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Existence and uniquenes solution of integral equations via common fixed point theorems

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Abstract

In this paper, we prove some common fixed point theorems on complex partial metric space. The presented results generalize and expand some of the literature well-known results. We also explore some of the application of our key results.

Key words: integral equations; complex partial metric space; common fixed point.
AMS Subject Classification: 47H10, 54H25.

1 Introduction

Azam et al. [1] introduced the concept of complex valued metric spaces and studied some fixed point theorems for mappings satisfying a rational inequality.

Two years after, in [2] Rao et al. discussed for the first time the idea of complex-valued b-metric spaces.

In 2017, Dhivya and Marudai [3] introduced the concept of complex partial metric space and suggested a plan to expand the results, as well as proving common fixed point theorems under the rational expression contraction condition. This idea has been followed by Gunaseelan [4], who introduced the concept of complex partial b-metric spaces and discussed some results of fixed point theory for self-mappings in these new spaces.

In [5], Prakasam and Gunaseelan proved an existence and uniqueness of common fixed point (with an illustrative example) theorem using (CLR) and (E.A.) properties in complex partial b-metric spaces. Their proved results generalize and extend some of the well known results in the literature.

In [6], Gunaseelan et al. proved a fixed point theorem in complex partial b-metric spaces under a contraction mapping. They also gave some applications of their main results.

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In this paper, we prove some common fixed point theorems on complex partial metric space.

2 Preliminaries

Let $\mathfrak{C}$ be the set of complex numbers and $\tau_1, \tau_2, \tau_3 \in \mathfrak{C}$. Define a partial order $\preceq$ on $\mathfrak{C}$ as follows:
$\tau_1 \preceq \tau_2$ if and only if $\mathcal{R}(\tau_1) \leq \mathcal{R}(\tau_2)$, $\mathcal{I}(\tau_1) \leq \mathcal{I}(\tau_2)$.
Consequently, one can infer that $\tau_1 \preceq \tau_2$ if one of the following conditions is satisfied:
(i) $\mathcal{R}(\tau_1) = \mathcal{R}(\tau_2)$, $\mathcal{I}(\tau_1) < \mathcal{I}(\tau_2)$,
(ii)$\mathcal{R}(\tau_1) < \mathcal{R}(\tau_2)$, $\mathcal{I}(\tau_1) = \mathcal{I}(\tau_2)$,
(iii)$\mathcal{R}(\tau_1) < \mathcal{R}(\tau_2)$, $\mathcal{I}(\tau_1) < \mathcal{I}(\tau_2)$,
(iv)$\mathcal{R}(\tau_1) = \mathcal{R}(\tau_2)$, $\mathcal{I}(\tau_1) = \mathcal{I}(\tau_2)$.
In particular, we write $\tau_1 \precnsim \tau_2$ if $\tau_1 \neq \tau_2$ and one of $(i), (ii)$ and (iii) is satisfied and we write $\tau_1 \prec \tau_2$ if only (iii) is satisfied. Notice that
(a) If $0 \preceq \tau_1 \precnsim \tau_2$, then $|\tau_1| < |\tau_2|$,
(b) If $\tau_1 \preceq \tau_2$ and $\tau_2 \prec \tau_3$ then $\tau_1 \prec \tau_3$,
(c) If $\eta, \gamma \in \mathbb{R}$ and $\eta \leq \gamma$ then $\eta\tau_1 \preceq \gamma\tau_1$ for all $0 \preceq \tau_1 \in \mathfrak{C}$.
Here $\mathfrak{C}_+ (= \{(\aleph, \mathfrak{y}) | \aleph, \mathfrak{y} \in \mathbb{R}_+\})$ and $\mathbb{R}_+ (= \{\aleph \in \mathbb{R} | \aleph \geq 0\})$ denote the set of non negative complex numbers, and the set of non negative real numbers, respectively.
Now, let us recall some basic concepts and notations, which will be used in the sequel.

Definition 2.1. *[3] A complex partial metric on a non-void set G is a function $\varrho_{cb} : G \times G \to \mathbb{C}^+$ such that for all $\theta, \omega, \vartheta \in G$:*
(i) $0 \preceq \varrho_{cb}(\theta, \theta) \preceq \varrho_{cb}(\theta, \omega)$(small self-distances)
(ii) $\varrho_{cb}(\theta, \omega) = \varrho_{cb}(\omega, \theta)$(symmetry)
(iii) $\varrho_{cb}(\theta, \theta) = \varrho_{cb}(\theta, \omega) = \varrho_{cb}(\omega, \omega)$ if and only if $\theta = \omega$(equality)
(iv) $\varrho_{cb}(\theta, \omega) \preceq \varrho_{cb}(\theta, \vartheta) + \varrho_{cb}(\vartheta, \omega) - \varrho_{cb}(\vartheta, \vartheta)$(triangularity).
A complex partial metric space is a pair (G, ϱ_{cb}) such that G is a non-void set and ϱ_{cb} is the complex partial metric on G.

Definition 2.2. *[3] Let $(G, \wp_{cb})$ be a complex partial metric space. Let $\{\theta_n\}$ be any sequence in $\theta \in G$. Then*

(i) The sequence $\{\theta_n\}$ is said to be converges to $\aleph$, if $\lim_{n\to\infty} \wp_{cb}(\theta_n, \theta) = \wp_{cb}(\theta, \theta)$.

(ii) The sequence $\{\theta_n\}$ is said to be Cauchy sequence in $(G, \wp_{cb})$ if $\lim_{n,m\to\infty} \wp_{cb}(\theta_n, \theta_m)$ exists and is finite.

(iii) $(G, \wp_{cb})$ is said to be a complete complex partial metric space if for every Cauchy sequence $\{\theta_n\}$ in G there exists $\theta \in G$ such that $\lim_{n,m\to\infty} \wp_{cb}(\theta_n, \theta_m) = \lim_{n\to\infty} \wp_{cb}(\theta_n, \theta) = \wp_{cb}(\theta, \theta)$.

(iv) *A mapping* $\Pi : G \to G$ *is said to be continuous at* $\theta_0 \in G$ *if for every* $\epsilon > 0$*, there exists* $\delta > 0$ *such that* $\Pi(B_{\wp_{cb}}(\theta_0, \delta)) \subset B_{\wp_{cb}}(\Pi(\theta_0, \epsilon))$.

Definition 2.3. *[3] Let* Π *and* Ψ *be self mappings of non-void set* G*. A point* $\aleph \in G$ *is called a common fixed point of* Π *and* Ψ *if* $\aleph = \Pi\aleph = \Psi\aleph$.

Theorem 2.1. *[3] Let* $(G, \preceq)$ *be a partially ordered set and suppose that there exists a complex partial metric* ϱ_{cb} *in* G *such that* (G, ϱ_{cb}) *is a complete complex partial metric space. Let* $\Pi, \Psi : G \to G$ *be a pair of weakly increasing mapping and suppose that for every comparable* $\aleph, \mathfrak{y} \in G$ *we have either*

$$\varrho_{cb}(\Pi\aleph, \Psi\mathfrak{y}) \preceq a\frac{\varrho_{cb}(\aleph, \Pi\aleph)\varrho_{cb}(\mathfrak{y}, \Psi\mathfrak{y})}{\varrho_{cb}(\aleph, \mathfrak{y})} + b\varrho_{cb}(\aleph, \mathfrak{y})$$

for $\varrho_{cb}(\aleph, \mathfrak{y}) \neq 0$ *with* $a \geq 0, b \geq 0$, $a + b < 1$, *or*

$$\varrho_{cb}(\Pi\aleph, \Psi\mathfrak{y}) = 0 \text{ if } \varrho_{cb}(\aleph, \mathfrak{y}) = 0.$$

If Π *or* Ψ *is continuous then* Π *and* Ψ *have a common fixed point* $\propto \in G$ *and* $\varrho_{cb}(\propto, \propto) = 0$.

Inspired by Theorem 2.1, here we prove some common fixed point theorems on complex partial metric space with an application. For complex partial metric space, we will use the CPMS notation.

