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Quantum Interference in Classical Physics & and its Impact on the Foundations of Quantum Mechanics

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Abstract: It is claimed that classical mechanics (CM) in terms of its equivalent Koopman-von Neumann formulation (KvN) is indeed a 'real' quantum theory since the formalism of quantum mechanics (FQM) does not, nor need it, exclude it as a such. This claim is made manifest by suggesting a common structure of which both KvN and ordinary quantum mechanics (OQM) correspond to unitary representations of, although unitarily inequivalent such. It is shown that OQM is obtained by enforcing the extra physical condition of quantized energy levels, which by itself does not constitute the 'quantum mystery'. It is furthermore shown that KvN in a physically realizable sense contain the hallmark quantum behavior of quantum interference. It is hence concluded that both CM and OQM are subjected to the same foundational issues. As the central part of both KvN and OQM is the introduction of probabilities via FQM it is suggested that the foundational issues regarding FQM are less about FQM itself and more about the concept of probability.

Keywords: The Koopman-von Neumann formulation of classical mechanics; THe measurement problem; Bell's theorem; The EPR paradox; Schrödinger's cat; The double-slit experiment; quantum interference; Probability theory

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1. Introduction

The discussion on the foundations of quantum mechanics typically revolves around the topics of Bell-type inequalities, Schrödinger's cat and quantum interference[1] . These topics are typically discussed in terms of FQM alone, without any particular regard to what physics has been put into it from 'the outside'.[2] However, FQM itself contains no more physical content other than that a physical observable A is in FQM represented as a self-adjoint operator whose range of possible outcomes corresponds to its spectrum and where the associated probability distribution over this range given a particular state ψ is given by the Born rule and that time-evolution is dictated by SE.[3][4] Neither does FQM itself includes no prescription of how to associate a physical observable to a particular self-adjoint operator, or sets of such. Canonical quantization[5] is an example of such a prescription. Note however that it is not part of FQM, nor need it be either. So, as the general discussion regarding the foundations of quantum mechanics is not dependent on a specific physical model and since canonical quantization itself is not part of FQM, it

[1] Also referred to as *double slit-type interference* or *phenomenon* [1].

[2] Regardless of the actual survey's result the choice and formulation of the questions themselves in [2] gives a good indication of what the research community on the foundations of quantum mechanics as a whole deem to be the issues.

[3] Note here that in FQM an observable is by necessity a self-adjoint operator but being a self-adjoint operator is not sufficient for being a physical observable, i.e not all self-adjoint operators correspond to physically meaningful observables.

[4] Note also that it is not a necessity to identify the generator of time evolution $\widehat{T}$ in SE,

$$-i\partial_t\psi_t = \widehat{T}\psi_t, \tag{1}$$

as the observable of energy. This is also the reason why $\widehat{T}$ here is referred to as the *generator of time evolution* rather than as the *Hamiltonian*.

[5] *Canonical quantization* is here a collective name for all mathematical procedures formulated such that

$$\{\cdot,\cdot\} \mapsto \frac{1}{i\hbar}[\cdot,\cdot] \tag{2}$$

in some way or another.

follows that any physical model expressible in FQM is a 'real' quantum theory.[6] On these grounds KvN [3–5] cannot hence simply be deemed as 'not a real' quantum theory. In this article KvN will be deemed as a real quantum theory. This will not only be because it cannot be excluded as a such. *Quantization* will in this article be understood in a wider sense, as a prescription for unitarily representing Hamiltonian dynamics in FQM. Both KvN and OQM will correspond to such unitary representations, although inequivalent such.. In addition, KvN will be shown to contain the hallmark quantum effect of quantum interference, and as such KvN will have been shown to indeed be a real quantum theory. It will also be shown that KvN is an equivalent formulation of CM[7], and thus CM and OQM are subjected to the same foundational issues. As these issues are not widely considered as problems in CM it is suggested that the foundational issues are not about FQM itself but about the concepts of probability. After all the main utility if FQM is as a framework for calculating probabilities, i.e as a probability model.

There are two key points here that need more elaboration. One is specifying what exactly OQM and KvN are unitary representations of. This will be discussed in detail in Section 4. The basic idea of this will however be discussed later in this introduction. The other point—which will be discussed first—is the claim that the hallmark quantum phenomena is quantum interference as the other typical ones —e.g Bell-type violations —are ultimately reducible to it. In Section 2 it will be specified in more detail how quantum interference corresponds to a purely probabilistic phenomena. But for our immediate purposes here it suffices to only mention that it occurs because a quantum state ψ can be equivalently be expanded into different orthonormal eigenbases $\{\psi_a\}_{a\in\mathbb{A}}$ and $\{\phi_b\}_{b\in\mathbb{B}}$ associated respectively to the quantum observables A and B, i.e

$$\sum_{a\in\mathbb{A}}\langle\psi_a,\psi\rangle\psi_a=\psi=\sum_{b\in\mathbb{B}}\langle\phi_b,\psi\rangle\phi_b \tag{3}$$

where $\langle\cdot,\cdot\rangle$ denotes the inner product on the considered Hilbert space $\mathcal{H}$.[8] Of course this is merely another way of saying that A and B are mutually non-commuting. Concerning the Bell inequality, we consider the Bell state

$$\Psi_+:=\frac{1}{\sqrt{2}}(\psi_{z+}\otimes\psi_{z+}+\psi_{z-}\otimes\psi_{z-}) \tag{4}$$

as expressed in the eigenbasis of the quantum observable

$$\sigma_z\otimes I+I\otimes\sigma_z \tag{5}$$

of spin in the z-direction. This state causes a violation of Bell's theorem [6] because it can be expressed completely equivalently[9] in terms of the observables of the other spin components as well, and in a way that preserves the given tensor decomposition. That is, Ψ_+ can equivalently be expressed in the x-direction as

$$\Psi_+=\frac{1}{\sqrt{2}}(\psi_{x+}\otimes\psi_{x+}+\psi_{x-}\otimes\psi_{x-}), \tag{6}$$

where the tensor decomposition is the same for both cases (4) and (6), i.e

$$\mathcal{H}_{\text{Alice}}\otimes\mathcal{H}_{\text{Bob}}. \tag{7}$$

[6] As will be discussed in part further down but more completely in Section 2, this physical model expressed in FQM must actually contain the phenomena of quantum interference in order to contain any hallmark quantum effect.

[7] CM in the form of Hamiltonian mechanics.

[8] Indeed, the whole point of decoherence is as a mechanism for enforcing a preferred basis.

[9] In terms of the considered Hilbert space $\mathcal{H}$ this equivalence is in fact of the strongest form, i.e in the form of an identity. However, as we can perform experiments that distinguish between these bases there exists a context where this is not a strict identity but just an equivalence relation. Hence the phrasing *equivalently* seems more prudent that *identically*.

