

ON THE DISTRIBUTION OF THE NONTRIVIAL ZEROS FOR THE DIRICHLET L -FUNCTIONS

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ABSTRACT. This paper addresses a variant of the product for the Dirichlet L -functions. We propose a completely detailed proof for the truth of the generalized Riemann conjecture for the Dirichlet L -functions, which states that the real part of the nontrivial zeros is $1/2$. The Wang and Hardy–Littlewood theorems are also discussed with removing the need for it. The results are applicable to show the truth of the Goldbach’s conjecture.

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1. INTRODUCTION

The Dirichlet L -function $L(s, \chi)$ was formulated by German mathematician Johann Peter Gustav Lejeune Dirichlet in 1837 for the meromorphic continuation of the function defined by the series [1]

$$(1) \quad L(s, \chi) = \sum_{n=1}^{\infty} \chi(n) s^{-n},$$

where $\chi(n)$ is a Dirichlet character (mod $q > 1$), $Re(s) > 1$, $n \in \mathbb{N}$ and $s \in \mathbb{C}$. Here, q is the prime, \mathbb{N} and \mathbb{C} are the sets of the natural numbers and complex numbers, and $Re(s) = \sigma \in \mathbb{R}$ and $Im(s) = t \in \mathbb{R}$ are the real and imaginary parts of the complex variable $s = \sigma + it \in \mathbb{C}$, where $i = \sqrt{-1}$.

For $Re(s) > 1$ the Euler product representation for Eq. (1) can be expressed in the form [2]:

$$(2) \quad L(s, \chi) = \prod_p (1 - \chi(p) p^{-s})^{-1}.$$

Let the primitive character χ^* with the modulus $q > 1$ such that [3]

$$(3) \quad \chi(n) = \begin{cases} \chi^*(n) & \text{if } \gcd(n, q) = 1, \\ 0 & \text{if } \gcd(n, q) \neq 1. \end{cases}$$

By Eq. (3) and $Re(s) > 1$, we rewrite Eq. (1) as [3]:

$$(4) \quad L(s, \chi) = L(s, \chi^*) \prod_{p|q} \left(1 - \frac{\chi^*(p)}{p^s}\right),$$

where

$$(5) \quad L(s, \chi^*) = \prod_p (1 - \chi^*(p) p^{-s})^{-1}.$$

Let χ_0 be the principal character with the modulus $q > 1$ such that [2, 3]

$$(6) \quad L(s, \chi_0) = \zeta(s) \prod_{p|q} \left(1 - \frac{1}{p^s}\right) \quad (Re(s) > 1),$$

where the Riemann zeta function is denoted by [4]

$$(7) \quad \zeta(s) = \sum_{n=1}^{\infty} s^{-n} \quad (Re(s) > 1).$$

Let χ^* be a primitive character modulo q with $q > 1$. Then the product for $L(s, \chi)$ can be represented as follows [3]:

$$(8) \quad L(s, \chi^*) = \left(\frac{q}{\pi}\right)^{-\frac{s+h(\chi^*)}{2}} \cdot \frac{\xi(s, \chi^*)}{\Gamma\left(\frac{s+h(\chi^*)}{2}\right)}$$

subject to

$$(9) \quad h(\chi^*) = \begin{cases} 0 & \text{if } \chi(-1) = 1, \\ 1 & \text{if } \chi(-1) = -1, \end{cases}$$

and

$$(10) \quad \xi(s, \chi^*) = \xi(0, \chi^*) \cdot e^{B(\chi^*)s} \cdot \prod_{k=1}^{\infty} \left(1 - \frac{s}{\rho_k}\right) e^{s/\rho_k}$$

where $s \in \mathbb{C}$, $\xi(0, \chi^*) \neq 0$, $B(\chi^*) = \xi^{(1)}/\xi(0, \chi^*)$, Γ is the gamma function [5], and ρ_k run all zeros of $\xi(s, \chi^*)$.

As the above results, we may present the followings [3]:

Cases 1. For $q = 1$,

$$(11) \quad L(s, \chi) = \zeta(s) \quad (Re(s) > 1);$$

Cases 2. For $\chi = \chi_0$ with the modulus $q > 1$,

$$(12) \quad L(s, \chi) = L(s, \chi_0) = \zeta(s) \prod_{p|q} \left(1 - \frac{1}{p^s}\right) \quad (Re(s) > 1);$$

Cases 3. Let χ^* be a primitive character modulo q with $q > 1$. Then

$$(13) \quad L(s, \chi) = L(s, \chi^*) = \xi(0, \chi^*) \cdot \left(\frac{q}{\pi}\right)^{-\frac{s+h(\chi^*)}{2}} \cdot e^{B(\chi^*)s} \cdot \frac{\prod_{k=1}^{\infty} \left(1 - \frac{s}{\rho_k}\right) e^{s/\rho_k}}{\Gamma\left(\frac{s+h(\chi^*)}{2}\right)}.$$

For **Cases 1** it is well known that Eq. (11) has a pole at $s = 1$ with residue 1 [6], the trivial zeros $s = -2h$ with $h \in \mathbb{N}$ [7], and the nontrivial zeros $s_k = 1/2 + i\varpi_k$ for $k \in \mathbb{N}$ (proved by author in the different methods [8, 9]), where ϖ_k are the Riemann-Siegel zeros, confirmed by the Riemann-Siegel formula [10]. Let \mathbb{Z} be the set of the integral numbers. For **Cases 2** it is seen that Eq. (12) has the zeros $s = 2\pi ij/(\log p)$ with $p|q$ and $j \in \mathbb{Z}$ (see [11], p.460) and a pole at $s = 1$ with residues $v(q)/q$ (see [2], p.334), where $v(q)$ is the Euler's totient function. For **Cases 3** it is shown that Eq. (13) has simple zeros $s = -h(\chi^*) - 2h$ with $h \in \mathbb{N}$ (see [11], p.460; [2] p.333) and the nontrivial zeros $\rho_k = \alpha_k + i\beta_k$, where β_k are the

Siegel zeros, confirmed by the Siegel formula [12]. Let $N(T, \chi)$ denote the number of zeros of Eq. (13) for $\operatorname{Re}(s) = 1/2$, $T > 2$ and $|\operatorname{Im}(s)| \leq T$ such that [13]

$$(14) \quad N(T, \chi) = \frac{T}{\pi} \log \frac{qT}{2\pi} - \frac{T}{\pi} + O\left(\frac{\log(qT)}{\log \log(qT)}\right).$$

It is clearly seen that the nontrivial zeros of Eq. (13) exist and that Eq. (13) has infinitely many nontrivial zeros by using Eq. (14).

Conjecture 1. *The generalized Riemann conjecture for Eq. (1) states the real part of the nontrivial zeros is $1/2$.*

According to Davenport (see [2], p.124), *Conjecture 1* was formulated in 1884 by Adolf Piltz. There have been applicable to consider the Goldbach's conjecture by Hardy and Littlewood in 1923 [14, 15] and by Wang in 1962 [16]. Hardy and Littlewood [17, 18, 19] proved that if *Conjecture 1* is true, almost all even numbers are sums of two primes and that every large odd number is the sum of three primes.

Recently, author have proposed two classes of the Riemann zeta function, which can be represented by (the first term was by Hadamard [9, 11, 20] and the second term was by author [9])

$$(15) \quad \begin{aligned} \zeta(s) &= \xi(s) / \Re(s) \\ &= \frac{\xi(0)e^{s\Im}}{(s-1)\Gamma(\frac{s}{2}+1)} \prod_{k=1}^{\infty} \left(1 - \frac{s}{s_k}\right) e^{s/s_k} \\ &= \frac{\xi(1/2)e^{s\Im}}{(s-1)\Gamma(\frac{s}{2}+1)} \prod_{k=1}^{\infty} \left(1 - \frac{s-\frac{1}{2}}{s_k-\frac{1}{2}}\right) e^{s/s_k} \\ &= \frac{\xi(s)e^{s\Im}}{(s-1)\Gamma(\frac{s}{2}+1)} \prod_{k=1}^{\infty} e^{s/s_k}, \end{aligned}$$

provided that there exists the entire Riemann zeta function, given as (the first term was by Hadamard [9, 11, 20] and the second term was by author [9])

$$(16) \quad \xi(s) = \xi(0) e^{s\Im_0} \prod_{k=1}^{\infty} \left(1 - \frac{s}{s_k}\right) e^{s/s_k} = \xi(1/2) e^{s\Im_0} \prod_{k=1}^{\infty} \left(1 - \frac{s-\frac{1}{2}}{s_k-\frac{1}{2}}\right) e^{s/s_k},$$

where

$$(17) \quad \Re(s) = (s-1) \pi^{-s/2} \Gamma(s/2+1),$$

$$(18) \quad \Im = \log 2\pi - 1 - \frac{\tilde{a}}{2},$$

and

$$(19) \quad \Im_0 = \log 2\pi - 1 - \frac{\tilde{a}}{2} - \frac{1}{2} \log \pi,$$

with the Euler's constant \tilde{a} .