3 Main Results

Theorem 3.1. *Let* $(G, \wp_{cb})$ *be a complete CPMS and* $\Pi, \Psi : G \to G$ *be two continuous mappings such that*

$$\wp_{cb}(\Pi\theta, \Psi\omega) \preceq \curlywedge \max\{\wp_{cb}(\theta, \omega), \wp_{cb}(\theta, \Pi\theta), \wp_{cb}(\omega, \Psi\omega), \frac{1}{2}(\wp_{cb}(\theta, \Psi\omega) + \wp_{cb}(\omega, \Pi\theta))\}, \tag{1}$$

for all $\theta, \omega \in G$, *where* $0 \leq \curlywedge < 1$. *Then the pair* (Π, Ψ) *has a unique common fixed point and* $\wp_{cb}(\theta^*, \theta^*) = 0$.

Proof. Let θ_0 be arbitrary point in G and define a sequence $\{\theta_n\}$ as follows:

$$\theta_{2n+1} = \Pi\theta_{2n} \quad \textit{and} \quad \theta_{2n+2} = \Psi\theta_{2n+1}, n = 0, 1, 2, \ldots \tag{2}$$

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Then by (1) and (2), we obtain

$$\begin{aligned}
\wp_{cb}(\theta_{2n+1},\theta_{2n+2}) &= \wp_{cb}(\Pi\theta_{2n},\Psi\theta_{2n+1})\\
&\preceq \curlywedge \max\{\wp_{cb}(\theta_{2n},\theta_{2n+1}),\wp_{cb}(\theta_{2n},\Pi\theta_{2n}),\wp_{cb}(\theta_{2n+1},\Psi\theta_{2n+1}),\\
&\quad \frac{1}{2}(\wp_{cb}(\theta_{2n},\Psi\theta_{2n+1})+\wp_{cb}(\theta_{2n+1},\Pi\theta_{2n}))\}\\
&\preceq \curlywedge \max\{\wp_{cb}(\theta_{2n},\theta_{2n+1}),\wp_{cb}(\theta_{2n},\theta_{2n+1}),\wp_{cb}(\theta_{2n+1},\theta_{2n+2}),\\
&\quad \frac{1}{2}(\wp_{cb}(\theta_{2n},\theta_{2n+2})+\wp_{cb}(\theta_{2n+1},\theta_{2n+1}))\}\\
&\preceq \curlywedge \max\{\wp_{cb}(\theta_{2n},\theta_{2n+1}),\wp_{cb}(\theta_{2n+1},\theta_{2n+2}),\\
&\quad \frac{1}{2}(\wp_{cb}(\theta_{2n},\theta_{2n+1})+\wp_{cb}(\theta_{2n+1},\theta_{2n+2})-\wp_{cb}(\theta_{2n+1},\theta_{2n+1})\\
&\quad +\wp_{cb}(\theta_{2n+1},\theta_{2n+1}))\}\\
&= \curlywedge \max\{\wp_{cb}(\theta_{2n},\theta_{2n+1}),\wp_{cb}(\theta_{2n+1},\theta_{2n+2}),\\
&\quad \frac{1}{2}(\wp_{cb}(\theta_{2n},\theta_{2n+1})+\wp_{cb}(\theta_{2n+1},\theta_{2n+2}))\}
\end{aligned}$$

Case I: If $\max\{\wp_{cb}(\theta_{2n},\theta_{2n+1}),\wp_{cb}(\theta_{2n+1},\theta_{2n+2}),$
$\frac{1}{2}(\wp_{cb}(\theta_{2n},\theta_{2n+1})+\wp_{cb}(\theta_{2n+1},\theta_{2n+2}))\} = \wp_{cb}(\theta_{2n+1},\theta_{2n+2})$, then we have

$$\wp_{cb}(\theta_{2n+1},\theta_{2n+2}) \preceq \curlywedge\wp_{cb}(\theta_{2n+1},\theta_{2n+2}).$$

This implies $\curlywedge \geq 1$, which is a contradiction.
Case II: If $\max\{\wp_{cb}(\theta_{2n},\theta_{2n+1}),\wp_{cb}(\theta_{2n+1},\theta_{2n+2}),$
$\frac{1}{2}(\wp_{cb}(\theta_{2n},\theta_{2n+1})+\wp_{cb}(\theta_{2n+1},\theta_{2n+2}))\} = \wp_{cb}(\theta_{2n},\theta_{2n+1})$, then we have

$$\wp_{cb}(\theta_{2n+1},\theta_{2n+2}) \preceq \curlywedge\wp_{cb}(\theta_{2n},\theta_{2n+1}). \tag{3}$$

From the next step, we have

$$\begin{aligned}
\wp_{cb}(\theta_{2n+2},\theta_{2n+3}) \preceq \curlywedge \max\{&\wp_{cb}(\theta_{2n+1},\theta_{2n+2}),\wp_{cb}(\theta_{2n+2},\theta_{2n+3}),\\
&\frac{1}{2}(\wp_{cb}(\theta_{2n+1},\theta_{2n+2})+\wp_{cb}(\theta_{2n+2},\theta_{2n+3}))\}.
\end{aligned}$$

The following three cases arises, we have
Case IIa:

$$\wp_{cb}(\theta_{2n+2},\theta_{2n+3}) \preceq \curlywedge\wp_{cb}(\theta_{2n+2},\theta_{2n+3}),$$

which implies $\curlywedge \geq 1$, is a contradiction.
Case IIb:

$$\wp_{cb}(\theta_{2n+2},\theta_{2n+3}) \preceq \curlywedge\wp_{cb}(\theta_{2n+1},\theta_{2n+2}). \tag{4}$$

From (3) and (4), $\forall n = 0,1,2,\ldots$, we get

$$\wp_{cb}(\theta_{n+1},\theta_{n+2}) \preceq \curlywedge\wp_{cb}(\theta_{n},\theta_{n+1}) \preceq \ldots \preceq \curlywedge^{n+1}\wp_{cb}(\theta_0,\theta_1).$$

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For $m, n \in \mathbb{N}$, with $m > n$, we have

$$\begin{aligned}
\wp_{cb}(\theta_n, \theta_m) &\preceq \wp_{cb}(\theta_n, \theta_{n+1}) + \wp_{cb}(\theta_{n+1}, \theta_m) - \wp_{cb}(\theta_{n+1}, \theta_{n+1}) \\
&\preceq \wp_{cb}(\theta_n, \theta_{n+1}) + \wp_{cb}(\theta_{n+1}, \theta_m) \\
&\preceq \wp_{cb}(\theta_n, \theta_{n+1}) + \wp_{cb}(\theta_{n+1}, \theta_{n+2}) + \wp_{cb}(\theta_{n+2}, \theta_m) \\
&\quad - \wp_{cb}(\theta_{n+2}, \theta_{n+2}) \\
&\preceq \wp_{cb}(\theta_n, \theta_{n+1}) + \wp_{cb}(\theta_{n+1}, \theta_{n+2}) + \wp_{cb}(\theta_{n+2}, \theta_m) \\
&\preceq \wp_{cb}(\theta_n, \theta_{n+1}) + \wp_{cb}(\theta_{n+1}, \theta_{n+2}) + \wp_{cb}(\theta_{n+2}, \theta_{n+3}) \\
&\quad + \ldots + \wp_{cb}(\theta_{m-2}, \theta_{m-1}) + \wp_{cb}(\theta_{m-1}, \theta_m).
\end{aligned}$$

Moreover, by using (4), we get

$$\begin{aligned}
\wp_{cb}(\theta_n, \theta_m) &\preceq 人^n \wp_{cb}(\theta_0, \theta_1) + 人^{n+1} \wp_{cb}(\theta_0, \theta_1) + 人^{n+2} \wp_{cb}(\theta_0, \theta_1) \\
&\quad + \ldots + 人^{m-2} \wp_{cb}(\theta_0, \theta_1) + 人^{m-1} \wp_{cb}(\theta_0, \theta_1) \\
&= \sum_{i=1}^{m-n} 人^{i+n-1} \wp_{cb}(\theta_0, \theta_1).
\end{aligned}$$