The analogous statement holds for the spin in the y-direction as well. We note that in FQM there is not only the ambiguity of 'which eigenbasis' but also one in terms of the tensor product decomposition of the considered Hilbert space. In this Bell situation, this ambiguity is actually of no concern as the one only considered the decomposition (7) is set by the observers Alice and Bob. But this decomposition is not given a priori by FQM alone but plugged in from 'the outside' by the given by the actual physical context of application, i.e the set-up of the experiment.[10] Without the ambiguity of 'which tensor decomposition' the tensor product itself causes no non-classical behavior as it through Born's rule merely gives a probability distribution

$$\mathbb{P}_{\Psi_+}(\sigma_z = \cdot, \sigma_z = \cdot) = |\langle \psi_{z\cdot} \otimes \psi_{z\cdot}, \Psi_+ \rangle|^2, \tag{8}$$

which is fully compatible with usual Kolmogorovean probability theory. As the ambiguity of tensor product decompositions is not present in the Bell-type situations there is no need to take them as the subjects of interpretation.[11][12] FQM alone does not give us a reason to favor certain tensor decomposition over others. This favoring comes from 'the outside'. As stated before, in the particular context of Bell-type experiments, it is given by the actual experiment set-up considered, i.e by the actual question to be investigated. All in all, it is the fact that the Bell states are equivalently expandable in terms of different eigenbases in a way that respects the 'Alice-Bob'-tensor decomposition that is responsible for the violation of Bell's inequality. Now, in Section 2 it will be demonstrated that quantum interference corresponds to a 'perturbation' of the formula of total probability (FTP). As the proof of Bell's inequality relies on that formula[13], its violation is really due to the violation of the formulara of total probability. Hence we see that is its quantum interference that is also the driving phenomena behind Bell-type phenomena. Regarding the measurement problem —i.e Schrödinger's cat—it is not a problem without quantum interference. For if no non-commutative observables to the 'dead/alive cat'-observable exists, then this superposition is nothing but an ordinary probability distribution. In which case the measurement problem would a be as much of a problem for coin tosses as well. Quantum interference is the quantum phenomena. This atatement is not news. Most notably it has also been stated by Richard Feynman [7] as the '...only mystery.'. As it will be shown that KvN is not only an equivalent formulatipn of CM but also that it contains quantum interference as a necessary feature, it will be concluded that KvN is a real quantum theory. In contrast to what is commonly held as true, KvN is subjected to the same foundational issues as OQM.

In Section 3 it will be shown that quantum interference is a necessity in KvN. This since: The KvN-SE have canonically equivalent solutions as the corresponding classical Liouville equation (LE),

$$\frac{d}{dt}\rho = \{\rho, H\}. \tag{12}$$

10 Note that this ambiguity of tensor product decomposition really is a special case of the issue of the ambiguity of 'which eigenbasis'.

11 We note here that a violation of Kolmogorovean probability theory is not a violation of probability itself as has also been noted by others [1,7,8]

12 This in contrast to the tensor product itself usually being taken as the essential part of the Bell-type situations.

13 We note that conditional probabilities in the sense of Bayes' formula

$$\mathbb{P}(U|V) := \frac{\mathbb{P}(U \cap V)}{\mathbb{P}(V)} \tag{9}$$

is in Kolmogorovian probability theory is a definition and not a theorem, although it sometimes carries that name [1]. Hence the FTP

$$\mathbb{P}(U) = \mathbb{P}(U|V)\mathbb{P}(V) + \mathbb{P}(U|V^c)\mathbb{P}(V^c) \tag{10}$$

is simply just another notation for

$$\mathbb{P}(U) = \mathbb{P}(U \cap V) + \mathbb{P}(U \cap V^c), \tag{11}$$

which holds by definition for a probability measure. It is in the proof of Bell's theorem assumed that this can be done for all observables involved. As such violations of Bell's theorem is a demonstration of the inability to jointly represent all the considered observables on the same probability space given those specific associated distributions [1]. To then interpret this as representing non-locality or non-realism is by no means necessary and is even missleading [9,10].

Specifically , the eigenstates ψ_λ of the KvN-generator of time evolution $\widehat{T}$ via Born's rule ,

$$\rho_\lambda := |\psi_\lambda|^2, \tag{13}$$

satisfy

$$\{\rho_\lambda, H\} = 0. \tag{14}$$

That is, the eigenvector ψ_λ satisfy the equilibrium condition of statistical mechanics [11]. This provides the possibility of giving $\widehat{T}$ a physical meaning as an observable, as the observable statistical equilibria. Furthermore, the non-stationary states of classical statistical mechanics are those solution ρ_t of LE for which

$$\partial_t \rho_t \neq 0. \tag{15}$$

These will be shown to correspond to superpositions of the eigenstates ψ_λ. Hence the existence of such states necessitates superposition. Moreover, KvN by necessity also contains physically realizable observable which are non-commuting with $\widehat{T}$. In KvN,

$$\begin{cases} i\left[\widehat{T}, \hat{q}\right] = \frac{\hat{p}}{m} \\ i\left[\widehat{T}, \hat{p}\right] = -V'(\hat{q}) \end{cases}, \tag{16}$$

where $\hat{p}$ and $\hat{q}$ respectively correspond to the observables of linear momentum and position. Hence, for any non-trivial Hamiltonian dynamics, $\hat{p}$ and $\hat{q}$ are both examples of physically meaningful observables that non-commute with $\widehat{T}$.

In Section 4 the precise structure to which both OQM and KvN correspopnd to unitary representations of will be presented. Here we provide some intuition for the precise definition of that structure which will be given in that section. By Stone's theorem [12], in FQM any observable $\widehat{A}$ evolves in time according to

$$\frac{d}{dt}\widehat{A} = i\left[\widehat{A}, \widehat{T}\right]. \tag{17}$$

As the classical Hamilton equations of motion read

$$\begin{cases} \dot{q} = \frac{p}{m} \\ \dot{p} = -V'(q) \end{cases}, \tag{18}$$

we can see that (16) simply is the FQM version of (18). As is well known, (16) holds in OQM but then with a different $\widehat{T}$. KvN and OQM can in this sense both be viewed as such unitary representations of the classical dynamics, though unitarily inequivalent such. Thus blurring the line between classical and quantum physics even further. In line with this representation theoretic view, KvN will sometimes be referred to as the *KvN representation* and OQM as the *OQM representation*.

This article is structured as follows: In Section 2 it will be shown that quantum interference is a purely probabilistic phenomena, not located solely to the actual double-slit experiments as performed with photons and electrons e.t.c. In Section 3 it is shown that KvN representation is equivalent to classical (statistical) mechanics and that it naturally also contains the phenomena of quantum interference. In Section 4 is demonstrated that both OQM and KvN can be seen as merely two unitarily inequivalent representations of the same structure, this inequivalence being induced by further physical constraints which by themselves are not 'quantum'. In Section 5 this article is put into context, i.e compared to the typical view that KvN is 'completely classical' in spite its formulation in FQM. The article ends with Section 6 where the argument and its conclusion of this article are summarized.

2. Quantum Interference

The main purpose of this article is to demonstrate that KvN and OQM are both quantum theories containing quantum interference, hence having shown that in terms of foundations what is meant by *classical* and *quantum physics* is more ambiguous than typically thought. This is by necessity an argument for a particular interpretation of FQM, but one further issue that any such must deal with. As such we need not be very specific about what *superposition*, *quantum state*, *probability* e.t.c. mean as these are concepts associated to a particular interpretation of FQM.