As is well known, *Conjecture 1* is an unsolved important mathematical problem in analytic number theory up to now. By inspired by the above results to structure the variant of the product for the Riemann zeta function, the main targets of the paper are to propose a variant of the product for Eq. (13), to prove *Conjecture 1* for Eq. (1) and to present a detailed account of applications of the Hardy–Littlewood and Wang theorems to the Goldbach's conjecture directly removing the need for *Conjecture 1*. The structure of the paper is given as follows. In Section 2, we introduce the results related to the Riemann zeta function and Dirichlet L -functions. In Section 3, we give the detailed proof of *Conjecture 1*. In Section 4 we apply to obtain the properties via Dirichlet L -functions, get the representations for Wang theorems, and apply the Hardy–Littlewood theorems to obtain the Goldbach's conjecture. Finally, we draw the conclusion in Section 5.

2. FUNDAMENTAL RESULTS

We now consider the variants of the products for the Riemann zeta function and Dirichlet L -functions.

2.1. A variant of the product for the Riemann zeta function. We now consider the variant of the product for the Riemann zeta function.

Lemma 1. *There exists*

$$(20) \quad \zeta(1) \neq 0,$$

$$(21) \quad \zeta(0) = -\frac{1}{2} \neq 0,$$

and

$$(22) \quad \zeta\left(\frac{1}{2}\right) \neq 0.$$

Proof. See the results of Landau [11]. □

Remark. Eq. (20) was proved in 1896 by Hadamard [21] and Vallee-Poussin [22]. Eqs. (21) and (22) were discussed in Landau [11].

It is proved by Riemann [4] that

$$(23) \quad \zeta(s) = \frac{\xi(s)}{\Re(s)},$$

where [11, 20]

$$(24) \quad \zeta(s) = \xi(s) / \Re(s) = \frac{\xi(0) e^{s\Im}}{(s-1) \Gamma\left(\frac{s}{2} + 1\right)} \prod_{k=1}^{\infty} \left(1 - \frac{s}{s_k}\right) e^{s/s_k}$$

and [8, 9]

$$(25) \quad \zeta(s) = \xi(s) / \Re(s) = \frac{\xi(1/2) e^{s\Im}}{(s-1) \Gamma\left(\frac{s}{2} + 1\right)} \prod_{k=1}^{\infty} \left(1 - \frac{s - \frac{1}{2}}{s_k - \frac{1}{2}}\right) e^{s/s_k},$$

where

$$(26) \quad \Re(s) = (s-1) \pi^{-s/2} \Gamma(s/2 + 1),$$

and

$$(27) \quad \Im = \log 2\pi - 1 - \tilde{a}/2$$

with the Euler's constant \tilde{a} .

Lemma 2. *There exists*

$$(28) \quad \xi(1) = \xi(0) = \frac{1}{2} \neq 0,$$

and

$$(29) \quad \xi\left(\frac{1}{2}\right) \neq 0.$$

Proof. See the results of Broughan (see [23], p.49). □

Lemma 3. *Let $\Im_0 = \log 2\pi - 1 - \frac{\tilde{a}}{2} - \frac{1}{2} \log \pi$, where \tilde{a} is the Euler's constant. Then two classes of the entire Riemann zeta function are equivalent:*

$$(30) \quad \xi(s) = \xi(0) e^{s\Im_0} \prod_{k=1}^{\infty} \left(1 - \frac{s}{s_k}\right) e^{s/s_k}$$

and

$$(31) \quad \xi(s) = \xi(1/2) e^{s\Im_0} \prod_{k=1}^{\infty} \left(1 - \frac{s - \frac{1}{2}}{s_k - \frac{1}{2}}\right) e^{s/s_k},$$

where s_k run the zeros for $\xi(s)$, $s \in \mathbb{C}$ and $k \in \mathbb{N}$.

Proof. According to Hadamard [20] and Landau [11], we have

$$\begin{aligned}
 & \xi(s) \\
 &= \xi(0) e^{s\Im_0} \prod_{k=1}^{\infty} \left(1 - \frac{s}{s_k}\right) e^{s/s_k} \\
 &= \xi(0) e^{s\Im_0} \prod_{k=1}^{\infty} \frac{s_k - s}{s_k} e^{s/s_k} \\
 &= \xi(0) e^{s\Im_0} \prod_{k=1}^{\infty} \left(\frac{s_k - \frac{1}{2}}{s_k - \frac{1}{2}} \cdot \frac{s_k - s}{s_k}\right) e^{s/s_k} \\
 (32) \quad &= \xi(0) e^{s\Im_0} \prod_{k=1}^{\infty} \frac{s_k - \frac{1}{2}}{s_k} \prod_{k=1}^{\infty} \left(\frac{s_k - s}{s_k - \frac{1}{2}}\right) e^{s/s_k} \\
 &= \xi(0) e^{s\Im_0} \prod_{k=1}^{\infty} \frac{s_k - \frac{1}{2}}{s_k} \prod_{k=1}^{\infty} \left(1 - \frac{s - \frac{1}{2}}{s_k - \frac{1}{2}}\right) e^{s/s_k} \\
 &= \xi(0) e^{s\Im_0} \prod_{k=1}^{\infty} \left(1 - \frac{1}{2s_k}\right) \prod_{k=1}^{\infty} \left(1 - \frac{s - \frac{1}{2}}{s_k - \frac{1}{2}}\right) e^{s/s_k} \\
 &= \xi(0) e^{(s - \frac{1}{2})\Im_0} \prod_{k=1}^{\infty} \left(1 - \frac{s - \frac{1}{2}}{s_k - \frac{1}{2}}\right) e^{s/s_k}
 \end{aligned}$$

where

$$(33) \quad \xi\left(\frac{1}{2}\right) = \xi(0) e^{\frac{\Im_0}{2}} \prod_{k=1}^{\infty} \left(1 - \frac{1}{2s_k}\right) e^{1/2s_k} = \xi(0) \prod_{k=1}^{\infty} \left(1 - \frac{1}{2s_k}\right),$$

which is derived by [6, 11, 20]

$$(34) \quad \xi(s) = \xi(0) e^{s\Im_0} \prod_{k=1}^{\infty} \left(1 - \frac{s}{s_k}\right) e^{s/s_k} = \xi(0) \prod_{k=1}^{\infty} \left(1 - \frac{s}{s_k}\right)$$

and

$$(35) \quad e^{\frac{\Im_0}{2}} \prod_{k=1}^{\infty} e^{1/2s_k} = 1.$$

Thus, the proof of Lemma 3 is completed. \square

Lemma 4. Assume that $s \in \mathbb{C}$, $k \in \mathbb{N}$, and $\Im_0 = \log 2\pi - 1 - \tilde{a}/2$, where \tilde{a} is the Euler's constant. Then two classes of the Riemann zeta function are equivalent:

$$(36) \quad \zeta(s) = \frac{\xi(0) e^{s\Im}}{(s-1) \Gamma\left(\frac{s}{2} + 1\right)} \prod_{k=1}^{\infty} \left(1 - \frac{s}{s_k}\right) e^{s/s_k}$$

and [5, 9]

$$(37) \quad \zeta(s) = \frac{\xi(1/2) e^{s\Im}}{(s-1) \Gamma\left(\frac{s}{2} + 1\right)} \prod_{k=1}^{\infty} \left(1 - \frac{s - \frac{1}{2}}{s_k - \frac{1}{2}}\right) e^{s/s_k},$$

Proof. By Eq. (23), Lemma 3 and $\Re(s) = (s-1)\pi^{-s/2}\Gamma(s/2+1)$, we obtain the required results. \square

Remark. For the detailed proofs of Lemmas 3 and 4, see [8, 9].

According to Patterson [23] we have

$$(38) \quad s(s-1)\Gamma\left(\frac{s}{2}\right)\zeta(s) = \prod_{k=1}^{\infty} \left(1 - \frac{s}{s_k^*}\right) \left(1 - \frac{s}{1-s_k^*}\right)$$

such that [8]

$$(39) \quad \xi(s) = \frac{1}{2} \prod_{k=1}^{\infty} \left(1 - \frac{s}{s_k^*}\right) \left(1 - \frac{s}{1-s_k^*}\right) = \xi(0) \prod_{k=1}^{\infty} \left(1 - \frac{s}{s_k^*}\right) \left(1 - \frac{s}{1-s_k^*}\right),$$

where $s \in \mathbb{C}$, $k \in \mathbb{N}$, and

$$(40) \quad s_k^* = \operatorname{Re}(s_k) + i|\operatorname{Im}(s_k)|.$$

It is proved by author [8] that for $s \in \mathbb{C}$,

$$(41) \quad \xi(s) = \xi(0) \prod_{k=1}^{\infty} \left(1 - \frac{s}{s_k^*}\right) \left(1 - \frac{s}{1-s_k^*}\right) = \xi\left(\frac{1}{2}\right) \prod_{k=1}^{\infty} \left[1 - \frac{\left(s - \frac{1}{2}\right)^2}{\left(s_k^* - \frac{1}{2}\right)^2}\right],$$

which implies from Eq. (23) that [8]