Therefore

$$\begin{aligned}
|\wp_{cb}(\theta_n, \theta_m)| &\leq \sum_{i=1}^{m-n} 人^{i+n-1} |\wp_{cb}(\theta_0, \theta_1)| = \sum_{t=n}^{m-1} 人^t |\wp_{cb}(\theta_0, \theta_1)| \\
&\leq \sum_{i=n}^{\infty} |\wp_{cb}(\theta_0, \theta_1)| \\
&= \frac{人^n}{1 - 人} |\wp_{cb}(\theta_0, \theta_1)|.
\end{aligned}$$

Then, we have

$$|\wp_{cb}(\theta_n, \theta_m)| \leq \frac{人^n}{1 - 人} |\wp_{cb}(\theta_0, \theta_1)| \to 0 \quad as \quad n \to \infty.$$

Hence $\{\theta_n\}$ is a Cauchy sequence in G.
Case IIc:

$$\wp_{cb}(\theta_{2n+2}, \theta_{2n+3}) \preceq 人 \frac{1}{2} (\wp_{cb}(\theta_{2n+1}, \theta_{2n+2}) + \wp_{cb}(\theta_{2n+2}, \theta_{2n+3})).$$

This implies that

$$\wp_{cb}(\theta_{2n+2}, \theta_{2n+3}) \preceq \frac{人}{(2 - 人)} \wp_{cb}(\theta_{2n+1}, \theta_{2n+2}). \tag{5}$$

Since $a := \frac{人}{2 - 人} < 1$, we get $\wp_{cb}(\theta_{n+1}, \theta_{n+2}) \preceq a \wp_{cb}(\theta_n, \theta_{n+1})$. Therefore $\{\theta_n\}_{n \in \mathbb{N}}$ is a Cauchy sequence in G.
Case III:

If $\max\{\wp_{cb}(\theta_{2n},\theta_{2n+1}),\wp_{cb}(\theta_{2n+1},\theta_{2n+2}),$
$\frac{1}{2}(\wp_{cb}(\theta_{2n},\theta_{2n+1})+\wp_{cb}(\theta_{2n+1},\theta_{2n+2}))\} = \frac{1}{2}(\wp_{cb}(\theta_{2n},\theta_{2n+1})+\wp_{cb}(\theta_{2n+1},\theta_{2n+2}))$.
Then, we have

$$\wp_{cb}(\theta_{2n+1},\theta_{2n+2}) \preceq \frac{\curlywedge}{2}(\wp_{cb}(\theta_{2n},\theta_{2n+1})+\wp_{cb}(\theta_{2n+1},\theta_{2n+2}))$$

Hence,

$$\wp_{cb}(\theta_{2n+1},\theta_{2n+2}) \preceq \frac{\curlywedge}{2-\curlywedge}\wp_{cb}(\theta_{2n},\theta_{2n+1}). \tag{6}$$

For the next step, we have

$$\begin{aligned}\wp_{cb}(\theta_{2n+2},\theta_{2n+3}) \preceq \curlywedge \max\{&\wp_{cb}(\theta_{2n+1},\theta_{2n+2}),\wp_{cb}(\theta_{2n+2},\theta_{2n+3}),\\ &\frac{1}{2}(\wp_{cb}(\theta_{2n+1},\theta_{2n+2})+\wp_{cb}(\theta_{2n+2},\theta_{2n+3}))\}.\end{aligned}$$

Then, we have the following three cases:
Case IIIa:

$$\wp_{cb}(\theta_{2n+2},\theta_{2n+3}) \preceq \curlywedge\wp_{cb}(\theta_{2n+2},\theta_{2n+3}),$$

which implies $\curlywedge \geq 1$, which is a contradiction.
Case IIIb:

$$\wp_{cb}(\theta_{2n+2},\theta_{2n+3}) \preceq \curlywedge\wp_{cb}(\theta_{2n+1},\theta_{2n+2}). \tag{7}$$

Then by (6) and (7), we get $\wp_{cb}(\theta_{n+1},\theta_{n+2}) \preceq \gamma\wp_{cb}(\theta_n,\theta_{n+1})$, where $\gamma = \max\left\{\curlywedge, \frac{\curlywedge}{2-\curlywedge}\right\} < 1$. Hence $\{\theta_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in G.
Case IIIc:

$$\wp_{cb}(\theta_{2n+2},\theta_{2n+3}) \preceq \frac{1}{2}(\wp_{cb}(\theta_{2n+1},\theta_{2n+2})+\wp_{cb}(\theta_{2n+2},\theta_{2n+3})).$$

Hence, we obtain

$$\wp_{cb}(\theta_{2n+2},\theta_{2n+3}) \preceq \frac{\curlywedge}{(2-\curlywedge)}\wp_{cb}(\theta_{2n+1},\theta_{2n+2}). \tag{8}$$

By using (6) and (8) yields

$$\wp_{cb}(\theta_{n+1},\theta_{n+2}) \preceq \wr\wp_{cb}(\theta_n,\theta_{n+1}), \tag{9}$$

where $0 \leq \wr = \frac{\curlywedge}{2-\curlywedge} < 1$.
Then $\forall n = 0,1,2,\ldots$, we get

$$\wp_{cb}(\theta_{n+1},\theta_{n+2}) \preceq \wr\wp_{cb}(\theta_n,\theta_{n+1}) \preceq \ldots \preceq \wr^{n+1}\wp_{cb}(\theta_0,\theta_1).$$

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For $m, n \in \mathbb{N}$, with $m > n$, we have

$$\begin{aligned}
\wp_{cb}(\theta_n, \theta_m) &\preceq \wp_{cb}(\theta_n, \theta_{n+1}) + \wp_{cb}(\theta_{n+1}, \theta_m) - \wp_{cb}(\theta_{n+1}, \theta_{n+1}) \\
&\preceq \wp_{cb}(\theta_n, \theta_{n+1}) + \wp_{cb}(\theta_{n+1}, \theta_m) \\
&\preceq \wp_{cb}(\theta_n, \theta_{n+1}) + \wp_{cb}(\theta_{n+1}, \theta_{n+2}) + \wp_{cb}(\theta_{n+2}, \theta_m) \\
&\quad - \wp_{cb}(\theta_{n+2}, \theta_{n+2}) \\
&\preceq \wp_{cb}(\theta_n, \theta_{n+1}) + \wp_{cb}(\theta_{n+1}, \theta_{n+2}) + \wp_{cb}(\theta_{n+2}, \theta_m) \\
&\preceq \wp_{cb}(\theta_n, \theta_{n+1}) + \wp_{cb}(\theta_{n+1}, \theta_{n+2}) + \wp_{cb}(\theta_{n+2}, \theta_{n+3}) \\
&\quad + \ldots + \wp_{cb}(\theta_{m-2}, \theta_{m-1})) + \wp_{cb}(\theta_{m-1}, \theta_m).
\end{aligned}$$

Using (9), we get

$$\begin{aligned}
\wp_{cb}(\theta_n, \theta_m) &\preceq \wr^n \wp_{cb}(\theta_0, \theta_1) + \wr^{n+1} \wp_{cb}(\theta_0, \theta_1) + \wr^{n+2} \wp_{cb}(\theta_0, \theta_1) \\
&\quad + \ldots + \wr^{m-2} \wp_{cb}(\theta_0, \theta_1) + \wr^{m-1} \wp_{cb}(\theta_0, \theta_1) \\
&= \sum_{i=1}^{m-n} \wr^{i+n-1} \wp_{cb}(\theta_0, \theta_1).
\end{aligned}$$

Therefore,

$$\begin{aligned}
|\wp_{cb}(\theta_n, \theta_m)| \leq \sum_{i=1}^{m-n} \wr^{i+n-1} |\wp_{cb}(\theta_0, \theta_1)| &= \sum_{t=n}^{m-1} \wr^t |\wp_{cb}(\theta_0, \theta_1)| \\
&\leq \sum_{i=n}^{\infty} \wr^t |\wp_{cb}(\theta_0, \theta_1)| \\
&= \frac{\wr^n}{1 - \wr} |\wp_{cb}(\theta_0, \theta_1)|.
\end{aligned}$$