Let A and B be two quantum observables on some Hilbert space $\mathcal{H}$. For the sake of the argument, it suffices to assume that B is non-degenerate with orthonormal eigenbasis $\{\phi_b\}_{b\in\mathbb{B}}$, where the eigenvectors are labeled by their respective eigenvalue b. We impose no such restriction on A. We let ψ_a denote an arbitrary (generalized) eigenvector of A with associated eigenvalue a. By Born's rule the probability distribution over A, given initial state ψ is

$$\mathbb{P}_{\psi,A}(a) := |\langle\psi_a,\psi\rangle|^2. \tag{19}$$

Analogously, for B,

$$\mathbb{P}_{\psi,B}(b) := |\langle\phi_b,\psi\rangle|^2. \tag{20}$$

By utilizing the completeness relation in terms of $\{\phi_b\}_{b\in\mathbb{B}}$, we obtain

$$\begin{aligned}
\mathbb{P}_{\psi,A}(a) &= \left|\sum_{b\in\mathbb{B}}\langle\psi_a,\phi_b\rangle\langle\phi_b,\psi\rangle\right|^2 && (21)\\
&= \sum_{b,b'\in\mathbb{B}}\langle\psi,\phi_{b'}\rangle\langle\phi_{b'}|\psi_a\rangle\langle\psi_a,\phi_b\rangle\langle\phi_b,\psi\rangle && (22)\\
&= \sum_{b\in\mathbb{B}}\mathbb{P}_{\phi_b,A}(a)\mathbb{P}_{\psi,B}(b) + \sum_{b\neq b'\in\mathbb{B}}\langle\psi,\phi_{b'}\rangle\langle\phi_{b'},\psi_a\rangle\langle\psi_a,\phi_b\rangle\langle\phi_b,\psi\rangle && (23)\\
&= \sum_{b\in\mathbb{B}}\mathbb{P}_{\phi_b,A}(a)\mathbb{P}_{\psi,B}(b) && (24)\\
&\quad + 2\sum_{b<b'\in\mathbb{B}}\mathfrak{Re}\{\langle\psi,\phi_{b'}\rangle\langle\phi_{b'},\psi_a\rangle\langle\psi_a,\phi_b\rangle\langle\phi_b,\psi\rangle\} && (25)\\
&= \sum_{b\in\mathbb{B}}\mathbb{P}_{\phi_b,A}(a)\mathbb{P}_{\psi,B}(b) && (26)\\
&\quad + 2\sum_{b<b'\in\mathbb{B}}\cos(\theta_{a,b,b'})\sqrt{\mathbb{P}_{\phi_b,A}(a)\mathbb{P}_{\psi,B}(b)\mathbb{P}_{\phi_{b'},A}(a)\mathbb{P}_{\psi,B}(b')}, && (27)
\end{aligned}$$

where the last line follows from the fact that for every complex number z there exists a unique number $\theta \in [0,2\pi)$ such that

$$z = |z|e^{i\theta}. \tag{28}$$

From this, it follows that

$$\mathfrak{Re}\{z\} = |z|\cos\theta, \tag{29}$$

θ being unique if we restrict it to values in $[0,\pi]$. As the meaning of $\mathbb{P}_{\phi_b,A}(a)$ is similar to that of the conditional probability $\mathbb{P}(A=a|B=b)$ in the Kolmogorov probability model, we can see that

$$\begin{aligned}
\mathbb{P}_{\psi,A}(a) = &\sum_{b\in\mathbb{B}}\mathbb{P}_{\phi_b,A}(a)\mathbb{P}_{\psi,B}(b)\\
&+\sum_{b<b'\in\mathbb{B}}2\cos(\theta_{a,b,b'})\sqrt{\mathbb{P}_{\phi_b,A}(a)\mathbb{P}_{\psi,B}(b)\mathbb{P}_{\phi_{b'},A}(a)\mathbb{P}_{\psi,B}(b')}
\end{aligned} \tag{30}$$

corresponds to FTP, but with an extra perturbation called the *interference term*. This identity is can be viewed as the in the larger category of contextual probability models [1] where (30) would be a particular form of what is called the *generalized formula of total probability* corresponding to the particular class of probability models called the *trigonometric non-Kolmogorovean probability models*.

Now, in specifying A as the observable of 'where on the screen the particle hit' and B as the observable of 'through which slit the particle went', the interference term is indeed the same interference as seen in the double-slit experiment [1,7]. Moreover, we see that the interference term vanishes if and only if A and B commute. This is why interference phenomena in FQM are indeed equivalent to t non-commutativity. The take-home point from all this is that quantum interference is a probabilistic phenomena not specific to the actual double-slit experiment alone.

3. The Koopman-von Neumann Representation of Classical Mechanics

We consider a phase space $\mathcal{P} \simeq \mathbb{R}^{6N}$ together with a set of fixed global canonical coordinates (p, q) on phase space $\mathcal{P}$ of dimension $6N$, where $N \in \mathbb{N}$. Typically these canonical coordinates are chosen such that q corresponds to the observable of position in Cartesian coordinates and p its corresponding conjugate momentum (CoM) [11] associated to q defined as

$$p := \frac{\partial L}{\partial \dot{q}}, \tag{31}$$

where L is a given Lagrangian function. We note that p defined in this way does not always correspond to linear momentum, e.g the charged particle moving in a magnetic field [8].[14] p does however do so for Hamiltonian functions of the typical form

$$H(p, q) = \frac{p^2}{2m} + V(q). \tag{32}$$

Furthermore, solely in terms of the formalism of Hamiltonian mechanics, there is nothing special about this particular choice of canonical coordinates. This choice of a coordinate q and its CoM does however play a distinguished role in going from the Lagrangian picture of classical mechanics into the Hamiltonian one via a Legendre transformation [11]. Note that it is not a necessity to interpret q as such. This is however how it is conventionally interpreted. We will stick to this convention here and consider a Hamiltonian function H which in terms of these have the typical form (31). To be explicit with the notation, we have defined

$$(p, q) := (p_1^1, p_2^1, p_3^1, \ldots, p_1^N, q_2^N, p_3^N, q_1^1, q_2^1, q_3^1, \ldots, q_1^N, q_2^N, q_3^N). \tag{33}$$

As we are assuming these to be global coordinates, we can without loss of generality assume that each p_i^n corresponds to the canonical projection[15] $\pi_{3(n-1)+1}$ and similarly each q_i^n as the projection $\pi_{3(N+n-1)+1}$. The time evolution of this Hamiltonian dynamical structure is given by a Hamiltonian flow

$$U : t \in \mathbb{R} \mapsto U_t \in \mathrm{Symp}(\mathcal{P}), \tag{35}$$

satisfying

$$(-\partial_q H, \partial_q H,) = \frac{d}{d\tau}\bigg|_{\tau=0} U_\tau. \tag{36}$$

We may, for all $t \in \mathbb{R}$ and every Schwartz function ψ on $\mathbb{R}^{6N}$, define the action

$$\widehat{U}_t \psi := \psi \circ U_{-t}. \tag{37}$$

[14] In this case the CoM p does not even correspond to an observable property as it is not gauge invariant [8].

