar{h}_{0a,}{}^{a}, \quad \bar{h}_{b0,}{}^{0} = -\bar{h}_{ba,}{}^{a}. \tag{16}$$

Thus, the zeroth derivative of $\bar{h}_{00}$ (which contains a $\frac{1}{c}$ as $x^0 = ct$) is the same order as the spatial derivative of $\bar{h}_{0a}$ meaning that $\bar{h}_{0a}$ is of order $\frac{v}{c}$ smaller than $\bar{h}_{00}$. And the zeroth derivative of $\bar{h}_{0a}$ (which appears in Equation (16)) is the same order as the spatial derivative of $\bar{h}_{ab}$. Meaning that $\bar{h}_{ab}$ is of order $\frac{v}{c}$ with respect to $\bar{h}_{0a}$ and of order $(\frac{v}{c})^2$ with respect to $\bar{h}_{00}$.

In the current approximation, the following results hold:

$$\bar{h} = \eta^{\mu\nu}\bar{h}_{\mu\nu} = \bar{h}_{00}. \tag{17}$$

$$h_{00} = \bar{h}_{00} - \frac{1}{2}\eta_{00}\bar{h} = \frac{1}{2}\bar{h}_{00}. \tag{18}$$

$$h_{\underline{aa}} = -\frac{1}{2}\eta_{\underline{aa}}\bar{h} = \frac{1}{2}\bar{h}_{00}. \tag{19}$$

(The underline $\underline{aa}$ signifies that the Einstein summation convention is not assumed).

$$h_{\mu\nu} = \bar{h}_{\mu\nu} = 0, \qquad \mu \neq \nu. \tag{20}$$

We can summarize the above results in a concise formula:

$$h_{\mu\nu} = h_{00}\delta_{\mu\nu}. \tag{21}$$

in which $\delta_{\mu\nu}$ is Kronecker's delta. It will be useful to introduce the gravitational potential ϕ which is defined below and can be calculated using Equation (13):

$$\phi \equiv \frac{c^2}{4}\bar{h}_{00} = -\frac{G}{c^2}\int \frac{T_{00}(\vec{x}', t - \frac{R}{c})}{R} d^3x' = -G\int \frac{\rho(\vec{x}', t - \frac{R}{c})}{R} d^3x' \tag{22}$$

from the above definition and equation (18) it follows that:

$$h_{00} = \frac{2}{c^2}\phi. \tag{23}$$

4. Linear Approximation of GR - The Lensing Trajectory

Let us start calculating the lensing trajectory by writing $u^\mu = \frac{dx^\mu}{dp}$ in terms of the notation introduced in equation (13):

$$u^0 = \frac{dx^0}{dp} = c\frac{dt}{dp}, \qquad u^a = \frac{dx^a}{dp} = \frac{dx^a}{dt}\frac{dt}{dp} = v^a\frac{dt}{dp} \tag{24}$$

it thus follows that:

$$\frac{du^a}{dp} = \frac{dv^a}{dp}\frac{dt}{dp} + v^a\frac{d^2t}{dp^2} \tag{25}$$

Taking into account equation (2) we obtain:

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$$\frac{dv^a}{dp}\frac{dt}{dp} + v^a \frac{d^2t}{dp^2} = -\Gamma^\alpha_{\mu\nu} u^\mu u^\nu \tag{26}$$

multiplying by $\left(\frac{dp}{dt}\right)^2$ and using the notation of equation (14) we obtain:

$$\frac{dv^a}{dt} + v^a \left(\frac{dp}{dt}\right)^2 \frac{d^2t}{dp^2} = -\Gamma^\alpha_{\mu\nu} v^\mu v^\nu. \tag{27}$$

However, according to equation (2):

$$\frac{d^2t}{dp^2} = \frac{1}{c}\frac{d^2x^0}{dp^2} = -\frac{1}{c}\Gamma^0_{\mu\nu} u^\mu u^\nu. \tag{28}$$

Inserting equation (28) into equation (27) we arrive at the form:

$$\frac{dv^a}{dt} = -\Gamma^\alpha_{\mu\nu} v^\mu v^\nu + \Gamma^0_{\mu\nu} v^\mu v^\nu \frac{v^a}{c}. \tag{29}$$

Let us now calculate the affine connection in the linear approximation:

$$\Gamma^\alpha_{\mu\nu} = \frac{1}{2}\eta^{\alpha\beta}(h_{\beta\mu,\nu} + h_{\beta\nu,\mu} - h_{\mu\nu,\beta}). \tag{30}$$

Taking into account equation (30) and equation (21) we obtain:

$$\Gamma^a_{\mu\nu} v^\mu v^\nu = \eta^{a\beta}\left(h_{\beta\mu,\nu} v^\mu v^\nu - \frac{1}{2}h_{\mu\nu,\beta} v^\mu v^\nu\right) = \eta^{a\beta}\left(\delta_{\beta\mu}\partial_\nu h_{00} v^\mu v^\nu - \frac{1}{2}\delta_{\mu\nu}\partial_\beta h_{00} v^\mu v^\nu\right). \tag{31}$$

The affine connection has only first order terms in $h_{\alpha\beta}$; hence, to the first order $\Gamma^\alpha_{\mu\nu} v^\mu v^\nu$ appearing in the geodesic, $v^\mu v^\nu$ is of the zeroth order. By definition $v^0 = \frac{dx^0}{dt} = c$. Also the null interval of the photon is given to zeroth order in h_{00} (that is in the absence of a gravitational field) by:

$$0 = ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu \Rightarrow 0 = \eta_{\mu\nu} v^\mu v^\nu = (v^0)^2 - (\vec{v})^2 \Rightarrow v = c. \tag{32}$$

It follows that in the zeroth order approximation:

$$\delta_{\mu\nu} v^\mu v^\nu = (v^0)^2 + \vec{v}^2 = 2c^2 \tag{33}$$

and also in the same approximation:

$$\eta^{a\beta}\delta_{\beta\mu} v^\mu v^\nu = \eta^{a\mu} v^\mu v^\nu = -v^a v^\nu. \tag{34}$$

We now combine the above results and write:

$$\Gamma^a_{\mu\nu} v^\mu v^\nu = -v^a v^\nu \partial_\nu h_{00} - c^2 \partial^a h_{00}. \tag{35}$$

We now turn our attention to the second term in equation (29)

$$\Gamma^0_{\mu\nu} v^\mu v^\nu \frac{v^a}{c} = \eta^{0\beta}\left(h_{\beta\mu,\nu} v^\mu v^\nu - \frac{1}{2}h_{\mu\nu,\beta} v^\mu v^\nu\right)\frac{v^a}{c} = \left(h_{0\mu,\nu} v^\mu v^\nu - \frac{1}{2}h_{\mu\nu,0} v^\mu v^\nu\right)\frac{v^a}{c} \tag{36}$$

inserting equation (21) into equation (36) we arrive at the result:

$$\Gamma^0_{\mu\nu} v^\mu v^\nu \frac{v^a}{c} = \left(\delta_{0\mu}\partial_\nu h_{00} v^\mu v^\nu - \delta_{\mu\nu}\frac{1}{2}\partial_0 h_{00} v^\mu v^\nu\right)\frac{v^a}{c} = \left(c v^\nu \partial_\nu h_{00} - c^2 \partial_0 h_{00}\right)\frac{v^a}{c} = v^a v^b \partial_b h_{00}. \tag{37}$$

Combining the results from equation (35) and equation (37) into equation (29) we arrive at

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the photon's equation of motion:

$$\frac{dv^a}{dt} = c^2 \partial^a h_{00} + v^a \left[v^\nu \partial_\nu h_{00} + v^b \partial_b h_{00} \right]. \tag{38}$$

In terms of the gravitational potential this can be rewritten using equation (23) as follows:

$$\frac{d\vec{v}}{dt} = -c^2 \vec{\nabla} h_{00} + 2\vec{v}\left(\vec{v} \cdot \vec{\nabla} h_{00}\right) + \vec{v}\partial_t h_{00} \quad \Rightarrow \quad \frac{d\vec{v}}{dt} = -2\vec{\nabla}\phi + 4\frac{\vec{v}}{c}\left(\frac{\vec{v}}{c} \cdot \vec{\nabla}\phi\right) + 2\frac{\vec{v}}{c^2}\partial_t \phi. \tag{39}$$

This equation is almost identical to equation 9.2.6 of Weinberg [28]. Notice, however, that Weinberg neglects the term $2\frac{\vec{v}}{c^2}\partial_t \phi$ in his post Newtonian approximation. Although this term can be neglected or shown to be small in specific circumstances such as case that ϕ is static or slowly varying, its removal may lead to inconsistencies as we explain below.