Hence, we have

$$|\wp_{cb}(\theta_n, \theta_m)| \leq \frac{\wr^n}{1 - \wr} |\wp_{cb}(\theta_0, \theta_1)| \to 0 \quad as \quad n \to \infty.$$

Hence $\{\theta_n\}$ is a Cauchy sequence in G. In all cases above discussed, we get the sequence $\{\theta_n\}_{n \in \mathbb{N}}$ is a Cauchy sequence. Since G is complete, there exists $\theta^* \in G$ such that $\theta_n \to \theta^*$ as $n \to \infty$ and

$$\wp_{cb}(\theta^*, \theta^*) = \lim_{n \to \infty} \wp_{cb}(\theta^*, \theta_n) = \lim_{n \to \infty} \wp_{cb}(\theta_n, \theta_n) = 0$$

By the continuity of Π it follows $\theta_{2n+1} = \Pi\theta_{2n} \to \Pi\theta^*$ as $n \to \infty$.

$$\text{i.e. } \wp_{cb}(\Pi\theta^*, \Pi\theta^*) = \lim_{n \to \infty} \wp_{cb}(\Pi\theta^*, \Pi\theta_{2n}) = \lim_{n \to \infty} \wp_{cb}(\Pi\theta_{2n}, \Pi\theta_{2n}).$$

But

$$\wp_{cb}(\Pi\theta^*, \Pi\theta^*) = \lim_{n \to \infty} \wp_{cb}(\Pi\theta_{2n}, \Pi\theta_{2n}) = \lim_{n \to \infty} \wp_{cb}(\theta_{2n+1}, \theta_{2n+1}) = 0.$$

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Next we have to prove that θ^* is a fixed point of Π.

$$\wp_{cb}(\Pi\theta^*, \theta^*) \preceq \wp_{cb}(\Pi\theta^*, \Pi\theta_{2n}) + \wp_{cb}(\Pi\theta_{2n}, \theta^*) - \wp_{cb}(\Pi\theta_{2n}, \Pi\theta_{2n}).$$

As $n \to \infty$, we obtain $|\wp_{cb}(\Pi\theta^*, \theta^*)| \leq 0$. Thus $\wp_{cp}(\Pi\theta^*, \theta^*) = 0$. Hence $\wp_{cb}(\theta^*, \theta^*) = \wp_{cb}(\theta^*, \Pi\theta^*) = \wp_{cb}(\Pi\theta^*, \Pi\theta^*) = 0$ and $\Pi\theta^* = \theta^*$. In the same way, we have $\theta^* \in G$ such that $\theta_n \to \theta^*$ as $n \to \infty$ and

$$\wp_{cb}(\theta^*, \theta^*) = \lim_{n\to\infty} \wp_{cb}(\theta^*, \theta_n) = \lim_{n\to\infty} \wp_{cb}(\theta_n, \theta_n) = 0$$

By the continuity of Π it follows $\theta_{2n+2} = \Psi\theta_{2n+1} \to \Psi\theta^*$ as $n \to \infty$.

$$\text{i.e. } \wp_{cb}(\Psi\theta^*, \Psi\theta^*) = \lim_{n\to\infty} \wp_{cb}(\Psi\theta^*, \Psi\theta_{2n+1}) = \lim_{n\to\infty} \wp_{cb}(\Psi\theta_{2n+1}, \Psi\theta_{2n+1}).$$

But

$$\wp_{cb}(\Psi\theta^*, \Psi\theta^*) = \lim_{n\to\infty} \wp_{cb}(\Psi\theta_{2n+1}, \Psi\theta_{2n+1}) = \lim_{n\to\infty} \wp_{cb}(\theta_{2n+2}, \theta_{2n+2}) = 0.$$

Next we have to prove that θ^* is a fixed point of Ψ.

$$\wp_{cb}(\Psi\theta^*, \theta^*) \preceq \wp_{cb}(\Psi\theta^*, \Psi\theta_{2n+1}) + \wp_{cb}(\Psi\theta_{2n+1}, \theta^*) - \wp_{cb}(\Psi\theta_{2n+1}, \Pi\theta_{2n+1}).$$

As $n \to \infty$, we obtain $|\wp_{cb}(\Psi\theta^*, \theta^*)| \leq 0$. Thus $\wp_{cp}(\Psi\theta^*, \theta^*) = 0$. Hence $\wp_{cb}(\theta^*, \theta^*) = \wp_{cb}(\theta^*, \Psi\theta^*) = \wp_{cb}(\Psi\theta^*, \Psi\theta^*) = 0$ and $\Psi\theta^* = \theta^*$. Therefore θ^* is a common fixed point of the pair (Π, Ψ).
To prove uniqueness, let us consider $\omega^* \in G$ is another common fixed point for the pair (Π, Ψ). Then

$$\begin{aligned}
\wp_{cb}(\theta^*, \omega^*) &= \wp_{cb}(\Pi\theta^*, \Psi\omega^*) \\
&\preceq \curlywedge \max\{\wp_{cb}(\theta^*, \omega^*), \wp_{cb}(\theta^*, \Pi\theta^*), \wp_{cb}(\omega^*, \Psi\omega^*), \\
&\quad \frac{1}{2}(\wp_{cb}(\theta^*, \Psi\omega^*) + \wp_{cb}(\omega^*, \Pi\theta^*))\} \\
&\preceq \curlywedge \max\{\wp_{cb}(\theta^*, \omega^*), \wp_{cb}(\theta^*, \theta^*), \wp_{cb}(\omega^*, \omega^*), \\
&\quad \frac{1}{2}(\wp_{cb}(\theta^*, \omega^*) + \wp_{cb}(\omega^*, \theta^*))\} \\
&\preceq \curlywedge \wp_{cb}(\theta^*, \omega^*).
\end{aligned}$$

This implies that $\theta^* = \omega^*$. □

In the absence of the continuity condition for the mappings Π and Ψ, we get the the following Theorem.

Theorem 3.2. *Let $(G, \wp_{cb})$ be a complete CPMS and $\Pi, \Psi\colon G \to G$ be two mappings such that*

$$\begin{aligned}
\wp_{cb}(\Pi\theta, \Psi\omega) \preceq \curlywedge \max\{&\wp_{cb}(\theta, \omega), \wp_{cb}(\theta, \Pi\theta), \wp_{cb}(\omega, \Psi\omega), \\
&\frac{1}{2}(\wp_{cb}(\theta, \Psi\omega) + \wp_{cb}(\omega, \Pi\theta))\},
\end{aligned} \tag{10}$$

for all $\theta, \omega \in G$, where $0 \leq \curlywedge < 1$. Then the pair (Π, Ψ) has a unique common fixed point and $\wp_{cb}(\theta^*, \theta^*) = 0$.