[15] The canonical projection π_k is defined as the map

$$\pi_k(x_1, \ldots, x_K) := x_k. \tag{34}$$

This action is bijective on the Schwartz space as all U_{-t} are smooth diffeomorphisms. As the Schwartz space is dense in $L^2(\mathbb{R}^{6N})$ we may uniquelly linearly extend this action (37) to all of $L^2(\mathbb{R}^{6N})$. That (37) defines a unitary action for each t follows from Liouville's theorem [13]. That is, by Liouville's theorem,

$$dp_t dq_t = dp dq, \tag{38}$$

where

$$(p_t, q_t) := U_{-t}(p, q). \tag{39}$$

Hence

$$\int \overline{\left(\widehat{U}_t\psi\right)}(p,q)\left(\widehat{U}_t\phi\right)(p,q)dpdq = \int \overline{\psi}(U_{-t}(p,q))\phi(U_{-t}(p,q))dpdq \tag{40}$$

$$= \int \overline{\psi}(p_t,q_t)\phi(p_t,q_t)dp_t dq_t \tag{41}$$

$$= \int \overline{\psi}(p,q)\phi(p,q)dpdq. \tag{42}$$

Thus each $\widehat{U}_t$ preserves the inner product indeed. Note that it has not here been shown that $\widehat{U}$ as defined is strongly continuous, something which must hold in order for it to classify as a time-evolution in the FQM sense. This issue is not so straightforwardly true as one might first think. The set of naive' eigenvectors of $\widehat{U}$ in $L^2(\mathbb{R}^{6N})$ might contain eigenvectors ψ_λ for which the condition of strong continuity,

$$\lim_{t\to 0} \|\widehat{U}_t\psi_\lambda - \psi_\lambda\| = 0, \tag{43}$$

fails to hold. In the appendix this is illustrated through the KvN representation of the harmonic oscillator.

Next we consider the multiplication operators:

$$\begin{cases} \left(\widehat{M}_{p_i^n}\psi\right)(x,y) = x_i^n\psi(x,y) \\ \left(\widehat{M}_{q_i^n}\psi\right)(x,y) = y_i^n\psi(x,y) \end{cases}, \tag{44}$$

where we similarly as in (33) have applied the notation

$$(x,y) := \left(x_1^1, x_2^1, x_3^1, \ldots, x_1^N, x_2^N, x_3^N, y_1^1, y_2^1, y_3^1, \ldots, y_1^N, y_2^N, y_3^N\right). \tag{45}$$

Since we have assumed that U acts on points (p,q) having the interpretation as kinetic momentum p and Cartesian position q, we naturally interpret the $\widehat{M}_{p_i^n}$'s as corresponding to the observable of the ith component of the kinetic momentum of the nth particle and $\widehat{M}_{q_i^n}$ as corresponding to the observable of the ith component of the position of the nth particle. Based on this the operator

$$\widehat{E} := H\left(\widehat{M}_p, \widehat{M}_q\right) \tag{46}$$

is interpreted as the observable of energy. Because of all this we adopt the conventional notation:

$$\begin{cases} \hat{p}_i^n := \widehat{M}_{p_i^n} \\ \hat{q}_i^n := \widehat{M}_{q_i^n} \end{cases}, \tag{47}$$

The sub-/supscripts may be dropped when they are not a necessity.

As $\widehat{U}$ is assumed to be a strongly continuous one-parameter group, we may apply Stone's theorem to obtain a generator of time evolution $\widehat{T}$. From (37), we get that $\widehat{T}$ acts as

$$\left(\widehat{T}\psi\right)(x,y) := i\frac{d}{d\tau}\Big|_{\tau=0}\left(\widehat{U}_\tau\psi\right)(x,y) \tag{48}$$
$$= i\partial_p H\big|_{(x,y)}\partial_y\psi(x,y) - i\partial_q H\big|_{(x,y)}\partial_x\psi(x,y), \tag{49}$$

on all $\psi \in L^2\left(\mathbb{R}^{6N}\right)$ in its domain $D(T)$. Notice that we may also write $\widehat{T}$ more concisely as

$$\widehat{T}\psi = i\{H,\psi\}, \tag{50}$$

$\{\cdot,\cdot\}$ denoting the Poisson bracket on $\mathcal{P}$. The corresponding SE is hence

$$i\partial_t\psi_t = \widehat{T}\psi_t. \tag{51}$$

Now, by Born's rule,

$$\rho_t := |\psi_t|^2 \tag{52}$$

is a probability distribution over phase space. Assuming that ψ_t is a solution to (51), it follows that

$$\begin{aligned}
\partial_t\rho_t &= \psi_t\overline{\partial_t\psi_t} + \overline{\psi_t}\partial_t\psi_t && (53)\\
&= i\psi_t\overline{i\partial_t\psi_t} - i\overline{\psi_t}i\partial_t\psi_t && (54)\\
&= i\psi_t\overline{\widehat{T}\psi_t} - i\overline{\psi_t}\widehat{T}\psi_t && (55)\\
&= i\psi_t\overline{i\{H,\psi_t\}} - i\overline{\psi_t}i\{H,\psi_t\} && (56)\\
&= \psi_t\overline{\{H,\psi_t\}} + \overline{\psi_t}\{H,\psi_t\} && (57)\\
&= \{H,\rho_t\}. && (58)
\end{aligned}$$

This means that ρ_t solves LE. Furthermore, from the method of characteristics —which allows us to apply the Picard-Lindelöf theorem —we get that any solution of LE is all of the form

$$\rho_t := \rho_0 \circ U_{-t}, \tag{59}$$

where ρ_0 is some initial distribution. We can always find $\psi \in L^2\left(\mathbb{R}^2\right)$ such that

$$\rho_0 = |\psi_0|^2. \tag{60}$$

For example,

$$\psi_0(x,y) = e^{i\omega(x,y)}\sqrt{\rho_0(x,y)}, \tag{61}$$

with ω being some real-valued function. Clearly,

$$\psi_t := \psi_0 \circ U_{-t} \tag{62}$$

solves the SE (51). Hence LE and the corresponding KvN-SE are equivalent, and hence KvN is contains classical statistical mechanics.[16] We note furthermore that, if ψ_λ is an eigenvector of $\widehat{T}$, i.e

$$\widehat{T}\psi_\lambda = \lambda\psi_\lambda, \tag{63}$$

then the probability distribution,

$$\rho_\lambda := |\psi_\lambda|^2, \tag{64}$$

[16] Solving LE is equivalent to solving the equations of motion. This makes LE, and hence the KvN-SE, hard to solve in practise.

satisfies

$$\{H,\rho_\lambda\} = \psi_\lambda\overline{\{H,\psi_\lambda\}} + \overline{\psi_\lambda}\{H,\psi_\lambda\} \tag{65}$$
$$= i\psi_\lambda\overline{\widehat{T}\psi_\lambda} - i\overline{\psi_t}\widehat{T}\psi_\lambda \tag{66}$$
$$= i\psi_\lambda\overline{\lambda\psi_\lambda} - i\overline{\psi_t}\lambda\psi_\lambda \tag{67}$$
$$= 0. \tag{68}$$