Let us inspect the null interval equation (4):

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = 0, \tag{40}$$

taking into account equation (9) and equation (21) this takes the form:

$$0 = ds^2 = (\eta_{\mu\nu} + h_{\mu\nu}) dx^\mu dx^\nu = (\eta_{\mu\nu} + h_{00}\delta_{\mu\nu}) dx^\mu dx^\nu, \tag{41}$$

Hence:

$$0 = (\eta_{\mu\nu} + h_{00}\delta_{\mu\nu}) v^\mu v^\nu \Rightarrow (1 + h_{00})c^2 - (1 - h_{00})\vec{v}^2 = 0 \Rightarrow v^2 = c^2 \frac{(1 + h_{00})}{(1 - h_{00})} \tag{42}$$

Or:

$$v = c\sqrt{\frac{(1 + h_{00})}{(1 - h_{00})}} = c\left((1 + h_{00} + O(h_{00}^2)\right) \tag{43}$$

compare to Weinberg [28] equation 9.2.5. Since by equation (22) and equation (23) h_{00} is a small negative number, it follows that $v < c$ (notice that this result holds for a global coordinate system, in the local flat coordinate system the velocity will of course be exactly c). Now:

$$v\frac{dv}{dt} = \vec{v} \cdot \frac{d\vec{v}}{dt} \tag{44}$$

To first order in h_{00}:

$$v\frac{dv}{dt} = c^2 \frac{dh_{00}}{dt}. \tag{45}$$

Taking into account equation (39):

$$\vec{v} \cdot \frac{d\vec{v}}{dt} = \vec{v} \cdot \left[-c^2 \vec{\nabla} h_{00} + 2\vec{v}\left(\vec{v} \cdot \vec{\nabla} h_{00}\right) + \vec{v}\partial_t h_{00} \right] = -c^2 \vec{v} \cdot \vec{\nabla} h_{00} + 2v^2 \left(\vec{v} \cdot \vec{\nabla} h_{00}\right) + v^2 \partial_t h_{00} \tag{46}$$

Thus to the same first order of h_{00} we have:

$$\vec{v} \cdot \frac{d\vec{v}}{dt} = c^2 \left(\vec{v} \cdot \vec{\nabla} h_{00}\right) + c^2 \partial_t h_{00} = c^2 \frac{dh_{00}}{dt} \tag{47}$$

which is dependent on not neglecting the $\partial_t h_{00}$ term. Thus this term is absolutely necessary to maintain the identity of equation (44) and without it we are led to a contradiction. We conclude that for a time dependent gravitational potential the $\partial_t h_{00}$ term is required.

Let us consider a photon travelling at a straight line in the direction $\hat{v}_0$ in the absence of gravity, the velocity of this photon would be $\vec{v}_0 = c\hat{v}_0$. Now let us assume that the photon passes near a weak gravitational source such that equation (39) is valid. We thus

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decompose the photon velocity field to two components one which parallel and one which is perpendicular to its original direction:

$$\vec{v}_{\parallel} = (\vec{v} \cdot \hat{v}_0)\hat{v}_0, \quad \vec{v}_{\perp} = \vec{v} - \vec{v}_{\parallel} = \vec{v} - (\vec{v} \cdot \hat{v}_0)\hat{v}_0, \quad \vec{v} = \vec{v}_{\perp} + \vec{v}_{\parallel} \tag{48}$$

Thus it follows from equation (39) that:

$$\frac{dv_{\parallel}}{dt} = \frac{d(\hat{v}_0 \cdot \vec{v})}{dt} = -c^2 \hat{v}_0 \cdot \vec{\nabla} h_{00} + 2\hat{v}_0 \cdot \vec{v}\left(\vec{v} \cdot \vec{\nabla} h_{00}\right) + \hat{v}_0 \cdot \vec{v}\partial_t h_{00} \tag{49}$$

Thus to first order in h_{00}:

$$\frac{dv_{\parallel}}{dt} = -c\vec{v} \cdot \vec{\nabla} h_{00} + 2c\left(\vec{v} \cdot \vec{\nabla} h_{00}\right) + c\partial_t h_{00} = c\left(\partial_t h_{00} + \vec{v} \cdot \vec{\nabla} h_{00}\right) = c\frac{dh_{00}}{dt} \tag{50}$$

in which we remember that $\vec{v} = \vec{v}_0 + O(h_{00}) = c\hat{v}_0 + O(h_{00})$. This can be integrated as follows:

$$v_{\parallel} = ch_{00} + k \tag{51}$$

in which k is a constant. Far away from the gravitational source $\lim_{|\vec{x}|->\infty} v_{\parallel} = c, \quad \lim_{|\vec{x}|->\infty} h_{00} = 0$, hence $k = c$, and we may write:

$$v_{\parallel} = c(h_{00} + 1). \tag{52}$$

Now:

$$v_{\parallel}^2 = c^2(1 + 2h_{00}) + O(h_{00}^2). \tag{53}$$

And according to equation (43):

$$v^2 = c^2(1 + 2h_{00}) + O(h_{00}^2). \tag{54}$$

It now follows from equation (48):

$$v_{\perp}^2 = v^2 - v_{\parallel}^2 = O(h_{00}^2) \quad \Rightarrow \quad v_{\perp} = O(h_{00}). \tag{55}$$

The lensing angle is defined (see figure 2)

$$\tan\alpha = \frac{v_{\perp}}{v_{\parallel}} = \frac{v_{\perp}}{c} + O(h_{00}^2) \Rightarrow \alpha \simeq \frac{v_{\perp}}{c} \tag{56}$$

as the angle α is small, thus for lensing in the linear approximation only the perpendicular component $v_{\perp}$ is important. Now to the first order in h_{00} we may write equation (39) as:

$$\frac{d\vec{v}}{dt} = -c^2\left(\vec{\nabla} h_{00} - \hat{v}_0(\hat{v}_0 \cdot \vec{\nabla} h_{00})\right) + \vec{v}_0\left(\vec{v} \cdot \vec{\nabla} h_{00} + \partial_t h_{00}\right) = -c^2 \vec{\nabla}_{\perp} h_{00} + \vec{v}_0 \frac{dh_{00}}{dt} \tag{57}$$

in which we define the perpendicular gradient as: $\vec{\nabla}_{\perp} \equiv \vec{\nabla} - \hat{v}_0(\hat{v}_0 \cdot \vec{\nabla})$. Now taking into account equation (48) and equation (50) we arrive at the result:

$$\frac{d\vec{v}_{\perp}}{dt} = \frac{d(\vec{v} - \vec{v}_{\parallel})}{dt} = -c^2 \vec{\nabla}_{\perp} h_{00} = -2\vec{\nabla}_{\perp}\phi \tag{58}$$

in which we have used equation (23) to give the perpendicular acceleration in terms of a gravitational potential.

5. Other approaches to the problem of lensing

Another approach to the problem of lensing than the one given above is to start from a Schwarzschild metric as is done by Weinberg [28]. This metric describe a static spherically symmetric mass distribution and thus is less general than the approach taken

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in this paper. It does have one advantage, however, and this is the ability to take into account strong gravitational fields and not just weak ones. This advantage is irrelevant in most astronomical cases in which gravity is weak and must be only considered for light trajectories near compact objects (black holes & neutron stars). The Schwarzschild squared interval can be written as:

$$ds^2_{Schwarzschild} = \left(1 - \frac{r_s}{r'}\right) c^2 dt^2 - \left(1 - \frac{r_s}{r'}\right)^{-1} dr'^2 - r'^2 \left(d\theta^2 + \sin^2\theta\, d\phi^2\right) \quad (59)$$

In which r', θ, ϕ are spherical coordinates and the point massive body is located at $r' = 0$. The Schwarzschild radius is defined as:

$$r_s = \frac{2GM}{c^2} \quad (60)$$

in which M is the mass of the point particle. Comparing the g_{00} component of equation (59) and equation (41) it follows that we can identify:

$$h_{00} = -\frac{r_s}{r'} \quad (61)$$

provided $\frac{r_s}{r'} \ll 1$ in accordance with equation (9), hence the two approaches should coincide for:

$$r' \gg r_s \quad (62)$$

Going back to equation (41) we have:

$$ds^2 = (\eta_{\mu\nu} + h_{\mu\nu}) dx^\mu dx^\nu = (\eta_{\mu\nu} + h_{00}\delta_{\mu\nu}) dx^\mu dx^\nu = (1 + h_{00}) c^2 dt^2 - (1 - h_{00}) d\vec{x}^2, \quad (63)$$

it tempting to write $d\vec{x}^2 = dr^2 + r^2 (d\theta^2 + \sin^2\theta d\phi^2)$ as is usually done for spherical coordinates. But notice that r is not strictly a radial coordinate which is defined as the circumference, divided by 2π, of a sphere centered around the massive body. In fact from equation (63) it is clear that the appropriate radial coordinate is:

$$r' = r\sqrt{1 - h_{00}} \quad (64)$$

which is a small correction to r. Now calculating the differential dr' it follows that:

$$dr' = dr\sqrt{1 - h_{00}} + r d\sqrt{1 - h_{00}} = dr \frac{1 - h_{00} - \frac{1}{2} r \frac{dh_{00}}{dr}}{\sqrt{1 - h_{00}}}. \quad (65)$$

According to equation (61):

$$h_{00} = -\frac{r_s}{r'} = -\frac{r_s}{r\sqrt{1 - h_{00}}} \Rightarrow h_{00}\sqrt{1 - h_{00}} = -\frac{r_s}{r} \Rightarrow h_{00} = -\frac{r_s}{r} \quad (66)$$

where the last equality is correct to first order. Alternatively one can use equation (22) to calculate the gravitational potential for a static point mass to obtain:

$$\phi = -G\frac{M}{r} \quad (67)$$

and then plug this into equation (23) to obtain again:

$$h_{00} = \frac{2}{c^2}\phi = -\frac{2GM}{c^2 r} = -\frac{r_s}{r}. \quad (68)$$

It now follows that:

$$\frac{dh_{00}}{dr} = \frac{r_s}{r^2} = -\frac{h_{00}}{r} \quad (69)$$

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Plugging equation (69) into equation (65) leads to:

$$dr' = dr\frac{1 - h_{00} - \frac{1}{2}r\frac{dh_{00}}{dr}}{\sqrt{1 - h_{00}}} = dr\frac{1 - \frac{1}{2}h_{00}}{\sqrt{1 - h_{00}}}. \quad (70)$$

Hence to first order in h_{00}:

$$dr' = dr. \quad (71)$$

Using the results equation (64) and equation (71) the interval given in equation (63) can be rewritten as:

$$ds^2 = (1 + h_{00})c^2dt^2 - (1 - h_{00})d\vec{x}^2 = (1 + h_{00})c^2dt^2 - (1 - h_{00})dr'^2 - r'^2\left(d\theta^2 + \sin^2\theta d\phi^2\right). \quad (72)$$

As to first order in h_{00}:

$$1 - h_{00} = \frac{1}{1 + h_{00}}, \quad (73)$$

and taking into account equation (61) we obtain:

$$ds^2 = (1 - \frac{r_s}{r'})c^2dt^2 - (1 - \frac{r_s}{r'})^{-1}dr'^2 - r'^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) = ds^2_{Schwarzschild}. \quad (74)$$

Thus to first order our metric is identical to Schwarzschild's for the case of a static point particle. This makes our analysis superior as it addresses the case of a general density distribution and does not ignore the possibility of time dependence which is crucial for retardation effects to take place.