$$(42) \quad \zeta(s) = \frac{\xi(s)}{\Re(s)} = \frac{\xi(0)\pi^{s/2}}{(s-1)\Gamma(s/2+1)} \prod_{k=1}^{\infty} \left(1 - \frac{s}{s_k^*}\right) \left(1 - \frac{s}{1-s_k^*}\right),$$

$$(43) \quad \zeta(s) = \frac{\xi(s)}{\Re(s)} = \frac{\xi(0)\pi^{s/2}}{(s-1)\Gamma(s/2+1)} \prod_{k=1}^{\infty} \left[1 - \frac{s(1-s)}{s_k^*(1-s_k^*)}\right],$$

and [8]

$$(44) \quad \zeta(s) = \frac{\xi(s)}{\Re(s)} = \frac{\xi\left(\frac{1}{2}\right)\pi^{s/2}}{(s-1)\Gamma(s/2+1)} \prod_{k=1}^{\infty} \left[1 - \frac{\left(s - \frac{1}{2}\right)^2}{\left(s_k^* - \frac{1}{2}\right)^2}\right],$$

where $s \in \mathbb{C}$, $s \neq 1$, and

$$(45) \quad \Re(s) = (s-1)\pi^{-s/2}\Gamma(s/2+1).$$

Now, from Eq. (42), we see that s_k , $1-s_k$, s_k^* and $1-s_k^*$ are the zeros for $\xi(s)$ and the nontrivial zeros for $\zeta(s)$ [8]. Moreover, $\xi(s)$ is a meromorphic continuation to the entire complex plane s with pole of residue 1 [6], and $\xi(s)$ is an integral function of order 1 and simple in the entire complex plane s [8]. By Eqs. (31) and (44), it is proved that $\operatorname{Re}(s_k) = 1/2$, in other words that the Riemann conjecture is true (for the detailed proof, see [8, 9]).

2.2. A variant of the product for the Dirichlet L -functions. In this part we present the variant of the product for the Dirichlet L -functions. We now denote the primitive character modulo q with $q > 1$ by $\chi^* = \chi^*(n)$.

Lemma 5. *If χ^* is a primitive character modulo q with $q > 1$ and $\chi^*(n) \neq \chi_0$, then*

$$(46) \quad L(1, \chi^*) \neq 0,$$

$$(47) \quad L(0, \chi^*) \neq 0$$

and

$$(48) \quad L\left(\frac{1}{2}, \chi^*\right) \neq 0.$$

Proof. For the detailed proof of Eq. (46), see the work of Dirichlet [1, 3]. Moreover, Eqs. (47) and (48) were proved in [25]. \square

Lemma 6. *If χ^* is a primitive character modulo q with $q > 1$, then we have that*

$$(49) \quad \xi(s, \chi^*) = L(s, \chi^*) \cdot \left(\frac{q}{\pi}\right)^{\frac{s+\hbar(\chi^*)}{2}} \cdot \Gamma\left(\frac{s+\hbar(\chi^*)}{2}\right),$$

is an entire function of order 1, where $s \in \mathbb{C}$ and $\hbar(\chi^*)$ is defined in (9).

Proof. For the detailed proof of Lemma 6, see [3]. \square

Lemma 7. *If χ^* is a primitive character modulo q with $q > 1$, then we have*

$$(50) \quad \xi(1, \chi^*) \neq 0,$$

$$(51) \quad \xi(0, \chi^*) \neq 0$$

and

$$(52) \quad \xi\left(\frac{1}{2}, \chi^*\right) \neq 0.$$

Proof. For the detailed proofs of Eqs. (50), (51) and (52), see [11]. \square

Lemma 8. *Suppose that χ^* is a primitive character modulo q with $q > 1$ and ρ_k run the nontrivial zeros of $L(s, \chi^*)$. Then there is*

$$(53) \quad \xi(s, \chi^*) = \xi(0, \chi^*) e^{B(\chi^*)s} \prod_{k=1}^{\infty} \left(1 - \frac{s}{\rho_k}\right) e^{s/\rho_k},$$

where $s \in \mathbb{C}$ and

$$(54) \quad B(\chi^*) = \frac{\xi^{(1)}(0, \chi^*)}{\xi}.$$

Proof. For the detailed proof of Lemma 8, see [3]. \square

Lemma 9. Suppose that χ^* is a primitive character modulo q with $q > 1$ and ρ_k run the nontrivial zeros of $L(s, \chi^*)$. Then there is

$$(55) \quad \frac{\xi^{(1)}}{\xi}(s, \chi^*) = B(\chi^*) + \sum_{k=1}^{\infty} \left(\frac{1}{s - \rho_k} + \frac{1}{\rho_k} \right).$$

Proof. For the detailed proof of Lemma 9, see [3]. \square

Lemma 10. Suppose that χ^* is a primitive character modulo q with $q > 1$ and ρ_k run the nontrivial zeros of $L(s, \chi^*)$. Then we have for $s \in \mathbb{C}$,

$$(56) \quad L(s, \chi^*) = B(\chi^*) \Phi(s, \hbar(\chi^*)) L(1-s, \overline{\chi^*}),$$

and

$$(57) \quad \xi(s, \chi^*) = \varepsilon(\chi^*) \xi(1-s, \overline{\chi^*}),$$

provided that

$$(58) \quad \Phi(s, \hbar(\chi^*)) = 2^s \pi^{1/2-s} \Gamma(1-s) \sin \frac{\pi}{2} (s + \hbar(\chi^*)),$$

where $\hbar(\chi^*)$ is defined in (9),

$$(59) \quad \varepsilon(\chi^*) = \frac{\tau(\chi^*)}{i^{\hbar(\chi^*)} \sqrt{q}},$$

the Gauss sum $\tau(\chi^*)$ of χ^* is denoted by

$$(60) \quad \tau(\chi^*) = \sum_{\theta=1}^q \chi^*(\theta) e^{\theta/q},$$

and $\overline{\chi^*}$ is the complex conjugate character to χ^* .

Proof. For the detailed proof of Lemma 10, see [2, 3]. \square

Lemma 11. Suppose that χ^* is a primitive character modulo q with $q > 1$. Then we have

$$(61) \quad |\tau(\chi^*)| = \sqrt{q}.$$

Proof. For the detailed proof of Lemma 11, see [3]. \square

Thus, we see that

$$(62) \quad \tau(\chi^*) \neq 0$$

if χ^* is a primitive character modulo q with $q > 1$.

Theorem 1. Suppose that χ^* is a primitive character modulo q with $q > 1$ and ρ_k run the nontrivial zeros of $L(s, \chi^*)$ with $k \in \mathbb{N}$. Then there exist the equivalent representations:

$$(63) \quad \xi(s, \chi^*) = \xi(0, \chi^*) e^{B(\chi^*)s} \prod_{k=1}^{\infty} \left(1 - \frac{s}{\rho_k}\right) e^{s/\rho_k},$$

and

$$(64) \quad \xi(s, \chi^*) = \xi\left(\frac{1}{2}, \chi^*\right) e^{(s-\frac{1}{2})B(\chi^*)} \prod_{k=1}^{\infty} \left(1 - \frac{s-\frac{1}{2}}{\rho_k - \frac{1}{2}}\right) e^{(s-1/2)/\rho_k},$$

where $s \in \mathbb{C}$ and

$$(65) \quad B(\chi^*) = \frac{\xi^{(1)}}{\xi}(0, \chi^*).$$

Proof. From Eqs. (51) and (52),

$$(66) \quad \xi\left(\frac{1}{2}, \chi^*\right) = \xi(0, \chi^*) e^{\frac{B(\chi^*)}{2}} \prod_{k=1}^{\infty} \left(1 - \frac{1}{2\rho_k}\right) \prod_{k=1}^{\infty} e^{1/(2\rho_k)} \neq 0.$$