In classical statistical mechanics thermodynamic equilibria correspond to such stationary solutions ρ_λ of LE. So thermal equilibria correspond to eigenvectors of $\widehat{T}$. Moreover, the non-stationary states ρ_t, i.e those for which

$$\partial_t\rho_t \neq 0, \tag{69}$$

correspond to superpositions of eigenvectors ψ_λ,

$$\rho_t = \left|\sum_\lambda e^{i\lambda t}\psi_\lambda\right|^2, \tag{70}$$

as was shown in (53). So we have seen that the eigenstates of $\widehat{T}$ have a clear physical interpretation as statistical equilibria. Superpositions of such are also a necessity, as they correspond to non-stationary states.[17] Now, by (50) $\widehat{T}$ acts as a derivative. Hence, since $\hat{p}$ and $\hat{q}$ are multiplication operators, it follows that

$$[\widehat{T},\hat{p}_i^n]\psi = i\{H,\hat{p}_i^n\psi\} - i\hat{p}_i^n\{H,\psi\} \tag{73}$$
$$= i\{H,p_i^n\}\psi + i\hat{p}_i^n\{H,\psi\} - i\hat{p}_i^n\{H,\psi\} \tag{74}$$
$$= i\{H,p_i^n\}\psi \tag{75}$$
$$= -i\frac{\partial V}{\partial q_i^n}\psi. \tag{76}$$

That is,

$$[\widehat{T},\hat{p}_i^n] = -i\frac{\partial V}{\partial q_i^n}(\hat{p},\hat{q}), \tag{77}$$

and similarly that

$$[\widehat{T},\hat{q}_i^n] = -i\frac{\hat{p}_i^n}{m_n}. \tag{78}$$

Note that this is merely the equations of motion in operator form. The up-shot is however that we have two physically meaningful observables $\hat{p}$ and $\hat{q}$ which are non-commutative with the physically meaningful observable $\widehat{T}$. By Section 2 non-commuting physically meaningful quantum observables is the signpost of quantum interference. As shown in this section KvN-SE is via Born's rule equivalent to LE. Hence quantum interference is also present in classical (statistical) mechanics.

4. Quantization as Representation theory

In this section, it will be shown that both KvN and OQM can be considered as merely two mutually inequivalent unitary representations of the same Hamiltonian dynamics. It

[17] Note that, although

$$\psi(x,y) = e^{-\beta H(x,y)}, \tag{71}$$

with $\beta > 0$, is an eigenvector of $\widehat{T}$ indeed, β is not its eigenvalue. The eigenvalue of this vector is in fact zero, since

$$\{H,\psi\} = 0. \tag{72}$$

So the eigenstate of $\widehat{T}$ can not so simply be considered as thermodynamic equilibria at different temperatures β. So the physical interpretation of $\widehat{T}$ is not as straight forward as that.

will furthermore be shown that enforcing quantized energy levels implies OQM representation as the only viable choice. As it is quantum interference that constitutes the hallmark quantum phenomena and not the quantization of energy levels, this alone does not make OQM 'more quantum' than KvN.[18]

We will now be more general regarding the considered dynamics that we have been in the previous section, not restricting solely to Hamiltonians of the special form (31). Hence, even if q still corresponds to position, p may not correspond to linear momentum. p will however still correspond to the CoM of q. For ease of notation, we will restrict to only one dimension of space. Everything done here is generalizable to more dimensions.

Definition 1. *An* **FQM representation of a Hamiltonian dynamical system** *() on a Hilbert space $\mathcal{H}$ is a unitary representation of the corresponding Hamiltonian flow U,*

$$\widehat{\ }: U_t \mapsto \widehat{U}_t, \tag{79}$$

on $\mathcal{H}$ such that:

1.
$$t \mapsto \widehat{U}_t \tag{80}$$
defines a strongly continuous one-parameter unitary subgroup on $\mathcal{H}$.

2. *There exist two self-adjoint operators,*
$$\hat{p} \quad \text{and} \quad \hat{q}, \tag{81}$$
on $\mathcal{H}$, whose respective spectrum agrees with the range $\mathbb{R}$ of their classical counterparts p and q, and such that
$$\begin{cases} \frac{d}{dt}\Big|_{t=0} \widehat{U}_t \hat{p} \widehat{U}_{-t} = -\frac{\partial H}{\partial q}\Big|_{(\hat{p},\hat{q})} \\ \frac{d}{dt}\Big|_{t=0} \widehat{U}_t \hat{q} \widehat{U}_{-t} = \frac{\partial H}{\partial p}\Big|_{(\hat{p},\hat{q})} \end{cases} \tag{82}$$
are both well-defined as operator equations on $\mathcal{H}$.[19]

3. *There exist two unitary representations $\mathcal{V}$ and $\mathcal{U}$ of $(\mathbb{R}, +)$ on $\mathcal{H}$ such that*
$$\begin{cases} \mathcal{V}_s \hat{p} \mathcal{V}_s = \hat{p} + s \\ \mathcal{U}_s \hat{q} \mathcal{U}_s = \hat{q} + s \end{cases}, \tag{83}$$
but
$$\begin{cases} \mathcal{U}_s \hat{p} \mathcal{U}_s = \hat{p} \\ \mathcal{V}_s \hat{q} \mathcal{V}_s = \hat{q} \end{cases}. \tag{84}$$
It is additionally assumed that the group $\mathcal{G}$ generated from $\{\mathcal{U}_s\}_{s\in\mathbb{R}}$ and $\{\mathcal{V}_s\}_{s\in\mathbb{R}}$ under operator multiplication acts irreducibly on $\mathcal{H}$.[20]

As there is nothing in FQM that excludes KvN as a proper quantum theory. The point of introducing this representation theoretic view in Definition 1 is to exemplify a way in which this can be made manifest in the notion of *quantization* itself. It is perhaps not completely necessary to motivate Definition 1 more than through showing that it contains both KvN and OQM. There are however somewhat intuitive and reasonable motivations behind this choice of definition. These will presented next.

The first point on Definition 1 validates the use of Stone's theorem.Hence, together

[18] In this regard, the name *quantum mechanics* is unfortunate.
[19] Considering the definition of operator equation, this is equivalent to the Ehrenfest relations.
[20] Note that $\mathcal{G}$ is not necessarily subjected to the Stone-von Neumann theorem [14] since the Weyl form of the CCR do not hold by necessity. In fact, this is what separates OQM from KvN.

with the second point, we can conclude that there exists a generator of time evolution $\widehat{T}$ satisfying

$$\begin{cases} i[\widehat{T},\hat{p}] = -\left.\frac{\partial H}{\partial q}\right|_{(\hat{p},\hat{q})} \\ i[\widehat{T},\hat{q}] = \left.\frac{\partial H}{\partial p}\right|_{(\hat{p},\hat{q})} \end{cases} . \tag{85}$$

As, in FQM, the time evolution of an observable is given as

$$t : \widehat{A} \mapsto \widehat{U}_t \widehat{A} \widehat{U}_{-t}, \tag{86}$$

the second point is simply the equations of motion in FQM. The validity of these are viewed as the justifying the identification of the observables $\hat{p}$ and $\hat{q}$ as the correspondents to their classical counterparts.