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Proof. Following from Theorem 3.1, we get that the sequence $\{\theta_n\}$ is a Cauchy sequence. Since G is complete, there exists $\theta^* \in G$ such that $\theta_n \to \theta^*$ as $n \to \infty$. Since Π and Ψ are not continuous, we have $\wp_{cb}(\theta^*, \Pi\theta^*) = \vartheta > 0$.
Then we estimate

$$\begin{aligned}
\vartheta &= \wp_{cb}(\theta^*, \Pi\theta^*)\\
&\preceq \wp_{cb}(\theta^*, \theta_{2i+2}) + \wp_{cb}(\theta_{2i+2}, \Pi\theta^*) - \wp_{cb}(\theta_{2i+2}, \theta_{2i+2})\\
&\preceq \wp_{cb}(\theta^*, \theta_{2i+2}) + \wp_{cb}(\theta_{2i+2}, \Pi\theta^*)\\
&\preceq \wp_{cb}(\theta^*, \theta_{2i+2}) + \wp_{cb}(\Psi\theta_{2i+1}, \Pi\theta^*)\\
&\preceq \wp_{cb}(\theta^*, \theta_{2i+2}) + \curlywedge \max\{\wp_{cb}(\theta_{2i+1}, \theta^*), \wp_{cb}(\theta_{2i+1}, \Psi\theta_{2i+1}), \wp_{cb}(\theta^*, \Pi\theta^*),\\
&\quad \frac{1}{2}(\wp_{cb}(\theta_{2i+1}, \Pi\theta^*) + \wp_{cb}(\theta^*, \Psi\theta_{2i+1}))\}\\
&\preceq \wp_{cb}(\theta^*, \theta_{2i+2}) + \curlywedge \max\{\wp_{cb}(\theta_{2i+1}, \theta^*), \wp_{cb}(\theta_{2i+1}, \theta_{2i+2}), \wp_{cb}(\theta^*, \Pi\theta^*),\\
&\quad \frac{1}{2}(\wp_{cb}(\theta_{2i+1}, \Pi\theta^*) + \wp_{cb}(\theta^*, \theta_{2i+2}))\}\\
&\preceq \wp_{cb}(\theta^*, \theta_{2i+2}) + \curlywedge \wp_{cb}(\theta^*, \Pi\theta^*)\\
&\preceq s\wp_{cb}(\theta^*, \theta_{2i+2}) + \curlywedge\vartheta.
\end{aligned}$$

This yields,

$$|\vartheta| \leq |\wp_{cb}(\theta^*, \theta_{2i+2})| + \curlywedge|\vartheta|.$$

Hence, $\curlywedge \geq 1$, which is a contradiction. Then $\theta^* = \Pi\theta^*$. In the same way, we obtain $\theta^* = \Psi\theta^*$. Hence θ^* is a common fixed point for the pair (Π, Ψ) and $\wp_{cb}(\theta^*, \theta^*) = \wp_{cb}(\theta^*, \Psi\theta^*) = \wp_{cb}(\Psi\theta^*, \Psi\theta^*) = 0$. For uniqueness of the common fixed point θ^* follows from Theorem 3.1. □

For $\Pi = \Psi$, we get the following fixed points results on CPMS.

Theorem 3.3. *Let $(G, \wp_{cb})$ be a complete CPMS and $\Pi\colon G \to G$ be a continuous mapping such that*

$$\begin{aligned}
\wp_{cb}(\Pi\theta, \Pi\omega) \preceq \curlywedge \max\{&\wp_{cb}(\theta, \omega), \wp_{cb}(\theta, \Pi\theta), \wp_{cb}(\omega, \Pi\omega),\\
&\frac{1}{2}(\wp_{cb}(\theta, \Pi\omega) + \wp_{cb}(\omega, \Pi\theta))\},
\end{aligned} \tag{11}$$

for all $\theta, \omega \in G$, where $0 \leq \curlywedge < 1$. Then the pair Π have a unique fixed point and $\wp_{cb}(\theta^*, \theta^*) = 0$.

Remark 3.4. *Similarly, we get a fixed point result in the absence of continuity condition for the mapping Π.*

Corollary 3.5. *Let $(G, \wp_{cb})$ be a complete CPMS and $\Psi\colon G \to G$ be a continuous mapping such that*

$$\begin{aligned}
\wp_{cb}(\Psi^n\theta, \Psi^n\omega) \preceq \curlywedge \max\{&\wp_{cb}(\theta, \omega), \wp_{cb}(\theta, \Psi^n\theta), \wp_{cb}(\omega, \Psi^n\omega),\\
&\frac{1}{2}(\wp_{cb}(\theta, \Psi^n\omega) + \wp_{cb}(\omega, \Psi^n\theta))\},
\end{aligned}$$

for all $\theta, \omega \in G$*, where* $0 \leq \curlywedge < 1, n \in \mathbb{N}$*. Then* Ψ *has a unique fixed point and* $\wp_{cb}(\theta^*, \theta^*) = 0$.

Proof. By Theorem 3.1, we get $\theta^* \in G$ such that $\Psi^n\theta^* = \theta^*$ and $\wp_{cb}(\theta^*, \theta^*) = 0$. Then we get

$$\begin{aligned}\wp_{cb}(\Psi\theta^*, \theta^*) = \wp_{cb}(\Psi\Psi^n\theta^*, \Psi^n\theta^*) &= \wp_{cb}(\Psi^n\Psi\theta^*, \Psi^n\theta^*)\\
&\preceq \curlywedge \max\{\wp_{cb}(\Psi\theta^*, \theta^*), \wp_{cb}(\Psi\theta^*, \Psi^n\Psi\theta^*), \wp_{cb}(\theta^*, \Psi^n\theta^*),\\
&\quad \frac{1}{2}(\wp_{cb}(\Psi\theta^*, \Psi^n\theta^*) + \wp_{cb}(\theta^*, \Psi^n\Psi\theta^*))\}\\
&\preceq \curlywedge \max\{\wp_{cb}(\Psi\theta^*, \theta^*), \wp_{cb}(\Psi\theta^*, \Psi\theta^*), \wp_{cb}(\theta^*, \theta^*),\\
&\quad \frac{1}{2}(\wp_{cb}(\Psi\theta^*, \theta^*) + \wp_{cb}(\theta^*, \Psi\theta^*))\}\\
&= \curlywedge\wp_{cb}(\Psi\theta^*, \theta^*).\end{aligned}$$

Hence $\Psi^n\theta^* = \Psi\theta^* = \theta^*$. Then Ψ has a unique fixed point. □

Remark 3.6. *From the above corollary 3.5, similarly, we get a fixed point result in the absence of continuity condition for the mapping* Ψ*.*

Next we will present a new generalization of a common fixed point theorem on CPMS.

Theorem 3.7. *Let* $(G, \wp_{cb})$ *be a complete CPMS and* $\Pi, \Psi \colon G \to G$ *be two continuous mappings such that*

$$\wp_{cb}(\Pi\theta, \Psi\omega) \preceq \curlywedge \max\left\{\wp_{cb}(\theta, \omega), \frac{\wp_{cb}(\theta, \Pi\theta)\wp_{cb}(\omega, \Psi\omega)}{1 + \wp_{cb}(\theta, \omega)}, \frac{\wp_{cb}(\theta, \Pi\theta)\wp_{cb}(\Pi\theta, \Psi\omega)}{1 + \wp_{cb}(\theta, \omega)}\right\}, \tag{12}$$

for all $\theta, \omega \in G$*, where* $0 \leq \curlywedge < 1$*. Then the pair* (Π, Ψ) *has a unique common fixed point and* $\wp_{cb}(\theta^*, \theta^*) = 0$.

Proof. Let θ_0 be arbitrary point in G and define a sequence $\{\theta_n\}$ as follows:

$$\theta_{2n+1} = \Pi\theta_{2n} \quad \textit{and} \quad \theta_{2n+2} = \Psi\theta_{2n+1}, n = 0, 1, 2, \ldots \tag{13}$$

Then by (12) and (13), we obtain

$$\begin{aligned}\wp_{cb}(\theta_{2n+1}, \theta_{2n+2}) &= \wp_{cb}(\Pi\theta_{2n}, \Psi\theta_{2n+1})\\
&\preceq \curlywedge \max\{\wp_{cb}(\theta_{2n}, \theta_{2n+1}), \frac{\wp_{cb}(\theta_{2n}, \theta_{2n+1})\wp_{cb}(\Psi\theta_{2n+1}, \Pi\theta_{2n})}{1 + \wp_{cb}(\theta_{2n}, \theta_{2n+1})},\\
&\quad \frac{\wp_{cb}(\theta_{2n}, \Pi\theta_{2n},)\wp_{cb}(\Pi\theta_{2n}, \Psi\theta_{2n+1})}{1 + \wp_{cb}(\theta_{2n}, \theta_{2n+1})}\}\\
&\preceq \curlywedge \max\{\wp_{cb}(\theta_{2n}, \theta_{2n+1}), \frac{\wp_{cb}(\theta_{2n}, \theta_{2n+1})\wp_{cb}(\theta_{2n+1}, \theta_{2n+2})}{1 + \wp_{cb}(\theta_{2n}, \theta_{2n+1})},\\
&\quad \frac{\wp_{cb}(\theta_{2n}, \theta_{2n+1})\wp_{cb}(\theta_{2n+1}, \theta_{2n+2})}{1 + \wp_{cb}(\theta_{2n}, \theta_{2n+1})}\}\\
&\preceq \curlywedge \max\{\wp_{cb}(\theta_{2n}, \theta_{2n+1}), \wp_{cb}(\theta_{2n+1}, \theta_{2n+2})\}.\end{aligned}$$