By Lemma 8, we get

$$(67) \quad \begin{aligned} & \xi(s, \chi^*) \\ &= \xi(0, \chi^*) e^{B(\chi^*)s} \prod_{k=1}^{\infty} e^{s/\rho_k} \prod_{k=1}^{\infty} \left(1 - \frac{s}{\rho_k}\right) \\ &= \xi(0, \chi^*) e^{B(\chi^*)s} \prod_{k=1}^{\infty} e^{s/\rho_k} \prod_{k=1}^{\infty} \left(\frac{\rho_k - s}{\rho_k}\right) \\ &= \xi(0, \chi^*) e^{B(\chi^*)s} \prod_{k=1}^{\infty} e^{s/\rho_k} \prod_{k=1}^{\infty} \left(\frac{\rho_k - \frac{1}{2}}{\rho_k - \frac{1}{2}} \cdot \frac{\rho_k - s}{\rho_k}\right) \\ &= \xi(0, \chi^*) e^{B(\chi^*)s} \prod_{k=1}^{\infty} e^{s/\rho_k} \prod_{k=1}^{\infty} \left(\frac{\rho_k - \frac{1}{2}}{\rho_k} \cdot \frac{\rho_k - s}{\rho_k - \frac{1}{2}}\right) \\ &= \xi(0, \chi^*) e^{B(\chi^*)s} \prod_{k=1}^{\infty} \frac{\rho_k - \frac{1}{2}}{\rho_k} \prod_{k=1}^{\infty} e^{s/\rho_k} \prod_{k=1}^{\infty} \left(\frac{\rho_k - s}{\rho_k - \frac{1}{2}}\right) \\ &= \xi(0, \chi^*) e^{B(\chi^*)s} \prod_{k=1}^{\infty} \frac{\rho_k - \frac{1}{2}}{\rho_k} \prod_{k=1}^{\infty} e^{s/\rho_k} \prod_{k=1}^{\infty} \left[\frac{(\rho_k - \frac{1}{2}) - (s - \frac{1}{2})}{\rho_k - \frac{1}{2}}\right] \\ &= \xi(0, \chi^*) e^{B(\chi^*)s} \prod_{k=1}^{\infty} \frac{\rho_k - \frac{1}{2}}{\rho_k} \prod_{k=1}^{\infty} e^{s/\rho_k} \prod_{k=1}^{\infty} \left(1 - \frac{s - \frac{1}{2}}{\rho_k - \frac{1}{2}}\right) \\ &= \xi(0, \chi^*) e^{B(\chi^*)s} \prod_{k=1}^{\infty} \frac{\rho_k - \frac{1}{2}}{\rho_k} \prod_{k=1}^{\infty} \left(1 - \frac{s - \frac{1}{2}}{\rho_k - \frac{1}{2}}\right) e^{s/\rho_k} \\ &= \xi(0, \chi^*) e^{sB(\chi^*)} \prod_{k=1}^{\infty} \left(1 - \frac{1}{2\rho_k}\right) \prod_{k=1}^{\infty} \left(1 - \frac{s - \frac{1}{2}}{\rho_k - \frac{1}{2}}\right) e^{s/\rho_k} \\ &= \xi\left(\frac{1}{2}, \chi^*\right) e^{(s-\frac{1}{2})B(\chi^*)} \prod_{k=1}^{\infty} \left(1 - \frac{s - \frac{1}{2}}{\rho_k - \frac{1}{2}}\right) e^{(s-1/2)/\rho_k}, \end{aligned}$$

which is the required result. \square

Theorem 2. Suppose that χ^* is a primitive character modulo q with $q > 1$ and ρ_k run the nontrivial zeros of $L(s, \chi^*)$ with $k \in \mathbb{N}$. Then there exist the equivalent representations:

$$(68) \quad L(s, \chi^*) = \frac{\xi(0, \chi^*) \cdot \left(\frac{q}{\pi}\right)^{-\frac{s+h(\chi^*)}{2}} \cdot e^{B(\chi^*)s}}{\Gamma\left(\frac{s+h(\chi^*)}{2}\right)} \cdot \prod_{k=1}^{\infty} \left(1 - \frac{s}{\rho_k}\right) e^{s/\rho_k},$$

and

$$(69) \quad L(s, \chi^*) = \frac{\xi\left(\frac{1}{2}, \chi^*\right) \cdot \left(\frac{q}{\pi}\right)^{-\frac{s+h(\chi^*)}{2}} \cdot e^{(s-\frac{1}{2})B(\chi^*)}}{\Gamma\left(\frac{s+h(\chi^*)}{2}\right)} \cdot \prod_{k=1}^{\infty} \left(1 - \frac{s - \frac{1}{2}}{\rho_k - \frac{1}{2}}\right) e^{(s-1/2)/\rho_k},$$

where $s \in \mathbb{C}$ and

$$(70) \quad B(\chi^*) = \frac{\xi^{(1)}}{\xi}(0, \chi^*).$$

Proof. With use of Lemma 6 and Theorem 1, we present [11]

$$(71) \quad L(s, \chi^*) = \frac{\xi(0, \chi^*) \cdot \left(\frac{q}{\pi}\right)^{-\frac{s+h(\chi^*)}{2}} \cdot e^{B(\chi^*)s}}{\Gamma\left(\frac{s+h(\chi^*)}{2}\right)} \cdot \prod_{k=1}^{\infty} \left(1 - \frac{s}{\rho_k}\right) e^{s/\rho_k},$$

and

$$(72) \quad L(s, \chi^*) = \frac{\xi\left(\frac{1}{2}, \chi^*\right) \cdot \left(\frac{q}{\pi}\right)^{-\frac{s+h(\chi^*)}{2}} \cdot e^{(s-\frac{1}{2})B(\chi^*)}}{\Gamma\left(\frac{s+h(\chi^*)}{2}\right)} \cdot \prod_{k=1}^{\infty} \left(1 - \frac{s - \frac{1}{2}}{\rho_k - \frac{1}{2}}\right) e^{(s-1/2)/\rho_k},$$

which are the desired results, where $s \in \mathbb{C}$. \square

Remark. There exists (see [2], p.83)

$$(73) \quad B(\chi^*) = \frac{\xi^{(1)}}{\xi}(0, \chi^*) = -\frac{\xi^{(1)}}{\xi}(1, \chi^*) = -B(\overline{\chi^*}) - \sum_{k=1}^{\infty} \left(\frac{1}{1-\rho_k} + \frac{1}{\rho_k}\right).$$

Lemma 12. Let χ^* be a primitive character modulo q with $q > 1$. $L(s, \chi^*)$ has an infinity of zeros ρ_k in the critical strip $0 \leq \operatorname{Re}(s) \leq 1$.

Proof. See the result of Davenport (see [2], p.82). \square

2.3. The Goldbach's problems. We now introduce the Hardy-Littlewood and Wang's theorems with the need of *Conjecture 1*.

Lemma 13. (*Hardy-Littlewood Theorem I*) [14]

If Conjecture 1 is true, then every odd number $m > 5$ is the sum of three primes.

Proof. See the work of Hardy and Littlewood [14] and the paper of Deshouillers and coauthors [15] under the condition of truth of *Conjecture 1*. *Lemma 13* was proved in 2013 by Helfgott [26] without *Conjecture 1* and a detailed account of the numerical verification for *Lemma 13* was shown by Helfgott and Platt in 2013 [27]. \square

For the sake of brevity, we denote the following proposition by $(1, \mathbb{X})$ [16, 28]:

Every sufficiently large even integer is a sum of a prime and an almost prime of at most \mathbb{X} prime divisors.

Lemma 14. (*Wang Theorem I*) [16]

If Conjecture 1 is true, then $(1, 4)$ is valid, where $\delta_2 \geq 3.237/2.237$.

Proof. See the work of Wang [16]. \square

Lemma 15. (*Wang Theorem II*) [16]

If Conjecture 1 is true, then $(1, 3)$ is valid, where $\delta_1 \geq 2.475/1.475$.

Proof. See the work of Wang [16]. \square

In fact, Chen proved in 1973 and 1978 that $(1, 2)$ is true [28, 29]. The records of verification of the strong Goldbach's conjecture holds to $N_1 = 4 \times 10 \wedge 18$ [30].

Lemma 16. (*Hardy-Littlewood Theorem II*) [17]

If Conjecture 1 is true, every even number $m > 2$ is sums of two primes.

Proof. See the work of Hardy and Littlewood [17]. Based on it, the work of Granville [18] gives the detailed proof of *Lemma 16* on the condition of truth of *Conjecture 1*. \square

3. A DETAILED PROOF FOR THE GENERALIZED RIEMANN CONJECTURE

We now apply the variant of the product for the Dirichlet L -functions to present the complete proof for *Conjecture 1*.

3.1. Family 1: $\chi = \chi_0$ and $q = 1$. When $\chi = \chi_0$ and $q = 1$, we have [3]

$$(74) \quad \zeta(s) = \sum_{n=1}^{\infty} s^{-n} \quad (\operatorname{Re}(s) > 1).$$

According to *Lemma 3*, we have the representation

$$\begin{aligned}
 \zeta(s) &= \frac{\xi(0)e^{s\zeta}}{(s-1)\Gamma(\frac{s}{2}+1)} \prod_{k=1}^{\infty} \left(1 - \frac{s}{s_k}\right) e^{s/s_k} \\
 &= \frac{\xi(1/2)e^{s\zeta}}{(s-1)\Gamma(\frac{s}{2}+1)} \prod_{k=1}^{\infty} \left(1 - \frac{s-\frac{1}{2}}{s_k-\frac{1}{2}}\right) e^{s/s_k} \\
 &= \frac{\xi(s)e^{s\zeta}}{(s-1)\Gamma(\frac{s}{2}+1)} \prod_{k=1}^{\infty} e^{s/s_k}
 \end{aligned}
 \tag{75}$$

where $s \in \mathbb{C}$ and $s \neq 1$, and so we have

$$\xi(s) = \xi(0) \prod_{k=1}^{\infty} \left(1 - \frac{s}{s_k}\right) = \xi(1/2) \prod_{k=1}^{\infty} \left(1 - \frac{s-\frac{1}{2}}{s_k-\frac{1}{2}}\right)
 \tag{76}$$

and

$$\Xi(\beta) = \xi\left(\frac{1}{2} + i\beta\right) = \xi(0) \prod_{k=1}^{\infty} \left(1 - \frac{\frac{1}{2} + i\beta}{s_k}\right) = \xi(1/2) \prod_{k=1}^{\infty} \left(1 - \frac{i\beta}{s_k - \frac{1}{2}}\right),
 \tag{77}$$

where $\xi(0) \neq 0$ and $\xi(1/2) \neq 0$ (for the details, see *Lemma 2*).