The third point represents the mutual independence of $\hat{p}$ and $\hat{q}$. This since the translations $\mathcal{U}$ and $\mathcal{V}$ corresponds to being able to 'independently vary' $\hat{q}$ and $\hat{p}$.[21] That $\mathcal{G}$ acts irreducibly, is considered as corresponding to this independent varying only being applicable to $\hat{q}$ and $\hat{p}$. That one cannot in the considered representation add more observables which can be independently varied, a form of ' saturation' of the description of the system. The justification for the use of the terms *independently vary* and *saturation* is based on the following argument: In the Hamiltonian mechanical setting, (p,q) corresponds to a (global) coordinate chart. Then, given nothing else, the value of p poses no restriction on that value of q, and vice versa. This independence can also be stated as there existing two separate group actions, u and v of $(\mathbb{R},+)$ on the phase space $\mathcal{P}$, such that

$$\begin{cases} (p,q) \circ u_s = (p-s,q) \\ (p,q) \circ v_s = (p,q-s) \end{cases} . \tag{87}$$

Conversely, as (p,q) is a global coordinate chart, these group actions can be constructed as

$$\begin{cases} u_s := (p,q)^{-1} \circ (p-s,q) \\ v_s := (p,q)^{-1} \circ (p,q-s) \end{cases} . \tag{88}$$

Based on this and thinking as generally as possible —wishing to apply the notion of independence to pairs of observables which not necessarily can be considered as components of a coordinate chart —we may consider two observables f and g on $\mathcal{P}$ as mutually independent if there exist group actions u and v of $(\mathbb{R},+)$ on $\mathcal{P}$ such that

$$\begin{cases} (f,g) \circ u_s = (f-s,g) \\ (f,g) \circ v_s = (f,q-s) \end{cases} . \tag{89}$$

By analogy, the requirements (83,84) translate this same sense of independence to FQM. Now on to the 'saturation' aspect. That we cannot 'add more' independent observables means in the particular setting of Hamiltonian mechanical setting that (p,q) is indeed a coordinate chart. So in this setting, that p and q make up a saturated description in the same sense as them providing a 'complete' description. However, *saturation* is to be thought of as a more general concepts which outside of Hamiltonian mechanics not necessarily coincides with the concept of *complete description*. Note however that a complete description is always saturated. We will use this logical implication as a way for identifying how the concept of *saturation* can be stated in FQM. In doing this we first note that *independence* and *completeness* can be stated group theoretically as well. If the joint action of u and v is unique in satisfying (87), and assuming that this joint action is transitive, then it follows that (f,g)

[21] Note that this is not the same as space translations in the sense of Galileo. The difference can be made clear by considering the generalization to an arbitrary number of particles. Then the position observable $\hat{q}$ can be varied independently from the others. This is not true for space translations in the Galilean sense, where all position observables would be translated by the same amount.

is bijective. This since then for every $x, y \in \mathcal{P}$ there exists $s_1, s_2 \in \mathbb{R}$ such that

$$y = u_{s_1} v_{s_2} x, \tag{90}$$

so if

$$(f, g)(x) = (f, g)(y), \tag{91}$$

then

$$(f, g)(x) = (f, g)(u_{s_1} v_{s_2} x) \tag{92}$$
$$= (f - s_1, g - s_2)(x), \tag{93}$$

which can only hold if $s_1 = s_2 = 0$, i.e if $x = y$. Note that this furthermore shows that the action is in addition also be free, meaning that it in fact is regular. Hence that (f, g) constitutes a coordinate chart —is complete —can be stated as the requirement of the joint action of u and v being regular. We can by analogy impose the same for the action of the group $\mathcal{G}$ in FQM, which indeed would imply that the corresponding representation is irreducible. Note however that the opposite is not by necessity true. An associated group action to an irreducible representation on a Hilbert space is not necessarily free. All we required however, was that saturation is a necessary but not sufficient condition for completeness. Hence we may view the requirement of $\mathcal{G}$ to act irreducibly in point 3. as the enforcement of $\hat{p}$ and $\hat{q}$ as being saturately represented.[22]

Having motivated the structure a bit we move on to see what it implies. We note first that by assumptions (83,84), follows that $[\hat{p}, \hat{q}]$ is a $\mathcal{G}$-invariant map, i.e

$$\begin{cases} \mathcal{U}_s[\hat{p}, \hat{q}]\mathcal{U}_s = [\hat{p}, \hat{q}] \\ \mathcal{V}_s[\hat{p}, \hat{q}]\mathcal{V}_s = [\hat{p}, \hat{q}] \end{cases} . \tag{94}$$

If we assume that $[\hat{p}, \hat{q}]$ is non-trivial, then this means that $\mathcal{G}$ acts invariantly on it domain D, where

$$D := \mathrm{Dom}(\hat{p}\hat{q}) \cap \mathrm{Dom}(\hat{q}\hat{p}). \tag{95}$$

This together with the assumption of $\mathcal{G}$ being irreducibly represented on $\mathcal{H}$ implies that D is an $\mathcal{G}$-subrepresentation. Note that D is not by necessity equal to $\mathcal{H}$, as D need not be complete. In fact, D is not complete. However, it is true that D is dense in $\mathcal{H}$. Hence $i[\hat{p}, \hat{q}]$ is a self-adjoint operator. We may hence apply the Spectral theorem [14]. Let μ denote the spectral measure of $i[\hat{p}, \hat{q}]$. The, for every Borel $E \subseteq \mathbb{R}$, the spectral subspace,

$$V_E := \mathrm{Ran}(\mu(E)), \tag{96}$$

is a $\mathcal{G}$-subrepresentation. Hence, for every E, V_E must either be equal to $\mathcal{H}$ or $\{0\}$. That is, $\mu(E)$ is either the identify or trivial. If the spectrum of $i[\hat{p}, \hat{q}]$ were to contain more then one point, then we could find two disjoint Borel sets $E, F\mathbb{R}$ such that

$$\mu(E) = \mu(F) = I. \tag{97}$$

However, as μ is a projection-valued measure, by definition, it must satisfy

$$\mu(E \cap F) = \mu(E)\mu(F). \tag{98}$$

However, as the left-hand side would in this case be equal to zero and the right-hand side equal to I, this causes a contradiction. The spectrum can hence only contain one point $c \in \mathbb{R}$. This means that

$$[\hat{p}, \hat{q}] = icI. \tag{99}$$

22 We note that some some situation $\hat{p}$ and $\hat{q}$ are only 'saturately represented' together with the introduction of spin.

In this sense the CCR can be considered as a manifestation of $\hat{p}$ and $\hat{q}$ being mutually independent and collectively saturating.

Note that it automatically follows that the energy observable

$$E := H(\hat{p}, \hat{q}) \tag{100}$$

is time-invariant.