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If $\max\{\wp_{cb}(\theta_{2n},\theta_{2n+1}),\wp_{cb}(\theta_{2n+1},\theta_{2n+2})\}=\wp_{cb}(\theta_{2n+1},\theta_{2n+2})$, then

$$\wp_{cb}(\theta_{2n+1},\theta_{2n+2})\preceq \curlywedge\wp_{cb}(\theta_{2n+1},\theta_{2n+2}).$$

This shows that $\curlywedge\geq 1$, which is a contradiction. Therefore

$$\wp_{cb}(\theta_{2n+1},\theta_{2n+2})\preceq \curlywedge\wp_{cb}(\theta_{2n},\theta_{2n+1}). \tag{14}$$

Similarly, we obtain

$$\wp_{cb}(\theta_{2n+2},\theta_{2n+3})\preceq \curlywedge\wp_{cb}(\theta_{2n+1},\theta_{2n+2}). \tag{15}$$

From (14) and (15), $\forall n=0,1,2,\ldots$, we get

$$\wp_{cb}(\theta_{n+1},\theta_{n+2})\preceq \curlywedge\wp_{cb}(\theta_{n},\theta_{n+1})\preceq\ldots\preceq \curlywedge^{n+1}\wp_{cb}(\theta_0,\theta_1). \tag{16}$$

For $m,n\in\mathbb{N}$, with $m>n$, we have

$$\begin{aligned}\wp_{cb}(\theta_n,\theta_m)&\preceq \wp_{cb}(\theta_n,\theta_{n+1})+\wp_{cb}(\theta_{n+1},\theta_m)-\wp_{cb}(\theta_{n+1},\theta_{n+1})\\&\preceq \wp_{cb}(\theta_n,\theta_{n+1})+\wp_{cb}(\theta_{n+1},\theta_m)\\&\preceq \wp_{cb}(\theta_n,\theta_{n+1})+\wp_{cb}(\theta_{n+1},\theta_{n+2})+\wp_{cb}(\theta_{n+2},\theta_m)\\&\quad-\wp_{cb}(\theta_{n+2},\theta_{n+2})\\&\preceq \wp_{cb}(\theta_n,\theta_{n+1})+\wp_{cb}(\theta_{n+1},\theta_{n+2})+\wp_{cb}(\theta_{n+2},\theta_m)\\&\preceq \wp_{cb}(\theta_n,\theta_{n+1})+\wp_{cb}(\theta_{n+1},\theta_{n+2})+\wp_{cb}(\theta_{n+2},\theta_{n+3})\\&\quad+\ldots+\wp_{cb}(\theta_{m-2},\theta_{m-1})+s^{m-n}\wp_{cb}(\theta_{m-1},\theta_m).\end{aligned}$$

By using (16), we get

$$\begin{aligned}\wp_{cb}(\theta_n,\theta_m)&\preceq \curlywedge^{n}\wp_{cb}(\theta_0,\theta_1)+\curlywedge^{n+1}\wp_{cb}(\theta_0,\theta_1)+\curlywedge^{n+2}\wp_{cb}(\theta_0,\theta_1)\\&\quad+\ldots+\curlywedge^{m-2}\wp_{cb}(\theta_0,\theta_1)+\curlywedge^{m-1}\wp_{cb}(\theta_0,\theta_1)\\&=\sum_{i=1}^{m-n}\curlywedge^{i+n-1}\wp_{cb}(\theta_0,\theta_1).\end{aligned}$$

Therefore

$$\begin{aligned}|\wp_{cb}(\theta_n,\theta_m)|&\leq\sum_{i=1}^{m-n}\curlywedge^{i+n-1}|\wp_{cb}(\theta_0,\theta_1)|=\sum_{i=1}^{m-n}\curlywedge^{t}|\wp_{cb}(\theta_0,\theta_1)|\\&\leq\sum_{i=n}^{\infty}\curlywedge^{t}|\wp_{cb}(\theta_0,\theta_1)|\\&=\frac{\curlywedge^{n}}{1-\curlywedge}|\wp_{cb}(\theta_0,\theta_1)|.\end{aligned}$$

Hence, we have

$$|\wp_{cb}(\theta_n,\theta_m)|\leq\frac{\curlywedge^{n}}{1-\curlywedge}|\wp_{cb}(\theta_0,\theta_1)|\to 0\quad as\quad n\to\infty.$$

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Hence $\{\theta_n\}$ is a Cauchy sequence in G. Since G is complete, then there exists $\theta^* \in G$ such that $\theta_n \to \theta^*$ as $n \to \infty$ and

$$\wp_{cb}(\theta^*, \theta^*) = \lim_{n\to\infty} \wp_{cb}(\theta^*, \theta_n) = \lim_{n\to\infty} \wp_{cb}(\theta_n, \theta_n) = 0.$$

Since Ψ is continuous yields

$$\theta^* = \lim_{n\to\infty} \theta_{2n+2} = \lim_{n\to\infty} \Psi\theta_{2n+1} = \Psi \lim_{n\to\infty} \theta_{2n+1} = \Psi\theta^*.$$

Similarly, by the continuity of Π, we get $\theta^* = \Pi\theta^*$. Then the pair (Π, Ψ) has a common fixed point. To prove uniqueness, let us consider $\omega^* \in G$ is another common fixed point for the pair (Π, Ψ). Then

$$\begin{aligned}\wp_{cb}(\theta^*, \omega^*) &= \wp_{cb}(\Pi\theta^*, \Psi\omega^*) \\ &\preceq \curlywedge \max\{\wp_{cb}(\theta^*, \omega^*), \frac{\wp_{cb}(\theta^*, \Pi\theta^*)\wp_{cb}(\omega^*, \Psi\omega^*)}{1 + \wp_{cb}(\theta^*, \omega^*)}, \\ &\quad \frac{\wp_{cb}(\theta^*, \Pi\theta^*)\wp_{cb}(\Psi\omega^*, \Pi\theta^*)}{1 + \wp_{cb}(\theta^*, \omega^*)}\} \\ &\preceq \curlywedge \wp_{cb}(\theta^*, \omega^*)\end{aligned}$$

This implies that $\theta^* = \omega^*$. □

In the absence of the continuity condition for the mapping Π and Ψ in the Theorem 3.7, we obtain the following the result.

Theorem 3.8. *Let $(G, \wp_{cb})$ be a complete CPMS and $\Pi, \Psi \colon G \to G$ be two mappings such that*

$$\wp_{cb}(\Pi\theta, \Psi\omega) \preceq \curlywedge \max\left\{\wp_{cb}(\theta, \omega), \frac{\wp_{cb}(\theta, \Pi\theta)\wp_{cb}(\omega, \Psi\omega)}{1 + \wp_{cb}(\theta, \omega)}, \frac{\wp_{cb}(\theta, \Pi\theta)\wp_{cb}(\Pi\theta, \Psi\omega)}{1 + \wp_{cb}(\theta, \omega)}\right\}, \tag{17}$$

for all $\theta, \omega \in G$, where $0 \leq \curlywedge < 1$. Then the pair (Π, Ψ) has a unique common fixed point and $\wp_{cb}(\theta^*, \theta^*) = 0$.

Proof. Following from Theorem 3.7, we get that the sequence $\{\theta_n\}$ is a Cauchy sequence. Since G is complete, then there exists $\theta^* \in G$ such that $\theta_n \to \theta^*$ as $n \to \infty$ and

$$\wp_{cb}(\theta^*, \theta^*) = \lim_{n\to\infty} \wp_{cb}(\theta^*, \theta_n) = \lim_{n\to\infty} \wp_{cb}(\theta_n, \theta_n) = 0.$$

Since Π and Ψ are not continuous, we have $\wp_{cb}(\theta^*, \Pi\theta^*) = \vartheta > 0$.