It is well known that Eqs. (76) and (77) are the entire functions of order 1.

By Eqs. (75) and (76), we have

$$\zeta(s) = 0
 \tag{78}$$

such that

$$s - 1 \neq 0,
 \tag{79}$$

$$\xi(s) = 0
 \tag{80}$$

and

$$\frac{1}{\Gamma(\frac{s}{2}+1)} = 0.
 \tag{81}$$

With use of Eq. (77), we have [8, 9]

$$\Xi(\varpi_k) = \xi(1/2) \prod_{k=1}^{\infty} \left(1 - \frac{i\varpi_k}{s_k - \frac{1}{2}}\right) = 0.
 \tag{82}$$

Since Eq. (76) is an entire function of order 1, it follows that

$$1 - \frac{i\varpi_k}{s_k - \frac{1}{2}} = 0,
 \tag{83}$$

where $k \in \mathbb{N}$.

From Eq. (83), we have

$$(84) \quad s_k = 1/2 + i\varpi_k$$

and [8, 9]

$$(85) \quad \tilde{s}_k = \frac{1}{2} \pm i|\varpi_k|,$$

where $k \in \mathbb{N}$, $i = \sqrt{-1}$, and ϖ_k are the Riemann-Siegel zeros.

Putting Eq. (76) into Eq. (84), we get

$$(86) \quad \xi(s) = \xi(1/2) \prod_{k=1}^{\infty} \left(1 - \frac{s - \frac{1}{2}}{i\varpi_k}\right).$$

Substituting $s_k = \omega_k + i\varpi_k$ into Eq. (87), we get

$$(87) \quad \xi(\omega_k + i\varpi_k) = \xi(1/2) \prod_{k=1}^{\infty} \left(1 - \frac{\omega_k + i\varpi_k - \frac{1}{2}}{i\varpi_k}\right) = 0.$$

Since Eq. (77) is an entire function of order 1, it follows that

$$(88) \quad 1 - \frac{\omega_k + i\varpi_k - \frac{1}{2}}{i\varpi_k} = 0,$$

which leads to

$$(89) \quad \omega_k = 1/2,$$

or, alternatively,

$$(90) \quad s_k = 1/2 + i\varpi_k.$$

With use of Eq. (79), Eq. (75) has a pole at $s = 1$ with residue 1 [6]. By Eq. (81), Eq. (75) has the trivial zeros $s = -2h$ with $h \in \mathbb{N}$ [6]. In view of Eq. (89), we find that Eq. (75) has the nontrivial zeros $s_k = 1/2 + i\varpi_k$ [8, 9].

3.2. Family 2: $\chi = \chi_0$ and $q > 1$. When $\chi = \chi_0$ and $q > 1$, we get [3]

$$(91) \quad L(s, \chi_0) = \zeta(s) \prod_{p|q} \left(1 - \frac{1}{p^s}\right) \quad (\operatorname{Re}(s) > 1).$$

By Lemmas 3, 4, (75) and (91), we show that

$$\begin{aligned}
 (92) \quad & L(s, \chi_0) \\
 &= \zeta(s) \prod_{p|q} \left(1 - \frac{1}{p^s}\right) \\
 &= \frac{\xi(0)e^{s\Im}}{(s-1)\Gamma(\frac{s}{2}+1)} \prod_{k=1}^{\infty} \left(1 - \frac{s}{s_k}\right) e^{s/s_k} \prod_{p|q} \left(1 - \frac{1}{p^s}\right) \\
 &= \frac{\xi(1/2)e^{s\Im}}{(s-1)\Gamma(\frac{s}{2}+1)} \prod_{k=1}^{\infty} \left(1 - \frac{s-\frac{1}{2}}{s_k-\frac{1}{2}}\right) e^{s/s_k} \prod_{p|q} \left(1 - \frac{1}{p^s}\right) \\
 &= \frac{\xi(s)e^{s\Im}}{(s-1)\Gamma(\frac{s}{2}+1)} \prod_{p|q} \left(1 - \frac{1}{p^s}\right) \prod_{k=1}^{\infty} e^{s/s_k},
 \end{aligned}$$

where $s \in \mathbb{C}$ and $s \neq 1$.

By Eq. (92) and Lemma 2, we have

$$(93) \quad L(s, \chi_0) = 0$$

such that

$$(94) \quad 1 - \frac{1}{p^s} = 0$$

and

$$(95) \quad \zeta(s) = 0.$$

By **Family 1**, it is clearly seen that Eq. (95) has the nontrivial zeros $s_k = 1/2 + i\varpi_k$ for $k \in \mathbb{N}$ [8, 9] and the trivial zeros $s = -2h$ with $h \in \mathbb{N}$ [6], and a pole at $s = 1$ with residue $v(q)/q = \prod_{p|q} \left(1 - \frac{1}{p}\right)$ ([3], p.334), where $v(q)$ is the Euler's totient function.

From Eq. (94) it is shown that Eq. (93) has the pure imaginary zeros $s = 2\pi ij / (\log p)$ with $p|q$ and $j \in \mathbb{Z}$ (see [11], p.460).

3.3. Family 3: $\chi = \chi^*$ and $\chi^*(-1) = 1$. When $\chi \neq \chi_0$ and $\chi(-1) = 1$, we write [3]

$$(96) \quad \xi(s, \chi^*) = L(s, \chi^*) \Gamma\left(\frac{s + k(\chi^*)}{2}\right) \left(\frac{q}{\pi}\right)^{\frac{s+k(\chi^*)}{2}}$$

as

$$(97) \quad \xi(s, \chi^*) = L(s, \chi^*) \Gamma\left(\frac{s}{2}\right) \left(\frac{q}{\pi}\right)^{\frac{s}{2}}$$

with

$$(98) \quad \xi(s, \chi^*) = \varepsilon(\chi^*) \xi(1-s, \overline{\chi^*}),$$

where $s \in \mathbb{C}$,

$$(99) \quad k(\chi^*) = 0,$$

$$(100) \quad \varepsilon(\chi^*) = \frac{\tau(\chi^*)}{\sqrt{q}},$$

and the Gauss sum of χ^* by

$$(101) \quad \tau(\chi^*) = \sum_{\theta=1}^q \chi^*(\theta) e^{\theta/q}.$$

Since $\xi(s, \chi^*)$ is an integral function of order 1, we have

$$(102) \quad \xi(s, \chi^*) = \xi(0, \chi^*) e^{B(\chi^*)s} \prod_{k=1}^{\infty} \left(1 - \frac{s}{\rho_k}\right) e^{s/\rho_k}$$

such that (see *Theorem 1*)

$$(103) \quad \begin{aligned} & \xi(s, \chi^*) \\ &= \xi(0, \chi^*) e^{B(\chi^*)s} \prod_{k=1}^{\infty} e^{s/\rho_k} \prod_{k=1}^{\infty} \left(1 - \frac{s}{\rho_k}\right) \\ &= \xi(0, \chi) e^{B(\chi)s} \prod_{k=1}^{\infty} \left(1 - \frac{1}{2\rho_k}\right) \prod_{k=1}^{\infty} \left(1 - \frac{s - \frac{1}{2}}{\rho_k - \frac{1}{2}}\right) e^{s/\rho_k}. \end{aligned}$$

By Eq. (102), we have

$$(104) \quad \xi\left(\frac{1}{2}, \chi^*\right) = \xi(0, \chi^*) e^{\frac{B(\chi^*)}{2}} \prod_{k=1}^{\infty} \left(1 - \frac{1}{2\rho_k}\right) \prod_{k=1}^{\infty} e^{1/(2\rho_k)}$$

such that

$$(105) \quad \frac{\xi(s, \chi^*)}{\xi\left(\frac{1}{2}, \chi^*\right)} = \frac{\xi(0, \chi^*) e^{B(\chi^*)s} \prod_{k=1}^{\infty} \left(1 - \frac{1}{2\rho_k}\right) \prod_{k=1}^{\infty} \left(1 - \frac{s - \frac{1}{2}}{\rho_k - \frac{1}{2}}\right) e^{s/\rho_k}}{\xi(0, \chi^*) e^{\frac{B(\chi^*)}{2}} \prod_{k=1}^{\infty} \left(1 - \frac{1}{2\rho_k}\right) \prod_{k=1}^{\infty} e^{1/(2\rho_k)}},$$

where (see *Lemma 7* for the details)

$$(106) \quad \xi(0, \chi^*) \neq 0$$

and

$$(107) \quad \xi\left(\frac{1}{2}, \chi^*\right) \neq 0.$$

By *Theorem 1*, we arrive at

$$(108) \quad \xi(s, \chi^*) = \xi\left(\frac{1}{2}, \chi^*\right) e^{B(\chi^*)(s - \frac{1}{2})} \prod_{k=1}^{\infty} \left(1 - \frac{s - \frac{1}{2}}{\rho_k - \frac{1}{2}}\right) e^{(s - 1/2)/\rho_k}.$$

From (97), (108) and *Theorem 2*, we suggest that

$$\begin{aligned}
 (109) \quad & L(s, \chi^*) \\
 &= \left(\frac{q}{\pi}\right)^{-\frac{s}{2}} \frac{\xi(s, \chi^*)}{\Gamma\left(\frac{s}{2}\right)} \\
 &= \frac{\left(\frac{q}{\pi}\right)^{-\frac{s}{2}} \cdot \xi\left(\frac{1}{2}, \chi^*\right) \cdot e^{B(\chi^*)\left(s-\frac{1}{2}\right)}}{\Gamma\left(\frac{s}{2}\right)} \prod_{k=1}^{\infty} \left(1 - \frac{s-\frac{1}{2}}{\rho_k - \frac{1}{2}}\right) e^{(s-1/2)/\rho_k}.
 \end{aligned}$$

Since Eq. (108) is an integral function of order 1, Eq. (109) is an integral function of order 1.