As shown in Section 3, KvN constitutes a FQM representation of classical mechanics. It is also well-known from any textbook [15] that OQM indeed satisfy points one[23] and two. Regarding point three, in the case of OQM,

$$\begin{cases} \mathcal{U}_s = e^{i\hat{q}s} \\ \mathcal{V}_s = e^{i\hat{p}s} \end{cases}, \tag{101}$$

which, by the Baker-Campbell-Hausdorff formula, implies that

$$\mathcal{U}_s\mathcal{V}_t = e^{-icst}\mathcal{V}_t\mathcal{U}_s. \tag{102}$$

In this case, the generated group $\mathcal{G}$ corresponds to a unitary representation of the *Heisenberg Lie group* [16]. This representation-theoretic view can be considered as the 'grown-up' way of quantizing a classical system as it corresponds to the integrated version of the CCR.[24] The reason for considering quantization as such is that in Hamiltonian mechanics, given a group action on phase space, there is a Lie algebra homomorphism from the associated Lie algebra of the considered group into the Poisson algebra [16]. This map is referred to as the *moment map*. As such, the moment map allows us to consider classical observables —smooth function on phase space —as generators of symmetries. In particular, in this sense, the observable of momentum can be considered as the generator of translations in space, which via the moment map implies that

$$\{p, q\} = 1. \tag{103}$$

This identification of observables as generators of symmetries carries over well into FQM via the representation theory of groups. For instance, $\hat{p}$ can in OQM, by (99), be identified as the generator of spatial translations. Because of this, *linear momentum* is by some taken as 'ontically' correspodning to the generator of spatial translations [18]. The problem, hoever, is that it does not take the particular dynamics into account. As noted section 3, for the charged particle moving in a magnetic field, p does not correspond to linear momentum. Yet, in the OQM representation of this system, it is still $\hat{p}$ that corresponds to the generator of spatial translations and not the linear momentum [8].[25] This suggests that even in this sense of *quantization* it should really be the CoM associated to the position observable that should be identified as the generator of spatial translation. By Stone-von Neumann's theorem [14], $\hat{p}$ and $\hat{q}$ satisfying the CCR, would by this reasoning be the FQM-equivalent to $\hat{p}$ being CoM to $\hat{q}$. This analogy does however not hold up. For the coordinate chart (p, q) is not unique in satisfying (101), all canonical coordinates do. So the CCR alone cannot be considered as an FQM-version of $\hat{p}$ as being the CoM of $\hat{q}$. The particular dynamics has to be taken into account, just as it is in Definition 1.

As just stated, KvN and OQM are both FQM representations of the same classical system. So at that 'pre-representation level' they are completely equivalent. At the representation level, however, this equivalence is however broken. This breaking can for instance be introduced by enforcing extra physical constraints, which by themselves are

[23] It being equivalent to SE.
[24] See *algebraic quantum mechanics* [17].
[25] We note that this applies to the purely classical setting as well. There the CoM via the moment map corresponds to the generator of spatial translations as well.

not part of the 'quantum mystery'. For instance, if we for some reason[26] have quantized energy levels, then it follows that $\hat{p}$ and $\hat{q}$ cannot commute. To show this, we first note that both $\hat{p}$ and $\hat{q}$, by assumption, both have the real line as their respective spectrum. Hence, if $\hat{p}$ and $\hat{q}$ commute, then the spectrum of the energy observable $H(\hat{p}, \hat{q})$ is the entire positive real line. Equivalently, if the spectrum of $H(\hat{p}, \hat{q})$ is not the entire positive real line, then $\hat{p}$ and $\hat{q}$ cannot commute. Hence, the KvN representation is not an option. By (99) this leaves OQM as the only option.[27] Even though it is the origin of the name 'quantum mechanics' the enforcement of quantized energy levels is not what constitutes the quantum mystery. As stated before, quantum interference is. So this extra enforcement does not differentiate OQM from KvN in terms of foundation al issues.

5. Discussion

KvN is often deemed as 'classical' based soley on Heisenberg's uncertainty relation not being satisfied for $\hat{p}$ and $\hat{q}$, as they are mutually commuting [3–5]. However, this is not the only phenomenon induced by non-commutativity of observables which can be deemed as 'quantum'. This is apparent from the probabilistic nature of quantum interference as presented in Section 2, but also exemplified via the Stern-Gerlach experiment [15]. Hence, deeming KvN as classical because of this means that it corresponds to classical mechanics.

In FQM there is another notion of *classical*, which must be differentiated from the previous one. This notion refers to the notion of classically behaving states. Those are the mixed states, as they are claimed to correspond to 'classical' probability distributions [15]. In essence, the reason for considering them as classical is because they satisfy FTP. There is however a big caveat here, they only do so with respect to the observable A whose eigenbasis diagonalizes it and another B—which can be non-commuting to A—and then only in the measurement sequence 'first A, then B'. The role of A here is really only as a selection procedure, assigning a value of A to each sample in an ensemble of such. That is selection procedure is what defines the initial state under which B is then measured. Hence, that FTP holds is a bit 'artificial', as it does so by construction.[28] Furthermore, if there are no physical observables that non-commute with A, then it is in FQM impossible to tell whether the state is 'pure' or 'mixed'.[29] To see why, consider an arbitrary probability distribution,

$$\mathbb{P}(A = a), \tag{104}$$

over the possible outcomes of the observable A. Given nothing more, this probability distribution can be attributed to any quantum state

$$\psi := \sum_{a \in \mathbb{A}} e^{i\theta_a} \sqrt{\mathbb{P}(A = a)} \psi_a, \tag{105}$$

where the θ_a's are in $[0, 2\pi)$. But it can also be attributed to the mixed state,

$$\rho := \sum_{a \in \mathbb{A}} \mathbb{P}(A = a) P_{\psi_a}, \tag{106}$$

where P_a denotes the orthogonal projection onto the eigenspace of a. More is needed to be able to differentiate between these states. This 'more' is the existence of an to A non-commuting observable. It is only then that the effects of the relative phases $e^{i\theta_a}$ become observable.[30] Although it is a standard result, we demonstrate how explicitly as it simultaneously shows again the relevance of the formula total probability in these regards.

26 i.e experiments show it to be the case.

27 This can be compared to the 'old quantum theory' [19] where the discretization of certain properties was the basic objective.

28 In line with the empiricist interpretation of FQM [20], *classical state* is a particular way of preparation.

29 Note that not all self-adjoint operators are observables, so it is in FQM not a priori true that such an observable always can be found.

30 In fact, in a previous article [21] the author has demonstrated the centrality of this concept to quantum mechanics by showing that by instead postulating this unitary equivalence as central, Born's rule can be shown to be a theorem rather than a postulate. The foundational importance of the relative phase as it relates to Born's rule has also been shown in others work, e.g [22].

The latter point is perhaps not usually stressed enough. We let B be non-commuting with A, whose eigenvectors with associated eigenvalue b are denoted as ϕ_b. Then the different states ψ and ρ yield the following probability distributions over B:

$$\begin{cases} \mathbb{P}_\psi(B=b) = |\langle \phi_b, \psi \rangle|^2 \\ \mathbb{P}_\rho(B=b) = \langle \phi_b, \rho\phi_b \rangle \end{cases} . \tag{107}$$

By a straightforward calculation, a calculation which was done in Section 2 as well,

$$\begin{cases} \mathbb{P}_\psi(B=b) = \sum_{a,a'\in\mathbb{A}} e^{i(\theta_a-\theta_{a'})}\sqrt{\mathbb{P}(A=a)\mathbb{P}(A=a')}\langle \phi_b, \psi_a \rangle \langle \psi_{a'}, \phi_b \rangle \\ \mathbb{P}_\rho(B=b) = \sum_{a\in\mathbb{A}} |\langle \phi_b, \psi_a \rangle|^2 \mathbb{P}(A=a) \end{cases} . \tag{108}$$

Here we see that in $\mathbb{P}_\rho(B=b)$, there is no θ_a-dependence, while there is in $\mathbb{P}_\psi(B=b)$. We also see that the bottom relation reads as FTP. Notice however, this would not hold if we tried replacing A with another with respect to which ρ is not diagonal. This means that even if mixed states satisfy FTP in this manner, the roles of A and B cannot be interchanged, something which can always be done if they corresponded to random variables on a Kolmogorovean probability space. Hence A and B cannot be equivalently considered as random variables, meaning that this notion of 'classical behavior' of states in FQM is distinctly different from the notion of 'classical' in classical mechanics. Hence, although KvN is classical in the sense of classical mechanics, it is not classical in the sense of FQM[31], and there is no contradiction here. The conclusion still remains, KvN and OQM are equivalent in terms of interpretations of FQM.