6. Lensing in the static case

Newtonian theory dictates that a body with any mass (since inertial mass and gravitational mass are equal) moving under the influence of gravity alone must follow a trajectory which is dictated by the equation:

$$\frac{d\vec{v}}{dt} = -\vec{\nabla}\phi_N. \quad (75)$$

The Newtonian potential ϕ_N resembles ϕ given by equation (22) but neglects the retardation effect such that:

$$\phi_N = -G\int\frac{\rho(\vec{x}', t)}{R}d^3x' \quad (76)$$

In [9] we have shown that the geodesic equations reduce for slow moving test particles to:

$$\frac{d\vec{v}}{dt} = -\vec{\nabla}\phi. \quad (77)$$

this will coincide with equation (75) for a static density distribution or for a slowly changing mass distribution as is well known. For light rays we obtained equation (58), this also resembles equation (75) but carries some major differences even for a static mass distribution. First there is a factor 2 multiplying the potential which is missing in equation (75). Second this equation only describe motion perpendicular to the original direction of the light ray and not the propagation in the direction of the light ray itself.

Suppose the gravitational field is orthogonal to the light ray at least when the light ray is moving in proximity to the gravitating body, that is when the gravitational force is most significant. In this case we can write approximately:

$$\frac{d\vec{v}_\perp}{dt} \simeq -2\vec{\nabla}\phi. \quad (78)$$

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It is tempting to make a further step and write:

$$\frac{d\vec{v}}{dt} \simeq -2\vec{\nabla}\phi \tag{79}$$

but the parallel component of $\vec{v}$ satisfies equation (52) and thus equation (79) is not correct even to the first order in h_{00} at it simply states that this component is fixed (remember we assume the force to be orthogonal to the light ray). Notice, however, that according to equation (56) the lensing phenomena is not affected by the parallel component, hence, no harm is done if we calculate this component to zero order instead of first order. It follows that for a static mass distribution it suffices for the purpose of deducing the lensing effect to solve the equation:

$$\frac{d\vec{v}}{dt} \simeq -2\vec{\nabla}\phi_N \tag{80}$$

which is just the Newtonian trajectory but with double the gravitational force. Now, the trajectory of gravitating test particles in a Newtonian point mass gravitational field is known and is given by the equation (Goldstein [34] equation 3.55) :

$$\frac{1}{r} = \frac{k}{l^2}[1 + e\cos(\theta - \theta')] \tag{81}$$

in which r, θ are standard cylindrical coordinates. A static point gravitating body is assumed located at the origin of axis and generating a gravitational potential of the form:

$$\phi = -\frac{k}{r}. \tag{82}$$

The quantity k according to Newtonian theory is $k_N = GM$ which follows from equation (75) and equation (67). This would also be the value according to General Relativity for a slowly moving test particle (see equation (77)). However, for a light ray we should use equation (80) instead and thus $k_E = 2k_N = 2GM$.

The quantity l is a constant of motion (which can be thought as an angular momentum in the direction perpendicular to the plane of motion per unit mass) and is given by:

$$l = r^2\dot{\theta}. \tag{83}$$

The eccentricity e is given by the equation:

$$e = \sqrt{1 + \frac{2El^2}{k^2}}. \tag{84}$$

which is dependent on another constant of motion E (which can be thought as the energy of the test particle per unit mass):

$$E = \frac{1}{2}v^2 + \phi. \tag{85}$$

A light ray coming from far away will have $v = c$ at infinity while $\lim_{r->\infty}\phi = 0$ hence we may write:

$$E = \frac{1}{2}c^2. \tag{86}$$

Thus:

$$e = \sqrt{1 + \frac{c^2l^2}{k^2}}. \tag{87}$$

Consider a star is located far away from the sun at a distance r_∞. Let the line connecting this star and the sun which is located at $r = 0$ coincide with the x-axis. Assume that the earth is also located along the x-axis and has an x coordinate of $x_{earth} = -r_{earth}$ as depicted in figure 3.

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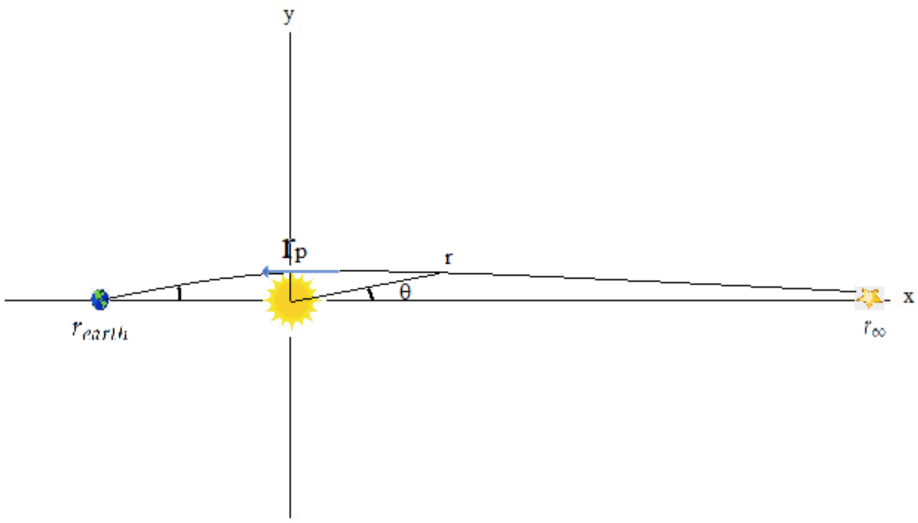

Figure 3. The trajectory of light from a distant star to the earth as affected by the sun's gravity

We consider a light ray orbit in which a star emits a photon which is detected on earth. At the closest point of the trajectory to the sun the photons travels at a direction which is purely in the $\hat{\theta}$ direction and has no radial component. At this point the photon is a distance r_p from the origin of axis and thus:

$$\vec{v} = r_p \dot{\theta}_p \hat{\theta} \Rightarrow r_p \dot{\theta}_p = v \simeq c. \tag{88}$$

This is inserted into equation (83) to yield:

$$l = r_p^2 \dot{\theta}_p \simeq c r_p. \tag{89}$$

Hence:

$$e = \sqrt{1 + \frac{c^4 r_p^2}{k^2}}. \tag{90}$$

Now according to general relativity $k = k_E = 2GM$:

$$e_E = \sqrt{1 + \frac{c^4 r_p^2}{4G^2 M^2}} = \sqrt{1 + \frac{r_p^2}{r_s^2}} \simeq \frac{r_p}{r_s} \gg 1. \tag{91}$$

In which r_s is the Schwarzschild radius defined in equation (60) and we assume the weak field approximation as described in equation (62), for the Newtonian theory we have $k = k_N = GM$ for which we obtain:

$$e_N = \sqrt{1 + \frac{c^4 r_p^2}{G^2 M^2}} = \sqrt{1 + 4\frac{r_p^2}{r_s^2}} \simeq 2\frac{r_p}{r_s} \gg 1. \tag{92}$$

In either relativistic or Newtonian theory the trajectories shape is a rather extreme hyperbola (see Goldstein [34] p. 94). Now the starting and ending points of the trajectory are known: The photon trajectory starts at the star with the cylindrical coordinates $(r, \theta) = (r_\infty, 0)$ and ends on earth with the cylindrical coordinates $(r, \theta) = (r_{earth}, \pi)$. Thus according to equation (81):

$$0 = \frac{1}{r_\infty} = \frac{k}{l^2}[1 + e\cos(\theta')] \Rightarrow e\cos(\theta') = -1 \Rightarrow \cos(\theta') = -\frac{1}{e} \tag{93}$$

$$\frac{1}{r_{earth}} = \frac{k}{l^2}[1 + e\cos(\pi - \theta')] \Rightarrow \frac{1}{r_{earth}} = \frac{k}{l^2}[1 - e\cos(\theta')] = \frac{2k}{l^2}. \tag{94}$$

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It follows that we may also estimate l by:

$$l = \sqrt{2kr_{earth}}. \tag{95}$$

We may insert the above results into equation (81) and write

$$\frac{1}{r} = \frac{1}{2r_{earth}}[1 + e\cos(\theta - \theta')] \Rightarrow r = \frac{2r_{earth}}{1 + e\cos(\theta - \theta')}. \tag{96}$$

We are now at a position to calculate the lensing angle given by equation (56):

$$\tan\alpha = \frac{v_\perp}{v_\parallel} = \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} \tag{97}$$

Now:

$$x(\theta) = r(\theta)\cos\theta, \quad y(\theta) = r(\theta)\sin\theta. \tag{98}$$

Calculating the derivative of the above quantities we have:

$$\frac{dx}{d\theta} = \frac{dr}{d\theta}\cos\theta - r\sin\theta, \qquad \frac{dy}{d\theta} = \frac{dr}{d\theta}\sin\theta + r\cos\theta. \tag{99}$$