Substituting $s = 1/2 + i\beta$ into Eq. (108), we have [11]

$$(110) \quad \Xi(\beta, \chi^*) = \xi(1/2 + i\beta, \chi^*) = \xi\left(\frac{1}{2}, \chi^*\right) e^{i\beta B(\chi^*)} \prod_{k=1}^{\infty} \left(1 - \frac{i\beta}{\rho_k - \frac{1}{2}}\right) e^{i\beta/\rho_k}.$$

It is shown that Eq. (66) is an integral function of order 1.

Taking $\beta = \beta_k$, we have from Eq. (66) that

$$(111) \quad \Xi(\beta_k, \chi^*) = \xi\left(\frac{1}{2}, \chi^*\right) e^{i\beta_k B(\chi^*)} \prod_{k=1}^{\infty} \left(1 - \frac{i\beta_k}{\rho_k - \frac{1}{2}}\right) e^{i\beta_k/\rho_k} = 0.$$

By Eq. (67) and *Lemma 7*, we may arrive at

$$(112) \quad 1 - \frac{i\beta_k}{\rho_k - \frac{1}{2}} = 0,$$

and by Eq. (112) we get

$$(113) \quad \rho_k = \frac{1}{2} + i\beta_k.$$

By Eq. (113), Eqs. (108), (109) and (110) can be rewritten as

$$(114) \quad \xi(s, \chi^*) = \xi\left(\frac{1}{2}, \chi^*\right) e^{B(\chi^*)\left(s-\frac{1}{2}\right)} \prod_{k=1}^{\infty} \left(1 - \frac{s-\frac{1}{2}}{i\beta_k}\right) e^{(s-1/2)/(1/2+i\beta_k)},$$

$$(115) \quad L(s, \chi^*) = \frac{\left(\frac{q}{\pi}\right)^{-\frac{s}{2}} \xi\left(\frac{1}{2}, \chi^*\right) e^{B(\chi^*)\left(s-\frac{1}{2}\right)}}{\Gamma\left(\frac{s}{2}\right)} \prod_{k=1}^{\infty} \left(1 - \frac{s-\frac{1}{2}}{i\beta_k}\right) e^{(s-1/2)/(1/2+i\beta_k)},$$

and

$$(116) \quad \Xi(\beta, \chi^*) = \xi(1/2 + i\beta, \chi^*) = \xi\left(\frac{1}{2}, \chi^*\right) e^{i\beta B(\chi^*)} \prod_{k=1}^{\infty} \left(1 - \frac{\beta}{\beta_k}\right) e^{i\beta/(1/2+i\beta_k)},$$

where $s \in \mathbb{C}$.

From Eq. (60) and *Lemma 6*, we observe that

$$(117) \quad L(s, \chi^*) = 0,$$

which leads to

$$(118) \quad \frac{1}{\Gamma\left(\frac{s}{2}\right)} = 0$$

and

$$(119) \quad \xi(s, \chi^*) = \xi\left(\frac{1}{2}, \chi^*\right) e^{B(\chi^*)(s-\frac{1}{2})} \prod_{k=1}^{\infty} \left(1 - \frac{s-\frac{1}{2}}{i\beta_k}\right) e^{(s-1/2)/(1/2+i\beta_k)} = 0.$$

Making use of Eq. (118), we find that [2, 3, 11]

$$(120) \quad s = -2k,$$

which are the trivial zeros of Eq. (109), where $k \in \mathbb{N} \cup \{0\}$.

Inserting $\rho_k = \alpha_k + i\beta_k$ into Eq. (119) implies that

$$(121) \quad \begin{aligned} & \xi(\alpha_k + i\beta_k, \chi^*) \\ &= \xi\left(\frac{1}{2}, \chi^*\right) e^{B(\chi^*)(\alpha_k + i\beta_k - \frac{1}{2})} \prod_{k=1}^{\infty} \left(1 - \frac{\alpha_k + i\beta_k - \frac{1}{2}}{i\beta_k}\right) e^{(\alpha_k + i\beta_k - 1/2)/(1/2 + i\beta_k)}, \\ &= 0. \end{aligned}$$

With Eq. (121) and *Lemma 6*, we arrive at

$$(122) \quad 1 - \frac{\alpha_k + i\beta_k - \frac{1}{2}}{i\beta_k} = 0,$$

and we get

$$(123) \quad \alpha_k = \frac{1}{2}.$$

Thus,

$$(124) \quad \rho_k = 1/2 + i\beta_k$$

or, alternatively,

$$(125) \quad \operatorname{Re}(\rho_k) = 1/2.$$

3.4. Family 4: $\chi = \chi^*$ and $\chi^*(-1) = -1$. When $\chi = \chi^*$ and $\chi^*(-1) = -1$, we write [3]

$$(126) \quad \xi(s, \chi^*) = L(s, \chi^*) \Gamma\left(\frac{s+k(\chi^*)}{2}\right) \left(\frac{q}{\pi}\right)^{\frac{s+k(\chi^*)}{2}}$$

as

$$(127) \quad \xi(s, \chi^*) = L(s, \chi^*) \Gamma\left(\frac{s+1}{2}\right) \left(\frac{q}{\pi}\right)^{\frac{s+1}{2}}$$

with

$$(128) \quad \xi(s, \chi^*) = \varepsilon(\chi^*) \xi(1-s, \overline{\chi^*}),$$

where $s \in \mathbb{C}$,

$$(129) \quad k(\chi^*) = 1,$$

$$(130) \quad \varepsilon(\chi^*) = \frac{\tau(\chi^*)}{\sqrt{q}},$$

and the Gauss sum of χ^* by [3]

$$(131) \quad \tau(\chi^*) = \sum_{\theta=1}^q \chi^*(\theta) e^{\theta/q}.$$

Since $\xi(s, \chi^*)$ is an integral function of order 1, by *Theorem 1*, we have from Eqs. (102) and (108) that

$$(132) \quad \begin{aligned} & \xi(s, \chi^*) \\ &= \xi(0, \chi^*) e^{B(\chi^*)s} \prod_{k=1}^{\infty} \left(1 - \frac{s}{\rho_k}\right) e^{s/\rho_k} \\ &= \xi\left(\frac{1}{2}, \chi^*\right) e^{B(\chi^*)(s-\frac{1}{2})} \prod_{k=1}^{\infty} \left(1 - \frac{s-\frac{1}{2}}{\rho_k-\frac{1}{2}}\right) e^{(s-1/2)/\rho_k}. \end{aligned}$$

Since Eq. (132) is an integral function of order 1, $L(s, \chi^*)$ is an integral function of order 1.

Taking $s = 1/2 + i\beta$ into Eq. (132), we get

$$(133) \quad \begin{aligned} & \Xi(\beta, \chi^*) \\ &= \xi(1/2 + i\beta, \chi^*) \\ &= \xi(0, \chi^*) e^{B(\chi^*)s} \prod_{k=1}^{\infty} \left(1 - \frac{\frac{1}{2}+i\beta}{\rho_k}\right) e^{s/\rho_k} \\ &= \xi\left(\frac{1}{2}, \chi^*\right) e^{i\beta B(\chi^*)} \prod_{k=1}^{\infty} \left(1 - \frac{i\beta}{\rho_k-\frac{1}{2}}\right) e^{i\beta/\rho_k}. \end{aligned}$$

We may find that Eq. (133) is an integral function of order 1.