Another counter point to this conclusion could be that OQM is 'more fundamental' than KvN, meaning that OQM can account for all physical phenomena that KvN can and more. It is true that KvN cannot account for all physical phenomena that OQM can. In particular, it cannot account for the actual double slit experiment as done with electrons [4]. However, as KvN is not (re)obtainded thorugh some $\hbar \to 0$ limit [23], OQM cannot account for all that KvN can. As such KvN cannot be reduced to OQM.[32] Hence one cannot employ reductionism to negate the conclusion that KvN and OQM are equivalent in terms of interpretations of FQM.

Now, classical mechanics typically is taken as the archetype of a realist theory, it hence follows that FQM itself can have nothing to do with realism. Furthermore, FQM itself says nothing about non-locality [9], which was also discussed in the introduction. As presented in [1], interpretations of FQM can be classified as being realist/non-realist, local/non-local with the third category being how probability is interpreted. This together with conclusion of this article—in terms of FQM, that the interpretational issues of OQM and KvN are the same—implies that <u>the</u> interpretational issue is *probability*.

6. Conclusions

In Section 2 it was shown that quantum interference is a purely probabilistic phenomena, not located solely to the actual double-slit experiments as performed with photons and electrons e.t.c. In Section 3 it was shown that KvN is equivalent to classical (statistical) mechanics and that it contains quantum interference. Specifically, statistical equilibria of the LE were shown to correspond to eigenstates of the KvN-generator of time-evolution $\widehat{T}$. As an effect the statistical non-equilibria corresponds to superpositions of these eigenstates. In non-trivial Hamiltonian systems both the observables of momentum and positions

[31] This latter statement does not even make any sense since the notion of classical refers to particular states.

[32] The view of classical mechanics as being reducible to OQM is common. This reductionistic view is often also applied on the larger lanscape of physics, and the natural sciences is in general. It is for instance claimed that thermodynamics is just statistical mechanics and that chemestry is just physics. However, as shown in [23,24], and also pointed out by Dyson [25], this type of reductionistic relation does not hold in general. Concrete demonstrations of the failure of reductionism can be found in [23,24]. In these refererences and in [26] it is furthermore discussed what this non-reductionism implies for the hierarchical ordering of theories in terms of a contextified view of emergence versus reduction and ontic versus epistemic.

are non-commuting with $\widehat{T}$ as they satisfy the equations of motion (81). It has hence been demonstrated that the necessity for quantum interference—that there existing non-commutative physical observables—is satisfied for KvN. In addition it was in Section 4 demonstrated that both OQM and KvN can be seen as merely two unitarily inequivalent representations of the same structure (1). This inequivalence being induced by further physical constraints which by themselves are not 'quantum', e.g enforcements of quantized energy levels. As such that it has been demonstrated that KvN is just as 'quantum' as OQM, and hence is subjected to the same foundational issues. From this it was in turn concluded that the real issue is not FQM but the concept of probability itself.

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Abbreviations

Abbreviations

The following abbreviations are used in this manuscript:

CCR	Canonical commutation relations
CoM	Canonical momentum
FTP	Formalism of total probability
FQM	Formalism of quantum mechanics
KvN	the Koopman-von Neumann formalism of classical mechanics
LE	Classical Liouville equation
OQM	Ordinary quantum mechanics
SE	Schrödinger equation

Appendix A

To show that it imposing $\widehat{U}$ to be strongly continuous by itself may impose restrictions that limit the set of possible eigenvectors. We consider the KvN representation of the simple harmonic oscillator. It is described by the Hamiltonian

$$H(p,q) = \frac{p^2}{2} + \frac{q^2}{2}. \tag{A1}$$

From this it follows that the corresponding Hamiltonian flow is given by

$$U_t(p,q) = (p\cos t - q\sin t, q\cos t + p\sin t), \tag{A2}$$

We consider the action-angle coordinates

$$(\omega, E) : (p,q) \in \mathbb{R}^2 \mapsto (-\Omega(p,q), H(p,q)) \in (-\pi,\pi] \times [0,\infty), \tag{A3}$$

where

$$\Omega(p,q) := \begin{cases} \arctan\left(\frac{p}{q}\right), & q \in (0,\infty) \\ \arctan\left(\frac{p}{q}\right) + \pi, & (p,q) \in [0,\infty) \times (-\infty,0) \\ \arctan\left(\frac{p}{q}\right) - \pi, & (p,q) \in (-\infty,0) \times (-\infty,0) \\ \frac{\pi}{2} & (p,q) \in (0,\infty) \times \{0\} \\ -\frac{\pi}{2} & (p,q) \in (-\infty,0) \times \{0\} \end{cases} \tag{A4}$$

and otherwise undefined. Naively, since indeed

$$\{H, -\Omega\} = 1, \tag{A5}$$

it follows that any eigenvector in the corresponding KvN representation is of the form

$$\psi_{n,f}(p,q) = \frac{1}{\sqrt{2\pi}} e^{in\Omega(p,q)} f(H(p,q)) \tag{A6}$$

where n is the eigenvalue, and where $f \in L^2([0,\infty))$ and normalized but otherwise unspecified. Furthermore, since

$$\Omega \circ U_{-t} = \Omega - (t - \pi k_t), \tag{A7}$$

where k_t is the integer such that

$$\frac{t}{\pi} - \frac{1}{2} < k_t < \frac{t}{\pi} + \frac{1}{2}, \tag{A8}$$

it follows that

$$\widehat{U}_t \psi_{n,f} = e^{ink_t\pi} e^{-int} \psi_{n,f}. \tag{A9}$$

Because of this and since $\widehat{U}_{2\pi} = I$ and $k_{2\pi} = 2i$, it follows that the eigenvalues n are the integers. Hence we must have

$$\widehat{U}_t \psi_{n,f} = (-1)^{nk_t} e^{-int} \psi_{n,f}. \tag{A10}$$

However, as

$$\widehat{U} : t \mapsto \widehat{U}_t \tag{A11}$$

is supposed to be a strongly continuous one-parameter group we must by necessity have that

$$\lim_{t \to 0} \|(-1)^{nk_t} e^{-int} \psi_{n,f} - \psi_{n,f}\| = 0 \tag{A12}$$

for all $n \in \mathbb{Z}$. This is equivalent to

$$\lim_{t \to 0} |(-1)^{nk_t} e^{-int} - 1| = 0, \tag{A13}$$

which in turn holds if and only if n is even. As such the KvN representation is on the sub-Hilbert spaces

$$\text{span}_{\mathbb{C}} \left\{ \psi_{2n,f} \right\}_{n \in \mathbb{Z}, f \in L^2([0,\infty))} \tag{A14}$$

of $L^2(\mathbb{R}^2)$.

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