The quantity $\frac{dr}{d\theta}$ is deduced from equation (96)

$$\frac{dr}{d\theta} = \frac{2r_{earth}e\sin(\theta - \theta')}{[1 + e\cos(\theta - \theta')]^2}. \tag{100}$$

Inserting equation (100) into equation (99) and using some basic trigonometric identities and equation (93) leads to the following results:

$$\frac{dx}{d\theta} = \frac{-2r_{earth}}{[1 + e\cos(\theta - \theta')]^2}[e\sin\theta' + \sin\theta], \qquad \frac{dy}{d\theta} = \frac{2r_{earth}}{[1 + e\cos(\theta - \theta')]^2}[\cos\theta - 1]. \tag{101}$$

It is now easy to insert the expression of equation (101) into equation (97) and obtain a simple expression:

$$\tan\alpha = \frac{1 - \cos\theta}{e\sin\theta' + \sin\theta}. \tag{102}$$

Thus, at the star $\theta = 0$ and thus the light ray satisfies $\tan\alpha = 0$, with a plausible physical solution of $\alpha = \pi$. This means that initially the light ray is propagating parallel to the x-axis in the direction of earth. On the other hand as the light ray approaches earth, we have $\theta = \pi$ leading to:

$$\tan\alpha = \frac{2}{e\sin\theta'} = \pm\frac{2}{e\sqrt{1 - \frac{1}{e^2}}} \simeq \pm\frac{2}{e}, \tag{103}$$

In which we used the fact that $e \gg 1$ as follows from equation (91) and equation (92). Choosing the physical plausible positive sign and taking into account that the lensing angle is small we have:

$$\alpha \simeq \frac{2}{e} \Rightarrow \alpha_E \simeq \frac{2}{e_E} = \frac{2r_s}{r_p} = 2\alpha_N. \tag{104}$$

The Schwarzschild radius of the sun can be calculated from equation (60) from the sun's mass which is $M_{sun} \simeq 1.99\ 10^{30}$ kg leading to $r_s \simeq 2950$ m. r_p was taken in the classical observation of Eddington [27] to be the Sun's radius, thus $r_p = 6.96\ 10^8$ m, it now follows that:

$$\alpha_E \simeq 8.47\ 10^{-6} \text{ radians} = 1.75 \text{ arcseconds}, \tag{105}$$

$$\alpha_N \simeq 4.27\ 10^{-6} \text{ radians} = 0.87 \text{ arcseconds}. \tag{106}$$

Eddington was able to show during a sun eclipse that the observed lensing angle is closer to α_E than to α_N thus adding an important empirical argument if favour of general relativity in addition to the theoretical arguments presented by Einstein.

To conclude this section we notice that the maximal h_{00} in the solar system is according to equation (61):

$$|h_{00\ max}| = \frac{r_s}{r_p} = \frac{1}{2}\alpha_E = \frac{1}{e_E} \simeq 4.27\ 10^{-6}. \tag{107}$$

Thus justifying the linear approximation of equation (9) easily at least for the Solar system.

7. A suggested experiment

The lensing angle measured by Eddington is not the only possible lensing angle in the Solar system. To see that other angles are possible consider the case depicted in figure 4.

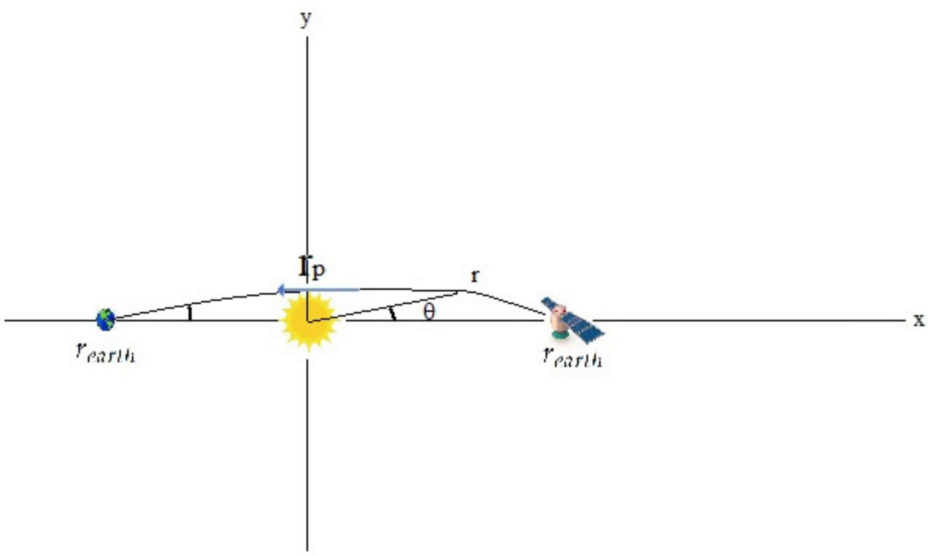

Figure 4. The trajectory of light from an antipodal satellite to the earth as affected by the sun's gravity

Here, the light ray is emitted from a satellite on the same trajectory as the earth but in the other side of the sun. Consider a line connecting the earth and the satellite, and assume that this line coincides with the x-axis. The sun is located at the origin of axis and both the satellite and the earth are in equal distance from the sun. Now the starting and ending points of the trajectory are known also in this case: The photon trajectory starts at the satellite with the cylindrical coordinates $(r, \theta) = (r_{earth}, 0)$ and ends on earth with the cylindrical coordinates $(r, \theta) = (r_{earth}, \pi)$. Thus according to equation (81):

$$\frac{1}{r_{earth}} = \frac{k}{l^2}[1 + e\cos(\theta')] \tag{108}$$

$$\frac{1}{r_{earth}} = \frac{k}{l^2}[1 + e\cos(\pi - \theta')] \Rightarrow \frac{1}{r_{earth}} = \frac{k}{l^2}[1 - e\cos(\theta')]. \tag{109}$$

The difference of the above two equations leads to:

$$2e\cos(\theta') = 0 \Rightarrow \theta' = \frac{\pi}{2}. \tag{110}$$

And their sum leads to:

$$\frac{2}{r_{earth}} = \frac{2k}{l^2} \Rightarrow \frac{1}{r_{earth}} = \frac{k}{l^2}. \tag{111}$$

Hence l can be also evaluated as:

$$l = \sqrt{kr_{earth}}. \tag{112}$$

We may insert the above results into equation (81) and write:

$$\frac{1}{r} = \frac{1}{r_{earth}}[1 + e\sin\theta] \Rightarrow r = \frac{r_{earth}}{1 + e\sin\theta}. \tag{113}$$

The quantity $\frac{dr}{d\theta}$ is deduced from equation (113)

$$\frac{dr}{d\theta} = -\frac{r_{earth} e\cos\theta}{[1 + e\sin\theta]^2} = -\frac{re\cos\theta}{1 + e\sin\theta}. \tag{114}$$

Inserting equation (113) and equation (114) into equation (99) will lead after some trigonometry to:

$$\frac{dx}{d\theta} = \frac{-r}{1 + e\sin\theta}[e + \sin\theta], \qquad \frac{dy}{d\theta} = \frac{r\cos\theta}{1 + e\sin\theta}. \tag{115}$$

It is now easy to insert the expression of equation (115) into equation (97) and obtain a simple expression:

$$\tan\alpha = \frac{-\cos\theta}{e + \sin\theta}. \tag{116}$$

The ray should be launched from $(r, \theta) = (r_{earth}, 0)$ at an angle:

$$\tan\alpha = \frac{-1}{e}. \tag{117}$$

that is at according to equation (91) a small angle to the negative x direction of $\frac{1}{e}$. And will arrive at earth at the angle:

$$\tan\alpha = \frac{1}{e}. \tag{118}$$

that is at a small angle $\frac{1}{e}$ below the negative x axis, causing the observer to see a deviation of the location of the satellite of the same angular magnitude, which is:

$$\alpha_E \simeq \frac{1}{e_E} = 4.27\ 10^{-6} \text{ radians} = 0.87 \text{ arcseconds}, \tag{119}$$

$$\alpha_N \simeq \frac{1}{e_N} = 2.13\ 10^{-6} \text{ radians} = 0.43 \text{ arcseconds}. \tag{120}$$

The relativistic α_E is expected rather than the Newtonian α_N. Notice that in this case the angular deviation is half of the one expected for a distant star. Of course one can use a laser source in a short wavelength that is not common in the suns natural radiation and modulate the laser such that a matched filter can be easily constructed in the receiver side to minimize noise interference.

8. Beyond the Newtonian Approximation

So far we have considered only the case of Newtonian potential which neglects retardation phenomena. This approach seem to suffice for the solar system in which despite the sun's slow change in mass through the solar wind, retardation effect seem to be negligible.