Substituting $\beta = \beta_k$ into Eq. (133), we have

$$(134) \quad \Xi(\beta_k, \chi^*) = \xi\left(\frac{1}{2}, \chi^*\right) e^{i\beta_k B(\chi^*)} \prod_{k=1}^{\infty} \left(1 - \frac{i\beta_k}{\rho_k - \frac{1}{2}}\right) e^{i\beta_k/\rho_k} = 0.$$

By the virtue of Eq. (134) and Lemma 6, we give

$$(135) \quad 1 - \frac{i\beta_k}{\rho_k - \frac{1}{2}} = 0,$$

and by Eq. (135), we arrive at

$$(136) \quad \rho_k = \frac{1}{2} + i\beta_k.$$

Combining Eqs. (127) and (132) and (136) and using Theorem 2, we have

$$(137) \quad \begin{aligned} L(s, \chi^*) &= \left(\frac{q}{\pi}\right)^{-\frac{s+1}{2}} \frac{\xi(s, \chi^*)}{\Gamma\left(\frac{s+1}{2}\right)} \\ &= \frac{\xi\left(\frac{1}{2}, \chi^*\right) \cdot \left(\frac{q}{\pi}\right)^{-\frac{s+1}{2}} \cdot e^{B(\chi^*)(s-\frac{1}{2})}}{\Gamma\left(\frac{s+1}{2}\right)} \prod_{k=1}^{\infty} \left(1 - \frac{s-\frac{1}{2}}{i\beta_k}\right) e^{(s-1/2)/(1/2+i\beta_k)}. \end{aligned}$$

In view of Eq. (137), we obtain

$$(138) \quad \frac{1}{\Gamma\left(\frac{s+1}{2}\right)} = 0$$

and

$$(139) \quad \xi(s, \chi^*) = \xi\left(\frac{1}{2}, \chi^*\right) \cdot e^{B(\chi^*)(s-\frac{1}{2})} \prod_{k=1}^{\infty} \left(1 - \frac{s-\frac{1}{2}}{i\beta_k}\right) e^{(s-1/2)/(1/2+i\beta_k)} = 0.$$

By Eq. (138), we present [2, 3, 11]

$$(140) \quad s = -2k - 1,$$

which is the trivial zeros of Eq. (137), where $k \in \mathbb{N} \cup \{0\}$.

Substituting $\rho_k = \alpha_k + i\beta_k$ into Eq. (139) we obtain

$$(141) \quad \begin{aligned} &\xi(\alpha_k + i\beta_k, \chi^*) \\ &= \xi\left(\frac{1}{2}, \chi^*\right) \cdot e^{B(\chi^*)(\alpha_k + i\beta_k - \frac{1}{2})} \prod_{k=1}^{\infty} \left(1 - \frac{\alpha_k + i\beta_k - \frac{1}{2}}{i\beta_k}\right) e^{(\alpha_k + i\beta_k - 1/2)/(1/2 + i\beta_k)} \\ &= 0, \end{aligned}$$

which implies, by Lemma 6, that

$$(142) \quad 1 - \frac{\alpha_k + i\beta_k - \frac{1}{2}}{i\beta_k} = 0.$$

Thus, by Eq. (123) we have

$$(143) \quad \alpha_k = 1/2$$

such that

$$(144) \quad \rho_k = 1/2 + i\beta_k.$$

By Lemma 12 and using the above results, we clearly see that the real part of the zeros in the critical trip is $1/2$.

It is shown that Conjecture 1 is true.

Hence, we finish the proof of Conjecture 1.

Remark. In short, we easily see the followings:

- When $\chi = \chi_0$ and $q = 1$, $L(s, \chi) = \zeta(s)$ is extended to be a meromorphic continuation to the entire complex plane s , and has a pole at $s = 1$ with residue 1, the trivial zeros $s = -2h$ with $h \in \mathbb{N}$, and the nontrivial zeros $s_k = 1/2 + i\varpi_k$ with $k \in \mathbb{N}$, which lie on the critical line $s = 1/2$ and in the critical trip $0 < \text{Re}(s) < 1$.

As shown in Section 1, Case 1 has been proved by authors in [8, 9].

- When $\chi = \chi_0$ and $q > 1$, $L(s, \chi) = L(s, \chi_0)$ is extended to be a meromorphic continuation to the entire complex plane s , and has a pole at $s = 1$ with residue $v(q)/q = \prod_{p|q} (1 - p^{-1})$, the pure imaginary zeros $s = 2\pi ij/(\log p)$ with $p|q$ and $j \in \mathbb{Z}$, the trivial zeros $s = -2h$ with $h \in \mathbb{N}$, and the nontrivial zeros $s_k = 1/2 + i\varpi_k$ for $k \in \mathbb{N}$.

It is seen that Cases 2 and 1 have the same as the nontrivial zeros $s_k = 1/2 + i\varpi_k$, a pole at $s = 1$ with different residues, and trivial zeros $s = -2h$ with $h \in \mathbb{N}$. Its nontrivial zeros for $L(s, \chi) = L(s, \chi_0)$ lie on the critical line $s = 1/2$ and in the critical trip $0 < \text{Re}(s) < 1$.

As shown in Case 3 of Section 1, we have followings:

- When $\chi = \chi^*$ and $\chi^*(-1) = 1$,

$$L(s, \chi) = L(s, \chi^*) = (q/\pi)^{-\frac{s}{2}} \xi(s, \chi^*) / \Gamma(s/2)$$

is an integral function of order 1, and has the zeros (all zeros are the nontrivial zeros) $\rho_k = 1/2 + i\beta_k$ with $k \in \mathbb{N}$.

- When $\chi = \chi^*$ and $\chi^*(-1) = -1$,

$$L(s, \chi) = L(s, \chi^*) = (q/\pi)^{-\frac{s+1}{2}} \xi(s, \chi) / \Gamma(s/2 + 1/2)$$

is an integral function of order 1, and has the zeros (all zeros are the nontrivial zeros) $\rho_k = 1/2 + i\beta_k$ with $k \in \mathbb{N}$. It is observed that in Case 3 of Section

1, they have the nontrivial zeros $\rho_k = 1/2 + i\beta_k$ with $k \in \mathbb{N}$, which lie on the critical line $s = 1/2$ and in the critical trip $0 < \text{Re}(s) < 1$.

4. NEW RESULTS AND APPLICATIONS

In this section we report the new formulas associated with the Dirichlet L -functions. Then we also give the new representations for the Wang theorems. Main target of the part is to present the applications of *Conjecture 1* and Hardy–Littlewood theorems to obtain the Goldbach’s conjecture.

4.1. New formulas for the Dirichlet L -functions. We now give the properties for the Dirichlet L -functions.

Theorem 3. Suppose that χ^* is a primitive character modulo q with $q > 1$. Let β_k be the Siegel zeros for the Dirichlet L -functions with the primitive character χ^* with $k \in \mathbb{N}$. Then there exist the equivalent representations:

$$(145) \quad \xi(s, \chi^*) = \xi(0, \chi^*) e^{sB(\chi^*)} \prod_{k=1}^{\infty} \left(1 - \frac{s}{\frac{1}{2} + i\beta_k}\right) e^{s/(1/2+i\beta_k)},$$

and

$$(146) \quad \xi(s, \chi^*) = \xi\left(\frac{1}{2}, \chi^*\right) e^{(s-\frac{1}{2})B(\chi^*)} \prod_{k=1}^{\infty} \left(1 - \frac{s-\frac{1}{2}}{i\beta_k}\right) e^{(s-1/2)/(1/2+i\beta_k)},$$

where $s \in \mathbb{C}$, and

$$(147) \quad B(\chi^*) = \frac{\xi^{(1)}}{\xi}(0, \chi^*).$$

Proof. By *Conjecture 1* and *Theorem 1*, we give the desired results. \square

Theorem 4. Suppose that χ^* is a primitive character modulo q with $q > 1$ and β_k run the Siegel zeros for the Dirichlet L -functions with the primitive character χ^* with $k \in \mathbb{N}$. Then there exist the equivalent representations:

$$(148) \quad L(s, \chi^*) = \frac{\xi(0, \chi^*) \cdot \left(\frac{q}{\pi}\right)^{-\frac{s+h(\chi^*)}{2}} \cdot e^{B(\chi^*)s}}{\Gamma\left(\frac{s+h(\chi^*)}{2}\right)} \cdot \prod_{k=1}^{\infty} \left(1 - \frac{s}{\frac{1}{2} + i\beta_k}\right) e^{s/(1/2+i\beta_k)},$$

and

$$(149) \quad L(s, \chi^*) = \frac{\xi\left(\frac{1}{2}, \chi^*\right) \cdot \left(\frac{q}{\pi}\right)^{-\frac{s+h(\chi^*)}{2}} \cdot e^{(s-\frac{1}{2})B(\chi^*)}}{\Gamma\left(\frac{s+h(\chi^*)}{2}\right)} \cdot \prod_{k=1}^{\infty} \left(1 - \frac{s-\frac{1}{2}}{i\beta_k}\right) e^{(s-1/2)/(1/2+i\beta_k)},$$

where $s \in \mathbb{C}$, $\hbar(\chi^*)$ is defined in Eq. (9), and

$$(150) \quad B(\chi^*) = \frac{\xi^{(1)}}{\xi}(0, \chi^*).$$

Proof. By Conjecture 1 and Theorem 2, we give the desired results. \square

Theorem 5. Suppose that χ^* is a primitive character modulo q with $q > 1$ and β_k are the Siegel zeros. Then there exist the equivalent representations:

$$(151) \quad \Xi(\beta, \chi^*) = \xi(0, \chi^*) e^{(1/2+i\beta)B(\chi^*)} \prod_{k=1}^{\infty} \left(1 - \frac{\frac{1}{2} + i\beta}{\frac{1}{2} + i\beta_k}\right) e^{(1/2+i\beta)/(1/2+i\beta_k)},$$

and

$$(152) \quad \Xi(\beta, \chi^*) = \xi\left(\frac{1}{2}, \chi^*\right) e^{i\beta B(\chi^*)} \prod_{k=1}^{\infty} \left(1 - \frac{\beta}{\beta_k}\right) e^{i\beta/(1/2+i\beta_k)},$$

where $\beta \in \mathbb{C}$, $\hbar(\chi^*)$ is defined in Eq. (9), and

$$(153) \quad B(\chi^*) = \frac{\xi^{(1)}}{\xi}(0, \chi^*).$$

Proof. By Theorem 3 and Conjecture 1, we obtain the desired results. \square

We now define the function by

$$(154) \quad \xi(s, \chi_0) = \Re(s) L(s, \chi_0) = (s-1) \pi^{-s/2} \Gamma(s/2+1) L(s, \chi_0),$$

where $s \in \mathbb{C}$ and $s \neq 1$.