Then we estimate

$$
\begin{aligned}
\vartheta &= \wp_{cb}(\theta^*, \Pi\theta^*) \\
&\preceq \wp_{cb}(\theta^*, \theta_{2i+2}) + \wp_{cb}(\theta_{2i+2}, \Pi\theta^*) - \wp_{cb}(\theta_{2i+2}, \theta_{2i+2}) \\
&\preceq \wp_{cb}(\theta^*, \theta_{2i+2}) + \wp_{cb}(\Pi\theta^*, \theta_{2i+2}) \\
&\preceq \wp_{cb}(\theta^*, \theta_{2i+2}) + \wp_{cb}(\Pi\theta^*, \Psi\theta_{2i+1}) \\
&\preceq \wp_{cb}(\theta^*, \theta_{2i+2}) + \curlywedge \max\left\{\wp_{cb}(\theta^*, \theta_{2i+1}), \frac{\wp_{cb}(\theta^*, \Pi\theta^*)\wp_{cb}(\theta_{2i+1}, \Psi\theta_{2i+1})}{1+\wp_{cb}(\theta^*, \theta_{2i+1})},\right. \\
&\quad \left.\frac{\wp_{cb}(\theta^*, \Pi\theta_*)\wp_{cb}(\Pi\theta^*, \Psi\theta_{2i+1})}{1+\wp_{cb}(\theta^*, \theta_{2i+1})}\right\} \\
&\preceq \wp_{cb}(\theta^*, \theta_{2i+2}) + \curlywedge \max\left\{\wp_{cb}(\theta^*, \theta_{2i+1}), \frac{\wp_{cb}(\theta^*, \Pi\theta^*)\wp_{cb}(\theta_{2i+1}, \theta_{2i+2})}{1+\wp_{cb}(\theta^*, \theta_{2i+1})}\right\}, \\
&\quad \left.\frac{\wp_{cb}(\theta^*, \Pi\theta_*)\wp_{cb}(\Pi\theta^*, \theta_{2i+2})}{1+\wp_{cb}(\theta^*, \theta_{2i+1})}\right\} \\
&\preceq \wp_{cb}(\theta^*, \theta_{2i+2}) + \curlywedge\wp_{cb}(\theta^*, \Pi\theta^*)^2 \\
&\preceq \wp_{cb}(\theta^*, \theta_{2i+2}) + \curlywedge\vartheta^2.
\end{aligned}
$$

This yields,

$$|\vartheta| \leq |\wp_{cb}(\theta^*, \theta_{2i+2})| + \curlywedge|\vartheta|^2.$$

Hence, $\curlywedge \geq 1$, which is a contradiction. Then $\theta^* = \Pi\theta^*$. In the same way, we obtain $\theta^* = \Psi\theta^*$. Hence θ^* is a common fixed point for the pair (Π, Ψ). For uniqueness of the common fixed point θ^* follows from Theorem 3.7. □

For $\Pi = \Psi$, we get the following fixed points results on CPMS.

Theorem 3.9. *Let $(G, \wp_{cb})$ be a complete CPMS and $\Pi\colon G \to G$ be a continuous mapping such that*

$$\wp_{cb}(\Pi\theta, \Pi\omega) \preceq \curlywedge \max\left\{\wp_{cb}(\theta, \omega), \frac{\wp_{cb}(\theta, \Pi\theta)\wp_{cb}(\omega, \Pi\omega)}{1+\wp_{cb}(\theta, \omega)}, \frac{\wp_{cb}(\theta, \Pi\theta)\wp_{cb}(\Pi\theta, \Pi\omega)}{1+\wp_{cb}(\theta, \omega)}\right\},$$

for all $\theta, \omega \in G$, where $0 \leq \curlywedge < 1$. Then Π has a unique fixed point and $\wp_{cb}(\theta^*, \theta^*) = 0$.

Remark 3.10. *Similarly, in the absence of continuity condition, we can get a fixed point result on Π.*

Corollary 3.11. *Let $(G, \wp_{cb})$ be a complete CPMS and $\Pi\colon G \to G$ be a continuous mapping such that*

$$
\begin{aligned}
\wp_{cb}(\Pi^n\theta, \Pi^n\omega) \preceq \curlywedge \max\Bigg\{&\wp_{cb}(\theta, \omega), \frac{\wp_{cb}(\theta, \Pi^n\theta)\wp_{cb}(\omega, \Pi^n\omega)}{1+\wp_{cb}(\theta, \omega)}, \\
&\frac{\wp_{cb}(\theta, \Pi^n\theta)\wp_{cb}(\Pi^n\theta, \Pi\omega)}{1+\wp_{cb}(\theta, \omega)}\Bigg\},
\end{aligned}
$$

for all $\theta, \omega \in G$, where $0 \leq \curlywedge < 1$. Then Π has a unique fixed point and $\wp_{cb}(\theta^*, \theta^*) = 0$.

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Proof. By Theorem 3.7, we get $\theta^* \in G$ such that $\Pi^n\theta^* = \theta^*$ and $\wp_{cb}(\theta^*, \theta^*) = 0$. Then we get

$$\begin{aligned}
\wp_{cb}(\Pi\theta^*, \theta^*) &= \wp_{cb}(\Pi\Pi^n\theta^*, \Pi^n\theta^*) = \wp_{cb}(\Pi^n\Pi\theta^*, \Pi^n\theta^*) \\
&\preceq \curlywedge \max\Big\{\wp_{cb}(\Pi\theta^*, \theta^*), \frac{\wp_{cb}(\Pi\theta^*, \Pi^n\Pi\theta^*)\wp_{cb}(\theta^*, \Pi^n\theta^*)}{1+\wp_{cb}(\Pi\theta^*, \theta^*)}, \\
&\quad \frac{\wp_{cb}(\Pi\theta^*, \Pi^n\Pi\theta^*)\wp_{cb}(\Pi^n\Pi\theta^*, \Pi^n\theta^*)}{1+\wp_{cb}(\Pi\theta^*, \theta^*)}\Big\} \\
&\preceq \curlywedge \max\Big\{\wp_{cb}(\Pi\theta^*, \theta^*), \frac{\wp_{cb}(\Pi\theta^*, \Pi\Pi^n\theta^*)\wp_{cb}(\theta^*, \Pi^n\theta^*)}{1+\wp_{cb}(\Pi\theta^*, \theta^*)}, \\
&\quad \frac{\wp_{cb}(\Pi\theta^*, \Pi\Pi^n\theta^*)\wp_{cb}(\Pi\Pi^n\theta^*, \Pi^n\theta^*)}{1+\wp_{cb}(\Pi\theta^*, \theta^*)}\Big\} \\
&= \curlywedge\wp_{cb}(\Pi\theta^*, \theta^*).
\end{aligned}$$

Hence $\Pi^n\theta^* = \Pi\theta^* = \theta^*$. Then Π has a unique fixed point. □

Remark 3.12. *From the above corollary 3.11, similarly, we get a fixed point result in the absence of continuity condition for the mapping* Π.