However, in other gravitating systems such as galaxies (not to mention galaxy clusters) there mere size and the nature of their mass exchange with the environment leads to a situation in which retardation effects cannot be neglected. Indeed it was shown [9] that the peculiar shape of galactic rotation curves can be explained by the retardation phenomena. This raises the question on the effect of the same on gravitation lensing. Let us reiterate our main results. The velocity of a photon in a gravitational field is given to the first order of h_{00} according to equation (52), equation (58), and equation (22):

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$$v_{\|} = c(h_{00}+1), \quad \frac{d\vec{v}_{\perp}}{dt} = -c^2\vec{\nabla}_{\perp}h_{00} = -2\vec{\nabla}_{\perp}\phi, \quad \phi = \frac{c^2}{2}h_{00} = -G\int\frac{\rho(\vec{x}', t-\frac{R}{c})}{R}d^3x' \tag{121}$$

The duration $\frac{R}{c}$ for galaxies may be a few tens of thousands of years, but can be considered short in comparison to the time taken for the galactic density to change significantly. Thus, we can write a Taylor series for the density:

$$\rho(\vec{x}', t-\frac{R}{c}) = \sum_{n=0}^{\infty}\frac{1}{n!}\rho^{(n)}(\vec{x}', t)(-\frac{R}{c})^n, \qquad \rho^{(n)} \equiv \frac{\partial^n\rho}{\partial t^n}. \tag{122}$$

By inserting Equations (122) into Equation (121) and keeping the first three terms, we will obtain:

$$\phi = -G\int\frac{\rho(\vec{x}', t)}{R}d^3x' + \frac{G}{c}\int\rho^{(1)}(\vec{x}', t)d^3x' - \frac{G}{2c^2}\int R\rho^{(2)}(\vec{x}', t)d^3x' \tag{123}$$

The Newtonian potential is the first term, the second term contributes only to $v_{\|}$ but no to $\vec{v}_{\perp}$, and the third term is the lower order correction to the Newtonian potential affecting $\vec{v}_{\perp}$ in equation (121) :

$$\phi_r = -\frac{G}{2c^2}\int R\rho^{(2)}(\vec{x}', t)d^3x' \tag{124}$$

We recall that according to equation (56) it is only $\vec{v}_{\perp}$ that affects the lensing angle, and thus despite the fact that second term $\frac{G}{c}\int\rho^{(1)}(\vec{x}', t)d^3x'$ has a physical measurable effect on $\vec{v}_{\|}$ it does not have any effect on the lensing phenomena to the first order in h_{00}.

We underline again that we are **not** considering a post-Newtonian approximation in this paper (see section 3), in which matter travels at nearly relativistic speeds, but we will be considering the retardation effects and finite propagation speed of the gravitational field. We emphasize that taking $\frac{v}{c} \simeq 10^{-3} << 1$ (as for galaxies $v \simeq 10^5$ m/s see figure 1 while $c \simeq 3\ 10^8$ m/s) is not the same as taking $\frac{R}{c} << t_r$ (with R being the typical size of a galaxy say about: $R \simeq 3\ 10^{20}$ m $\Rightarrow \frac{R}{c} \simeq 10^{12}$ s) and t_r is the typical time the mass of the same galaxy changes (to be discussed in section 9). Equation (58) can be rewritten in terms of a perpendicular "force" per unit mass such that:

$$\frac{d\vec{v}_{\perp}}{dt} = \vec{F}_{\perp}, \qquad \vec{F}_{\perp} \equiv -2\vec{\nabla}_{\perp}\phi \tag{125}$$

The total perpendicular force per unit mass is:

$$\begin{aligned}
\vec{F}_{\perp} &= \vec{F}_{N\perp} + \vec{F}_{r\perp} \\
\vec{F}_{N\perp} &= -2\vec{\nabla}_{\perp}\phi_N = -2G\int\frac{\rho(\vec{x}', t)}{R^2}\hat{R}_{\perp}d^3x', \qquad \hat{R} \equiv \frac{\vec{R}}{R}, \quad \hat{R}_{\perp} \equiv \hat{R} - \hat{v}_0(\hat{v}_0\cdot\hat{R}) \\
\vec{F}_{r\perp} &\equiv -2\vec{\nabla}_{\perp}\phi_r = \frac{G}{c^2}\int\rho^{(2)}(\vec{x}', t)\hat{R}_{\perp}d^3x'
\end{aligned} \tag{126}$$

Now if the lensing trajectory is far from the gravitating mass such that $r \equiv |\vec{x}| \gg r' \equiv |\vec{x}'|$ it follows that $R \simeq r$ and $\hat{R} \simeq \hat{r} \equiv \frac{\vec{x}}{|\vec{x}|}$. The force can now be written as:

$$\begin{aligned}
\vec{F}_{\perp} &= \vec{F}_{N\perp} + \vec{F}_{r\perp} \\
\vec{F}_{N\perp} &\simeq -2G\int\frac{\rho(\vec{x}', t)}{r^2}\hat{r}_{\perp}d^3x' = -\frac{2GM}{r^2}\hat{r}_{\perp}, \qquad \hat{r}_{\perp} \equiv \hat{r} - \hat{v}_0(\hat{v}_0\cdot\hat{r}) \\
\vec{F}_{r\perp} &\simeq \frac{G}{c^2}\int\rho^{(2)}(\vec{x}', t)\hat{r}_{\perp}d^3x' = \frac{G}{c^2}\ddot{M}\hat{r}_{\perp}.
\end{aligned} \tag{127}$$

In the above $M = \int\rho d^3x'$ is the total mass of the gravitating body, which may be a galaxy or a cluster of galaxies, and $\ddot{M}$ is the second derivative of the same. Now although $\ddot{M}$

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may be affected by many astrophysical processes we suggest that main contribution is the depletion of gas outsider the gravitating body (see section for a detailed discussion in section 9), hence $\ddot{M} = -|\ddot{M}| < 0$. It follows that according to equation (127):

$$\vec{F}_{\perp} \simeq -[\frac{2GM}{r^2} - \frac{G}{c^2}\ddot{M}]\hat{r}_{\perp} = -\frac{2G(M + M_d(r))}{r^2}\hat{r}_{\perp} \tag{128}$$

where the "dark matter" mass is defined as:

$$M_d(r) \equiv \frac{r^2|\ddot{M}|}{2c^2} \tag{129}$$

we observe that the "dark matter" mass associated with lensing is the same as the "dark matter" mass associated with galactic rotation curves (see equation (106) of [9]), thus explaining the observational results of [20].

9. A Dynamical Model

As mass is accumulated in the galaxy or galaxy cluster, it must be depleted in the surrounding medium. This is due to the fact that the total mass is conserved; still, it is of interest to see if this intuition is compatible with a model of gas dynamics. For simplicity, we assume that the gas is a barotropic ideal fluid and its dynamics are described by the Euler and continuity equations as follows:

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{v}) = 0 \tag{130}$$

$$\frac{d\vec{v}}{dt} \equiv \frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \vec{\nabla})\vec{v} = -\frac{\vec{\nabla} p(\rho)}{\rho} - \vec{\nabla}\phi \tag{131}$$

where the pressure $p(\rho)$ is assumed to be a given function of the density, $\frac{\partial}{\partial t}$ is a partial temporal derivative, $\vec{\nabla}$ has its standard meaning in vector analysis and $\frac{d}{dt}$ is the material temporal derivative. We have neglected viscosity terms due to the low gas density.

9.1. General considerations

Let us now take a partial temporal derivative of equation (130) leading to:

$$\frac{\partial^2 \rho}{\partial t^2} + \vec{\nabla} \cdot (\frac{\partial \rho}{\partial t}\vec{v} + \rho \frac{\partial \vec{v}}{\partial t}) = 0. \tag{132}$$

Using equation (130) again we obtain the expression:

$$\frac{\partial^2 \rho}{\partial t^2} = \vec{\nabla} \cdot \left(\vec{\nabla} \cdot (\rho \vec{v})\vec{v} - \rho \frac{\partial \vec{v}}{\partial t}\right). \tag{133}$$

We divide the left and right hand sides of the equation by c^2 as in equation (126) and obtain:

$$\frac{1}{c^2}\frac{\partial^2 \rho}{\partial t^2} = \vec{\nabla} \cdot \left(\frac{\vec{v}}{c}\left(\frac{\vec{v}}{c} \cdot \vec{\nabla}\rho + \rho \vec{\nabla} \cdot (\frac{\vec{v}}{c})\right) - \rho \frac{1}{c}\frac{\partial \frac{\vec{v}}{c}}{\partial t}\right). \tag{134}$$

Since $\frac{\vec{v}}{c}$ is rather small in galaxies and galaxy clusters it follows that $\frac{1}{c^2}\frac{\partial^2 \rho}{\partial t^2}$ is also small unless the density or the velocity have significant spatial derivatives. A significant acceleration $\frac{\partial \frac{\vec{v}}{c}}{\partial t}$ resulting from a considerable force can also have a decisive effect. The depletion of available gas can indeed cause such gradients as we describe below using a detailed model. Taking the volume integral of the left and right hand sides of equation (134) and using

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Gauss theorem we arrive at the following equation:

$$\frac{1}{c^2}\ddot{M} = \frac{1}{c^2}\int \frac{\partial^2 \rho}{\partial t^2} d^3x = \oint d\vec{S} \cdot \left(\frac{\vec{v}}{c} \left(\frac{\vec{v}}{c} \cdot \vec{\nabla}\rho + \rho \vec{\nabla} \cdot (\frac{\vec{v}}{c}) \right) - \rho \frac{1}{c} \frac{\partial \frac{\vec{v}}{c}}{\partial t} \right). \tag{135}$$

The surface integral is taken over a surface encapsulating the galaxy or galaxy cluster. This leads according to equation (129) to a "dark matter" effect of the form:

$$M_d(r) = \frac{r^2}{2} \oint d\vec{S} \cdot \left(\frac{\vec{v}}{c} \left(\frac{\vec{v}}{c} \cdot \vec{\nabla}\rho + \rho \vec{\nabla} \cdot (\frac{\vec{v}}{c}) \right) - \rho \frac{1}{c} \frac{\partial \frac{\vec{v}}{c}}{\partial t} \right). \tag{136}$$

Thus we obtain the order of magnitude estimation:

$$\left[\frac{M_d(r)}{M}\right] \approx \left(\frac{v}{c}\right)^2 \left[\frac{r}{l_\rho} + \frac{r}{l_v} + \frac{r}{l_d}\right] \tag{137}$$

In the above we define three gradient lengths:

$$l_\rho \equiv \frac{\rho}{|\vec{\nabla}\rho|}, \qquad l_v \equiv \frac{v}{|\vec{\nabla} \cdot v|}, \qquad l_d \equiv \frac{v^2}{|\partial_t v|} \tag{138}$$

We can also write:

$$\frac{1}{l_t} = \left[\frac{1}{l_\rho} + \frac{1}{l_v} + \frac{1}{l_d}\right] \tag{139}$$

in which the smallest gradient length will be the most significant one in terms of the "dark matter" phenomena. In the depletion model to be described below we assume that l_ρ associated with density gradients is the shortest length scale. For galaxies we have $\left(\frac{v}{c}\right)^2 \approx 10^{-6}$, hence the factor $\frac{r}{l_t}$ should be around 10^6 to have a significant "dark matter" effect.