Remark. Suppose that χ_0 is the principal character modulo q with $q > 1$. Then we have that

$$(155) \quad \xi(s, \chi_0) = \xi(0) e^{s\Im_0} \prod_{k=1}^{\infty} \left(1 - \frac{s}{s_k}\right) e^{s/s_k} \prod_{p|q} \left(1 - \frac{1}{p^s}\right)$$

and

$$(156) \quad \xi(s, \chi_0) = \xi(1/2) e^{s\Im_0} \prod_{k=1}^{\infty} \left(1 - \frac{s - \frac{1}{2}}{s_k - \frac{1}{2}}\right) e^{s/s_k} \prod_{p|q} \left(1 - \frac{1}{p^s}\right),$$

where $s \in \mathbb{C}$ and $s \neq 1$.

Because Conjecture 1 is true, $1 - \rho_k$, $\overline{\rho_k}$ and $1 - \overline{\rho_k}$ are the nontrivial zeros for the Dirichlet L -functions, which is in agreement with the results of Montgomery and Vaughan [3].

From Eq. (152), we have

$$(157) \quad \Xi(\beta, \chi^*) = \xi\left(\frac{1}{2}, \chi^*\right) e^{i\beta B(\chi^*)} \prod_{k=1}^{\infty} \left(1 - \frac{\beta}{\beta_k}\right) e^{i\beta/(1/2+i\beta_k)},$$

and

$$(158) \quad \Xi(-\beta, \chi^*) = \xi\left(\frac{1}{2}, \chi^*\right) e^{-i\beta B(\chi^*)} \prod_{k=1}^{\infty} \left(1 + \frac{\beta}{\beta_k}\right) e^{-i\beta/(1/2+i\beta_k)},$$

such that

$$(159) \quad \Xi(\beta, \chi^*) \Xi(-\beta, \chi^*) = \xi^2\left(\frac{1}{2}, \chi^*\right) \prod_{k=1}^{\infty} \left(1 - \frac{\beta^2}{\beta_k^2}\right).$$

Let

$$(160) \quad \rho_k^* = \operatorname{Re}(\rho_k) + i|\operatorname{Im}(\rho_k)|$$

such that $1 - \rho_k$ are the nontrivial zeros for the Dirichlet L -functions.

By Lemma 8 and Eq. (160), we have

$$(161) \quad \begin{aligned} & \xi(s, \chi^*) \\ &= \xi(0, \chi^*) e^{B(\chi^*)s} \prod_{k=1}^{\infty} \left(1 - \frac{s}{\rho_k^*}\right) e^{s/\rho_k^*} \\ &= \xi\left(\frac{1}{2}, \chi^*\right) e^{(s-1/2)B(\chi^*)} \prod_{k=1}^{\infty} \left(1 - \frac{s-\frac{1}{2}}{\rho_k^*-\frac{1}{2}}\right) e^{(s-1/2)/\rho_k^*} \end{aligned}$$

and

$$(162) \quad \begin{aligned} & \xi(s, \chi^*) \\ &= \xi(0, \chi^*) e^{B(\chi^*)s} \prod_{k=1}^{\infty} \left(1 - \frac{s}{1-\rho_k^*}\right) e^{s/(1-\rho_k^*)} \\ &= \xi\left(\frac{1}{2}, \chi^*\right) e^{(s-1/2)B(\chi^*)} \prod_{k=1}^{\infty} \left(1 + \frac{s-\frac{1}{2}}{\rho_k^*-\frac{1}{2}}\right) e^{(s-1/2)/(1-\rho_k^*)} \end{aligned}$$

such that

$$(163) \quad \begin{aligned} & \xi^2(s, \chi^*) \\ &= \xi^2(0, \chi^*) e^{2sB(\chi^*)} \prod_{k=1}^{\infty} \left(1 - \frac{s}{\rho_k^*}\right) \prod_{k=1}^{\infty} \left(1 - \frac{s}{1-\rho_k^*}\right) e^{\frac{s}{\rho_k^*(1-\rho_k^*)}} \\ &= \xi^2\left(\frac{1}{2}, \chi^*\right) e^{2(s-1/2)B(\chi^*)} \prod_{k=1}^{\infty} \left(1 - \left(\frac{s-\frac{1}{2}}{\rho_k^*-\frac{1}{2}}\right)^2\right) e^{\frac{s-1/2}{\rho_k^*(1-\rho_k^*)}}. \end{aligned}$$

According to Lemma 11 and Conjecture 1, we set up the following result.

Corollary 1. Let $\chi(n)$ be a Dirichlet character (mod $q > 1$). Suppose that $N(T, \chi)$ denotes the number of zeros of the Dirichlet L -function (1) for $\operatorname{Re}(s) = 1/2$, $T > 2$ and $0 \leq \operatorname{Im}(s) \leq T$ and $\tilde{N}(T, \chi)$ denotes the number of zeros of the Dirichlet L -function (1) for $0 < \operatorname{Re}(s) < 1$, $T > 2$ and $0 \leq \operatorname{Im}(s) \leq T$. Then we have

$$(164) \quad N(T, \chi) = \tilde{N}(T, \chi).$$

Proof. From the obtained results in Section 3, we find the followings:

When $\chi = \chi_0$ and $q = 1$, Eq. (164) is true (see [8, 9] for further details), where

$$L(s, \chi) = \zeta(s).$$

When $\chi = \chi_0$ and $q > 1$, we have

$$(165) \quad N(T, \chi_0) = \tilde{N}(T, \chi_0),$$

where $L(s, \chi) = L(s, \chi_0)$.

When $\chi = \chi^*$ and $\chi^*(-1) = 1$, we conclude that

$$(166) \quad N(T, \chi^*) = \tilde{N}(T, \chi^*).$$

where $L(s, \chi) = L(s, \chi^*) = (q/\pi)^{-\frac{s}{2}} \xi(s, \chi^*) / \Gamma(s/2)$.

When $\chi = \chi^*$ and $\chi^*(-1) = -1$, we arrive at

$$(167) \quad N(T, \chi^*) = \tilde{N}(T, \chi^*),$$

where $L(s, \chi) = L(s, \chi^*) = (q/\pi)^{-\frac{s+1}{2}} \xi(s, \chi) / \Gamma(s/2 + 1/2)$.

Thus, we finish the proof. \square

4.2. The truth of the Goldbach's conjecture. We now give new representations for the Wang theorems and present the applications of the Hardy–Littlewood theorems to the Goldbach's conjecture because *Conjecture 1* is proved and true.

Theorem 6. (*The weak Goldbach's conjecture*)

Every odd number $m > 5$ is the sum of three primes.

Proof. By Lemma 13, we have show the required result since *Conjecture 1* is true. \square

Theorem 7. (*Wang Theorem I*)

(1, 4) is valid, where $\delta_2 \geq 3.237/2.237$.

Proof. By using Lemma 14 and considering the fact *Conjecture 1* is proved and true, we get the result. \square

Theorem 8. (*Wang Theorem II*)

(1, 3) is valid, where $\delta_1 \geq 2.475/1.475$.

Proof. Similarly, by using *Lemma 15*, we have the result because *Conjecture 1* is true. \square

Theorem 9. (*The strong Goldbach's conjecture*)

Every even number $m > 2$ is sums of two primes.

Proof. By *Lemma 16*, *Theorem 9* is true because of true of *Conjecture 1*. \square

Remark. By the work of Deshouillers and coauthors [15], we have that the weak Goldbach's conjecture is true because *Conjecture 1* is valid. By the work of Granville [18], we also see that the strong Goldbach's conjecture is true because *Conjecture 1* is valid. Thus, the Goldbach's conjecture is true.

5. CONCLUSION

The present paper has proved that *Conjecture 1* is true with the variant of the product for the entire function related to the Dirichlet L -functions. We have presented the applications of it to the Wang theorems for the Goldbach's problems. By using the Hardy–Littlewood theorems, we have shown that every odd number $m > 5$ is the sum of three primes and that every even number $m > 2$ is sums of two primes. The obtained result is proposed to solve the mathematical problems under the assumption of the truth of *Conjecture 1*.

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