Example 3.13. *Let* $G = \{1, 2, 3, 4\}$ *be endowed with the order* $\theta \preceq \omega$ *if and only if* $\theta \leq \omega$. *Then* $\preceq$ *is a partial order in* G. *Define the complex partial metric space* $\wp_{cb} : G \times G \to \mathbb{C}^+$ *as follows:*

(θ, ω)	$\wp_{cb}(\theta, \omega)$
(1,1), (2,2)	0
(1,2),(2,1),(1,3),(3,1),(2,3),(3,2),(3,3)	e^{ix}
(1,4),(4,1),(2,4),(4,2),(3,4),(4,3),(4,4)	$3e^{ix}$

Obviously, $(G, \wp_{cb})$ *is a complete CPMS for* $x \in [0, \frac{\pi}{2}]$. *Define* $\Pi, \Psi : G \to G$ *by* $\Pi\theta = 1$,

$$\Psi(\theta) = \begin{cases} 1 & \text{if } \theta \in \{1,2,3\} \\ 2 & \text{if} \theta = 4. \end{cases}$$

Clearly Π *and* Ψ *are continuous functions. Now for* $\curlywedge = \frac{1}{3}$, *we consider the following cases:*

(A) If $\theta = 1$ *and* $\omega \in G - \{4\}$, *then* $\Pi(\theta) = \Psi(\omega) = 1$ *and the conditions of Theorem 3.1 satisfied.*

(B) If $\theta = 1$, $\omega = 4$, *then* $\Pi\theta = 1$, $\Psi\omega = 2$,

$$\begin{aligned}
\wp_{cb}(\Pi\theta, \Psi\omega) = e^{ix} &\preceq 3 \curlywedge e^{ix} \\
&= \curlywedge \max\{3e^{ix}, 0, 3e^{ix}, \frac{1}{2}(e^{ix} + 3e^{ix})\} \\
&= \curlywedge \max\{\wp_{cb}(\theta, \omega), \wp_{cb}(\theta, \Pi\theta), \wp_{cb}(\omega, \Psi\omega), \\
&\quad \frac{1}{2}(\wp_{cb}(\theta, \Psi\omega) + \wp_{cb}(\omega, \Pi\theta))\},
\end{aligned}$$

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(C) If $\theta = 2,\ \omega = 4,$ *then* $\Pi\theta = 1,\ \Psi\omega = 2,$

$$\begin{aligned}\wp_{cb}(\Pi\theta, \Psi\omega) = e^{ix} &\preceq 3 \curlywedge e^{ix} \\ &= \curlywedge \max\{3e^{ix}, e^{ix}, 3e^{ix}, \frac{1}{2}(0 + 3e^{ix})\} \\ &= \curlywedge \max\{\wp_{cb}(\theta, \omega), \wp_{cb}(\theta, \Pi\theta), \wp_{cb}(\omega, \Psi\omega), \\ &\quad \frac{1}{2}(\wp_{cb}(\theta, \Psi\omega) + \wp_{cb}(\omega, \Pi\theta))\},\end{aligned}$$

(D) If $\theta = 3,\ \omega = 4,$ *then* $\Pi\theta = 1,\ \Psi\omega = 2,$

$$\begin{aligned}\wp_{cb}(\Pi\theta, \Psi\omega) = e^{ix} &\preceq 3 \curlywedge e^{ix} \\ &= \curlywedge \max\{3e^{ix}, e^{ix}, 3e^{ix}, \frac{1}{2}(e^{ix} + 3e^{ix})\} \\ &= \curlywedge \max\{\wp_{cb}(\theta, \omega), \wp_{cb}(\theta, \Pi\theta), \wp_{cb}(\omega, \Psi\omega), \\ &\quad \frac{1}{2}(\wp_{cb}(\theta, \Psi\omega) + \wp_{cb}(\omega, \Pi\theta))\},\end{aligned}$$

(E) If $\theta = 4,\ \omega = 4,$ *then* $\Pi\theta = 2,\ \Psi\omega = 2,$

$$\begin{aligned}\wp_{cb}(\Pi\theta, \Psi\omega) = e^{ix} &\preceq 3 \curlywedge e^{ix} \\ &= \curlywedge \max\{3e^{ix}, 3e^{ix}, 3e^{ix}, \frac{1}{2}(3e^{ix} + 3e^{ix})\} \\ &= \curlywedge \max\{\wp_{cb}(\theta, \omega), \wp_{cb}(\theta, \Pi\theta), \wp_{cb}(\omega, \Psi\omega), \\ &\quad \frac{1}{2}(\wp_{cb}(\theta, \Psi\omega) + \wp_{cb}(\omega, \Pi\theta))\},\end{aligned}$$

Moreover for $\curlywedge = \frac{1}{3}$, *with* $\curlywedge < 1$, *the conditions of Theorem 3.1 are satisfied. Therefore,* 1 *is the unique common fixed point of* Π *and* Ψ.

4 Application

Consider the following systems of integral equations:

$$w(s) = \int_a^b T_1(s, p, w(p))dp, \tag{18}$$

and

$$z(s) = \int_a^b T_2(s, p, z(p))dp, \tag{19}$$

where

(i) $w(s)$ and $z(s)$ are unknown variables for each $s \in J = [a, b],\ b > a \geq 0$,

(ii) $T_1(s, p)$ and $T_2(s, p)$ are deterministic kernels defined for $s, p \in J = [a, b]$.

In this section, we present an existence theorem for a common solution to (18) and (19) that belongs to $G = (C(J), \mathbb{R}^n)$ (the set of continuous functions defined on J) by using the obtained result in Theorem 3.1. We consider the continuous mappings $\Pi, \Psi : G \to G$ given by

$$\Pi w(s) = \int_a^b T_1(s, p, w(p))dp, w \in G, s \in J,$$

and

$$\Psi z(s) = \int_a^b T_2(s, p, z(p))dp, z \in G, s \in J,$$

Then, the existence of a common solution to the integral equations (18) and (19) is equivalent to the existence of a common fixed point of T_1 and T_2. It is well known that G, endowed with the metric $\wp_{cb}$ defined by

$$\wp_{cb}(w, z) = |w(s) - z(s)| + 2,$$

forall $w, z \in G$ is a complete CPMS. G can also be equipped with the partial order $\preceq$ given by

$$w, z \in G, \ w \preceq z \text{ if and only } w(s) \geq z(s), \text{ for all } s \in J.$$

Further let us consider a system of integral equation as (18) and (19) under the following condition hold:

(A) $T_1, T_2 : J \times J \times \mathbb{R}^n \to \mathbb{R}^n$ are continuous functions satisfying

$$|T_1(s, p, w(p)) - T_2(s, p, z(s))| \preceq \frac{S(w, z)}{(b - a)e^t} - \frac{2}{b - a}, \forall t > 0,$$

where

$$S(w, z) = \max\{\wp_{cb}(w, z), \wp_{cb}(w, \Pi w), \wp_{cb}(z, \Psi z), \frac{1}{2}(\wp_{cb}(w, \Psi z) + \wp_{cb}(z, \Pi w))\}.$$

Theorem 4.1. *Let $(C(J), \mathbb{R}^n, \wp_{cb})$ be a complete CPMS, then the system (18) and (19) under the condition (A) have a unique common solution.*

Proof. For $w, z \in (C(J), \mathbb{R}^n)$ and $s \in J$, we define the continuous mappings $\Pi, \Psi : G \to G$ by

$$\Pi w(s) = \int_a^b T_1(s, p, w(p))dp,$$

and

$$\Psi z(s) = \int_a^b T_2(s, p, z(p))dp.$$

Then we have

$$\begin{aligned}
\wp_{cb}(\Pi w(s), \Psi z(s)) &= |\Pi w(s) - \Psi z(s)| + 2 \\
&= \int_a^b |T_1(s,p,w(p)) - T_2(s,p,z(s))| dp + 2 \\
&\preceq \int_a^b \left(\frac{S(w,z)}{(b-a)e^t} - \frac{2}{b-a} \right) dp + 2 \\
&= \frac{S(w,z)}{e^t} \\
&= \curlywedge S(w,z) \\
&= \curlywedge \max\{\wp_{cb}(w,z), \wp_{cb}(w,\Pi w), \wp_{cb}(z,\Psi z), \\
&\quad \frac{1}{2}(\wp_{cb}(w,\Psi z) + \wp_{cb}(z,\Pi w))\}.
\end{aligned}$$

Hence, all the conditions of Theorem 3.1 are satisfied for $0 < \curlywedge = \frac{1}{e^t} < 1$ with $t > 0$. Therefore the system of integral equations (18) and (19) have a unique common solution. □

5 Conclusion

In this paper, we proved some common fixed point theorems on complex partial metric space. An illustrative example and application on complex partial metric space is given.

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