9.2. Detailed Model

For simplicity we shall take a galactic model, we assume axial symmetry; hence, all variables are independent of θ. Moreover, the mass influx coming from above and below the galaxy is much more significant compared to the influx coming from the galactic edge. This is due to the large difference in the galaxy surfaces perpendicular to the z axis compared to the area of its edge. The area of the surface of the galaxy which is perpendicular to the z axis is:

$$S_z = S_{z+} + S_{z-} = \pi r_m^2 + \pi r_m^2 = 2\pi r_m^2 \tag{140}$$

in which S_z is the total surface area of the galaxy perpendicular to the z axis, S_{z+} is the upper area of the surface of the galaxy perpendicular to the z axis, S_{z-} is the lower area of the surface of galaxy perpendicular to the z axis and r_m is the galactic radius (see Figure 5).

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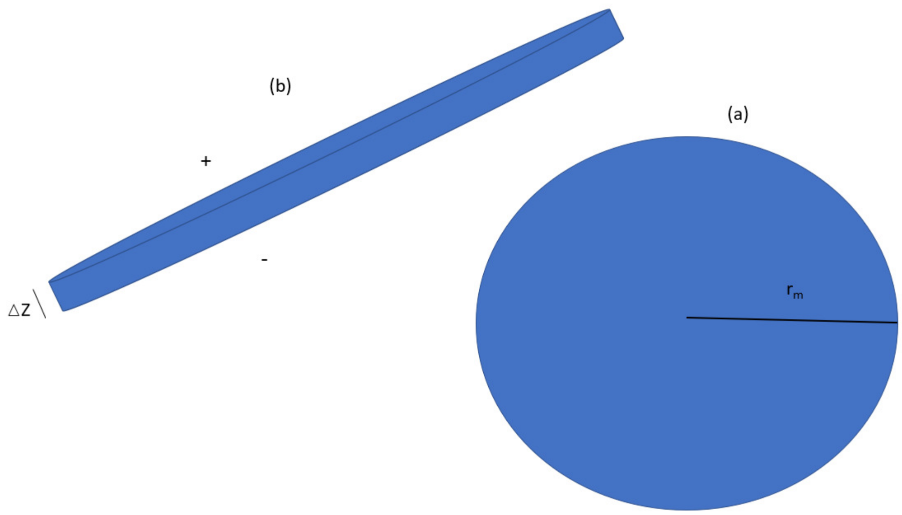

Figure 5. An idealized cylindrical galaxy from different perspectives. (**a**) From above; (**b**) tilted edge perspective.

The area of the surface of the galactic edge with thickness Δz is:

$$S_e = 2\pi r_m \Delta z. \tag{141}$$

Thus, the ratio of the surface area is:

$$\frac{S_e}{S_z} = \frac{\Delta z}{r_m}. \tag{142}$$

Typical values of Δz are about 0.4 kilo parsec and r_m is about 17 kilo parsec (for M33), giving an area ratio of about 1%. In such circumstances, the edge mass influx is less important and we can assume a velocity field of the form:

$$\vec{v} = v_z(\bar{r}, z, t)\hat{z} + v_\theta(\bar{r}, z, t)\hat{\theta}. \tag{143}$$

$\hat{z}$ and $\hat{\theta}$ are unit vectors in the z and θ directions, respectively. The influx is described schematically in Figure 6.

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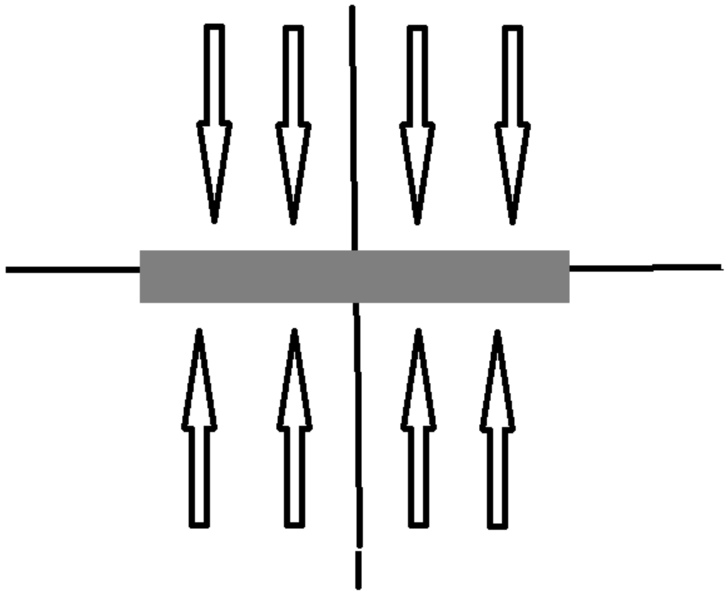

Figure 6. A schematic view of the galactic influx from the side.

In this case, the continuity Equation (130) will take the form:

$$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho v_z)}{\partial z} = 0 \tag{144}$$

By defining the quantity:

$$\gamma \equiv \rho v_z \Rightarrow \rho = \frac{\gamma}{v_z} \tag{145}$$

and using the above definition, Equation (144) takes the form:

$$\frac{\partial (\frac{\gamma}{v_z})}{\partial t} + \frac{\partial \gamma}{\partial z} = 0 \tag{146}$$

Assuming, for simplicity, that v_z is stationary and defining the auxiliary variable t_z:

$$t_z \equiv \int \frac{dz}{v_z} \tag{147}$$

we arrive at the equations:

$$\frac{\partial \gamma}{\partial t} + \frac{\partial \gamma}{\partial t_z} = 0. \tag{148}$$

This equation can be solved easily, as follows:

$$\gamma(\bar{r}, z, t) = f(t - t_z), \qquad f(-t_z) = \gamma(\bar{r}, z, 0) = v_z \rho(\bar{r}, z, 0) \tag{149}$$

for the function $f(x)$, which is fixed by the density initial conditions and the velocity profile.

Let us now turn our attention to the Euler Equation (131); for stationary flows, it takes the form:

$$(\vec{v} \cdot \vec{\nabla})\vec{v} = -\frac{\vec{\nabla} p(\rho)}{\rho} - \vec{\nabla}\phi \tag{150}$$

According to Equation (143):

$$\vec{v} \cdot \vec{\nabla} = v_z \frac{\partial}{\partial z} + \frac{v_\theta}{\bar{r}} \frac{\partial}{\partial \theta} \tag{151}$$

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Now, by writing Equation (150) in terms of its components, we arrive at the following equations:

$$v_z \frac{\partial v_z}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - \frac{\partial \phi}{\partial z} \tag{152}$$

$$-\frac{v_\theta^2}{\bar{r}} = -\frac{1}{\rho}\frac{\partial p}{\partial \bar{r}} - \frac{\partial \phi}{\partial \bar{r}}, \qquad \left(\frac{\partial \hat{\theta}}{\partial \theta} = -\hat{\bar{r}}\right). \tag{153}$$

It is usually assumed that the radial pressure gradients are negligible with respect to the gravitational forces and thus we arrive at the equation:

$$\frac{v_\theta^2}{\bar{r}} \simeq \frac{\partial \phi}{\partial \bar{r}}, \tag{154}$$

As for the z-component part of the Euler Equation (150), it can be easily written in terms of the specific enthalpy $w(\rho) = \int \frac{dP}{\rho}$ in the form:

$$\frac{\partial}{\partial z}\left(\frac{1}{2}v_z^2 + w(\rho) + \phi\right) = 0 \Rightarrow \frac{1}{2}v_z^2 + w(\rho) + \phi = C(r,t). \tag{155}$$

We recall that ρ depends on v_z through Equations (145) and (149):

$$\rho(r,z,t) = \frac{\gamma}{v_z} = \frac{f(t - \int \frac{dz}{v_z})}{v_z} \tag{156}$$

As both the specific enthalpy and the gravitational potential are dependent on the density, Equation (155) turns into a rather complicated nonlinear integral equation for v_z. However, many galaxies are flattened structures; hence, it can thus be assumed that the pressure z gradients are significant as one approaches the galactic plane. We will thus assume, for the sake of simplicity, that the pressure gradients balance the gravitational pull of the galaxy and thus v_z is just a function of r, in which case the convective derivative of v_z vanishes. The above assumption holds below and above the galactic plane, but not at the galactic plane itself. This suggests the following simple model for the velocity v_z (see Figure 6):

$$v_z = \begin{cases} -|v_z| & z > 0 \\ |v_z| & z < 0 \end{cases} \tag{157}$$

in which $|v_z|$ is a known function of $\bar{r}$. The velocity field is discontinuous at the galactic plane due to our simplification assumptions, but, of course, need not be so in reality. We also assume, for simplicity, that the velocity field $|v_z|$ is constant for $\bar{r} < r_m$ and vanishes for $\bar{r} > r_m$. According to Equation (149), the time-dependent density profile is fixed by the initial density conditions. In this section, we will deal with the density profile outside the galactic plane and will leave the discussion of the density profile in and near the galactic plane to the previous section. We consider an initial density profile as follows:

$$\begin{aligned} \rho_o(\bar{r},z,0) &= re(z)\left[\rho_1(\bar{r}) + \rho_2(\bar{r})e^{k|z|}\right], \\ re(z) &= \begin{cases} 1 & |z| < z_i \\ 0 & |z| \geq z_i \end{cases} \end{aligned} \tag{158}$$

in which the rectangular function $re(z)$ keeps the exponential function from diverging. The density profile is depicted in Figure 7.

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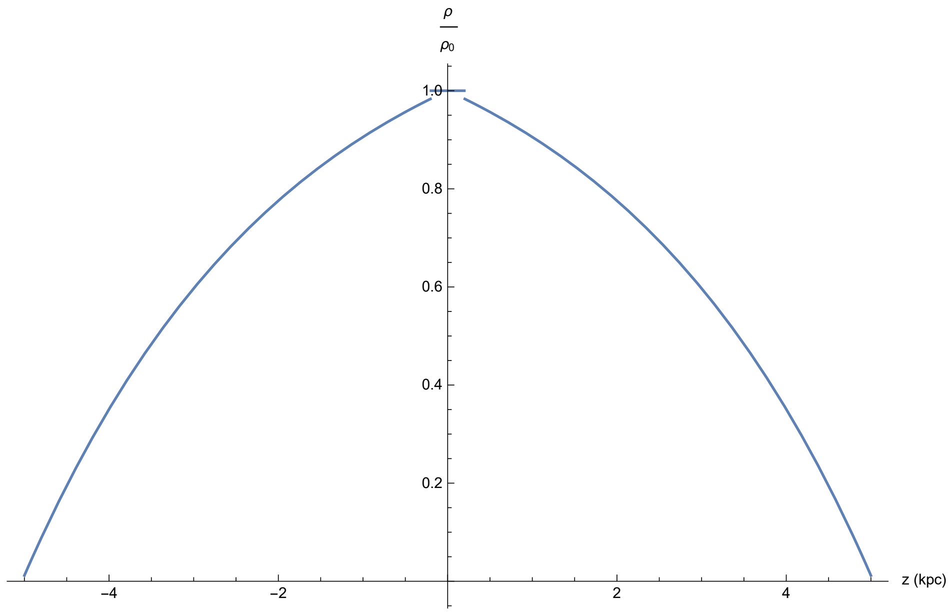

Figure 7. An initial density profile outside the galactic plane, in which $\rho_0 = \rho_1 + \rho_2$, $\frac{\rho_2}{\rho_1} = -0.2$ and $z_i = 5$ (kpc) and $k = 0.32$ (kpc^{-1}).

We assume that ρ_2 is negative and thus the density becomes dilute at distances far from the galactic plane. As v_z is constant both above and below the galactic plane, $t_z = \frac{z}{v_z}$ up to a constant. Therefore, it is easy to deduce from Equation (149) the functional form of $f(\beta)$ **(**$\beta = -t_z$**)**:

$$f(\beta) = v_z re(-v_z\beta)[\rho_1 + \rho_2 e^{k|v_z\beta|}] \tag{159}$$

Hence, according to Equation (156), the time-dependent density function for matter outside the galactic plane is obtained:

$$\rho_o(\bar{r}, z, t) = \frac{\gamma}{v_z} = re(z - v_z t)[\rho_1(\bar{r}) + \rho_2(\bar{r})e^{k|z-v_z t|}] \tag{160}$$

The density of matter outside the galactic plane will vanish for $t > t_m = \frac{z_i}{|v_z|}$; hence, we will discuss only the duration of $t < t_m$. Let us look at the mass contained in the cylinder defined by the galaxy (see Figure 8) and let us assume that the total mass in that cylinder is M_T. Now, the mass outside the galactic disk will be:

$$\begin{aligned} M_o(t) &= 2\pi\left[\int_{-z_i}^{-\frac{1}{2}\Delta z} dz \int_0^{r_m} d\bar{r}\bar{r}\rho_o(\bar{r}, z, t)\right. \\ &+ \left.\int_{\frac{1}{2}\Delta z}^{z_i} dz \int_0^{r_m} d\bar{r}\bar{r}\rho_o(\bar{r}, z, t)\right] \end{aligned} \tag{161}$$

Hence, the mass in the galactic disk is:

$$M(t) = M_T - M_o(t) \tag{162}$$

Moreover, the galactic mass derivatives are:

$$\dot{M}(t) = -\dot{M}_o(t), \qquad \ddot{M}(t) = -\ddot{M}_o(t) \tag{163}$$

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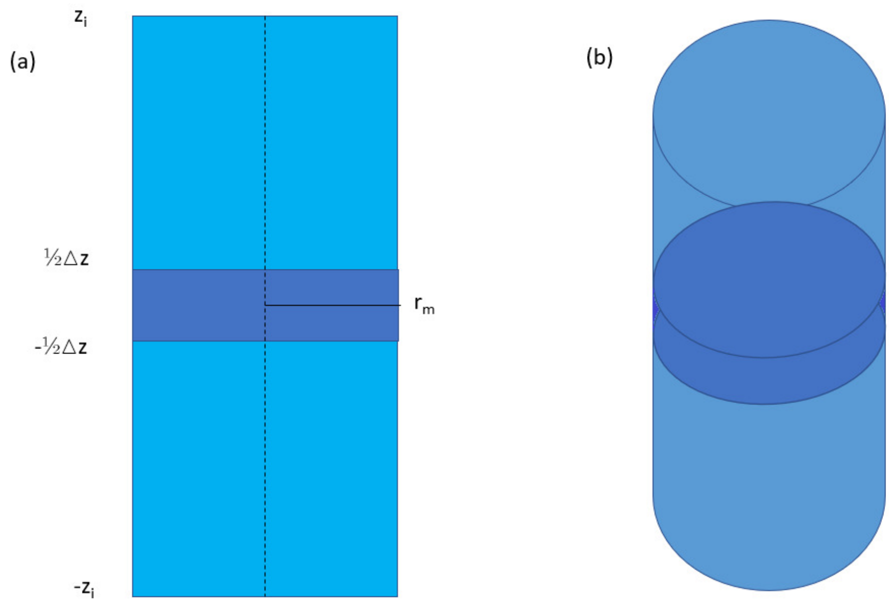

Figure 8. The mass column defined by the galaxy: (**a**) side view; (**b**) three-dimensional view.

By inserting Equation (160) into Equation (161), we may calculate $M_o(t)$:

$$M_o(t) = 2\left[\lambda_1\left(z_i - |v_z|t - \frac{1}{2}\Delta z\right) + \frac{\lambda_2}{k}\left(e^{kz_i} - e^{k(|v_z|t+\frac{1}{2}\Delta z)}\right)\right] \quad (164)$$

in which:

$$\lambda_1 \equiv 2\pi \int_0^{r_m} d\bar{r}\bar{r}\rho_1(\bar{r}), \qquad \lambda_2 \equiv 2\pi \int_0^{r_m} d\bar{r}\bar{r}\rho_2(\bar{r}). \quad (165)$$

Then, calculating the second derivative of $M_o(t)$ and using Equation (163) leads to the result:

$$\ddot{M}(t) = -\ddot{M}_o(t) = 2k|v_z|^2 e^{\frac{1}{2}\Delta z k}\lambda_2 e^{k|v_z|t}. \quad (166)$$

We then denote:

$$\alpha \equiv k|v_z|, \qquad \tau \equiv \frac{1}{\alpha}, \qquad \ddot{M}(0) = 2k|v_z|^2 e^{\frac{1}{2}\Delta z k}\lambda_2. \quad (167)$$

Thus:

$$\ddot{M}(t) = \ddot{M}(0)e^{\frac{t}{\tau}}. \quad (168)$$

It should be stressed that the current approach does not require that the velocity $|v_z|$ is high; in fact, the vast majority of galactic bodies (stars, gas) are substantially subluminal—in other words, the ratio of $\frac{|v_z|}{c} \ll 1$. The typical velocities in galaxies are 100 km/s, which makes this ratio 0.001 or smaller. However, we stress again the fact that every gravitational system, even if it is made of subluminal bodies, has a finite retardation distance, beyond which the retardation effect cannot be neglected. As demonstrated above, a galaxy exchanges mass with its environment. This means that a galaxy has a finite retardation distance. The question is thus quantitative: how large is the retardation distance? For the M33 galaxy, the velocity curve indicates that retardation effects cannot be neglected beyond a certain distance, which is calculated in [9] to be roughly 14,000 light years; similar analyses for other galaxies of different types have shown similar results [8]. We demonstrate here, using a detailed model, that this does not require a high velocity of gas or stars, in or out of the galaxy, and is perfectly consistent with the current observational

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knowledge of galactic and extragalactic material content and dynamics. Equation (167) means that we must have $\lambda_2 < 0$ according to Equation (128) in order to assure an attractive force. Next, we calculate $\dot{M}(t)$; using Equations (163) and (164), we obtain:

$$\dot{M}(t) = -\dot{M}_o(t) = 2|v_z|\lambda_1 + 2|v_z|e^{\frac{1}{2}\Delta zk}\lambda_2 e^{k|v_z|t}. \tag{169}$$

Hence:

$$\dot{M}(0) = 2|v_z|\lambda_1 + 2|v_z|e^{\frac{1}{2}\Delta zk}\lambda_2 = 2|v_z|\lambda_1 + \frac{\ddot{M}(0)}{\alpha} \tag{170}$$

Thus, λ_1 is:

$$\lambda_1 = \frac{1}{2|v_z|}\left[\dot{M}(0) - \frac{\ddot{M}(0)}{\alpha}\right] \tag{171}$$

Inserting Equations (171) and (167) into Equation (169) leads to:

$$\dot{M}(t) = \dot{M}(0) + \tau\ddot{M}(0)\left(e^{\frac{t}{\tau}} - 1\right) = \dot{M}(0) - \tau|\ddot{M}(0)|\left(e^{\frac{t}{\tau}} - 1\right). \tag{172}$$

Finally, by combining Equations (162), (164), (167) and (171) and noticing that:

$$M(0) = M_T - \frac{z_i - \frac{1}{2}\Delta z}{|v_z|}\left(\dot{M}(0) - \frac{\ddot{M}(0)}{\alpha}\right) - \frac{\ddot{M}(0)}{\alpha^2}\left(e^{k(z_i - \frac{1}{2}\Delta z)} - 1\right) \tag{173}$$

we arrive at:

$$M(t) = M(0) + \left(\dot{M}(0) - \tau\ddot{M}(0)\right)t + \tau^2\ddot{M}(0)\left(e^{\frac{t}{\tau}} - 1\right), \quad \tau > 0. \tag{174}$$

We shall now partition the mass into a linear and exponential growing part as follows:

$$M(t) = M_l(t) + M_e(t) = M_l(t) - |M_e(t)| > 0 \Rightarrow M_l(t) > |M_e(t)|. \tag{175}$$

$$\begin{aligned} M_l(t) &= M(0) + \tau^2|\ddot{M}(0)| + \left(\dot{M}(0) + \tau|\ddot{M}(0)|\right)t > 0, \\ M_e(t) &= \tau^2\ddot{M}(t) = -\tau^2|\ddot{M}(0)|e^{\frac{t}{\tau}} < 0. \end{aligned} \tag{176}$$

By dividing Equation (175) by $|\ddot{M}(t)|$ a, we have:

$$t_r^2 \equiv \frac{M(t)}{|\ddot{M}(t)|} = \frac{M_l(t)}{|\ddot{M}(t)|} - \tau^2. \tag{177}$$

in which t_r is the important time scale that determines the significance of retardation phenomena with respect to Newtonian gravity and the need to take retardation and "dark matter" into account.

10. Conclusions

In this paper we have deduced from general relativity a linear approximation. Under the said linear approximation we have solved Einstein field equations in term of retarded solutions. Those where used to derive the trajectory of a light ray (photon) both in the direction parallel to its original direction and perpendicular to it. Our current approach was compared to the approach based on the Schwarzschild metric [28] and was shown to be better in the sense that it is applicable to general mass distributions including ones that are changing in time. The light ray equations is the presence of a static point mass allowed us to re-derive the classical light deflection of Einstein and Eddington [17] and suggest a new experiment. This was followed by a discussion on the retardation effects on light ray trajectories, deriving an expression for "dark matter" which is the same as the one obtained in [9] for slowly moving bodies. Thus justifying results reported in the literature of the equivalence of "dark matter" for both galaxy rotation curve and gravitational lensing. This

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is followed by a discussion on the physical requirements needed in order to derive "dark matter" effects from retardation and a detailed model showing how those requirements are satisfied in a galactic scenario.

The need to satisfy the Lorentz symmetry group prevents the weak field approximation of GR from allowing action at distance potentials and thus only retarded solutions are allowed. Retardation is manifested more strongly when large distances and large second derivatives are involved. It should be stressed that the current approach does not require that velocities, v are high; in fact, the vast majority of galactic & galactic cluster bodies (stars, gas) are substantially subluminal—in other words, the ratio of $\frac{v}{c} \ll 1$. The typical velocities in galaxies are $100 \frac{\text{km}}{\text{s}}$ (see Figure 1), which makes this ratio 0.001 or smaller. However, one should consider the fact that every gravitational system, even if it is made of subluminal bodies, has a retardation distance, beyond which the retardation effect cannot be neglected. Every natural system, such as a star or a galaxy and even a galactic cluster, exchanges mass with its environment. For example, the sun loses mass through solar wind and galaxies accrete gas from the intergalactic medium. This means that all natural gravitational systems have a finite retardation distance. The question is thus quantitative: how large is the retardation distance? The change in the mass of the sun is quite small and thus the retardation distance of the solar system is huge, allowing us to neglect retardation effects within the solar system. However, for the M33 galaxy, the velocity curve indicates that the retardation effects cannot be neglected beyond a certain distance, which was calculated in [9] to be roughly $R_r = 4.54$ kpc; similar analyses for other galaxies of different types have shown similar results [8]. We demonstrated, using a detailed model, in Section 9, that this does not require a high velocity of gas or stars in or out of the galaxy and is perfectly consistent with the current observational knowledge of galactic and extragalactic material content and dynamics.

We point out that, if the mass outside the galaxy is still abundant (or totally consumed), $\ddot{M} \simeq 0$ and retardation force should vanish. This was reported [36] for the galaxy NGC1052-DF2.

We note that the same terms in the gravitation equation that are responsible for the gravitational radiation recently discovered are also responsible for the rotation curves of galaxies and gravitational lensing. The expansion given in Equation (123), being a Taylor series expansion up to the second order, is only valid for limited radii:

$$R < c\, T_{max} \equiv R_{max} \tag{178}$$

This means that current expansion is related to the near field case; this is acceptable since the extension of the rotation curve in galaxies and the distance of lensing trajectories from galaxies is the same order of magnitude as the size of the galaxy itself. An opposite case in which the size of the object is much smaller than the distance to the observer will result in a different approximation to Equation (13), leading to the famous quadruple equation of gravitational radiation, as predicted by Einstein [39] and verified indirectly in 1993 by Russell A. Hulse and Joseph H. Taylor, Jr., for which they received the Nobel Prize in Physics. The discovery and observation of the Hulse–Taylor binary pulsar offered the first indirect evidence of the existence of gravitational waves [40]. On 11 February 2016, the LIGO and Virgo Scientific Collaboration announced that they had made the first direct observation of gravitational waves. The observation was made five months earlier, on 14 September 2015, using Advanced LIGO detectors. The gravitational waves originated from the merging of a binary black hole system [41]. Thus, the current paper involves a near-field application of gravitational radiation while previous works discuss far- field results.

We regret that direct measurement of the second temporal derivative of the galactic mass is not available. What is available is the remarkable fit between the retardation model and the galactic rotation curve, as can be seen in Figure 1, which constitutes indirect evidence of the galactic mass second derivative. The reader is reminded that competing theories like dark matter do not supply any observational evidence either. Despite the work of thousands of people and the investment of billions of dollars, there is still no evidence of dark matter. Occam's razor dictates that when two theories compete, the one that makes less assumptions (ontological assumptions about the physical existence of an exotic form of matter included) has the upper hand. In the case of retardation theory, only baryonic matter and a large second temporal derivative of mass are assumed.

Problems that inflict dark matter theory, such as the cuspy halo problem (also known as the core-cusp problem) refers to a discrepancy between the inferred dark matter density profiles of low-mass galaxies and the density profiles predicted by cosmological N-body simulations. Nearly all simulations form dark matter halos, which have "cuspy" dark matter distributions, with density increasing steeply at small radii, while the rotation curves of most observed dwarf galaxies suggest that they have flat central dark matter density profiles ("cores"). This problem does not occur in the retardation model which denies the existence of dark matter. One cannot discuss flat or sharp profiles of dark matter if dark matter does not exist. The inherent problems with dark matter's dynamics further strengthen the claim of this work that dark matter does not exist and the gravitational lensing characteristics attributed to dark matter should be attributed to retardation.

To conclude this section, we would like to mention the remarkable theory of conformal gravity put forward by Mannheim [45,46]. The current retardation approach leads to a Newtonian potential plus a linear potential. Such potential types can also be derived from the completely different theoretical considerations of conformal gravity. Indeed, on purely phenomenological grounds, such fits have essentially already been published in the literature. While those fits looked very good, they had to treat the coefficient of the linear potential as a variable that changed from galaxy to galaxy; this element is not in line with conformal gravity in which the coefficient of the linear potential is a new universal constant of nature. This can be explained much more easily in the framework of retardation theory, in which a variable $\ddot{M}/M$ seems reasonable depending on the dynamical conditions of various galaxies. Indeed, $\ddot{M}/M$ has a theoretical basis, while a pure phenomenological approach does not. As far as we understand the work of Mannheim [44–46], it is related to conformal gravity, which is different from GR, and thus has to justify other results of GR (Big Bang Cosmology, etc.). We also underline that retardation theory does not contradict conformal gravity and, in reality, both effects may exist, although Occam's razor forbids us to add new universal constants if the existing ones suffice to explain observations.

Retardation theory's approach is minimalistic (in the sense that it satisfies the Occam's razor rule), does not affect observations that are beyond the near-field regime and, therefore, does not clash with GR theory and its observations (nor with Newtonian theory, as the retardation effect is negligible for small distances). We underline that a perfect fit to the rotation curve and gravitational lensing is achieved with a single parameter and we do not adjust the mass to light ratio in order to improve our fit as other authors do. Retardation effects in electromagnetic theory were discussed in [47–50].

In this paper we study the gravitational lensing scenario showing that retardation effects lead to the same "dark matter" mass for lensing as for galactic rotation curves.

Finally the paper does not discuss dark matter in a cosmological context; this is left for future works.

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