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New Insight into Quaternions and Their Matrices

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Abstract

The aim of this paper is to bring together quaternions and generalized complex numbers. Generalized quaternions with generalized complex number components are expressed and their algebraic structures are examined. Several matrix representations and computational results are introduced. As a crucial part, alternative approach for generalized quaternion matrix with elliptic number entries are developed.

Keywords: Generalized quaternion, generalized complex number, matrix representation.
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1 Introduction

Hamilton introduced the Hamiltonian quaternions for representing vectors in the space, [27, 28]. The real quaternion is written as $q = a_0 + a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$, where $a_0, a_1, a_2, a_3 \in \mathbb{R}$ are components and $\mathbf{i}, \mathbf{j}, \mathbf{k} \notin \mathbb{R}$ are versors, [26]. The set of real quaternions, as an extension of complex numbers, is an associative but non-commutative Clifford algebra and used in many fields in applied mathematics. The associative quaternions will be divided into two classes: in the first class there are the non-commutative quaternions (Hamiltonian, hyperbolic, split quaternions, generalized quaternions [1, 17, 18, 25, 31, 37, 43–45] etc.), in the second class there are the commutative quaternions (generalized Segre quaternions [13, 14], dual quaternions [16, 19, 20, 35, 36] etc.).

The algebra of generalized quaternions as non-commutative system, denoted by $\mathcal{Q}_{\alpha,\beta}$, includes a variety of well-known four-dimensional algebras as special cases. Conditions of the versors for them are given by:

$$\begin{array}{ccc} \mathbf{i}^2 = -\alpha, & \mathbf{j}^2 = -\beta, & \mathbf{k}^2 = -\alpha\beta, \\ \mathbf{ij} = -\mathbf{ji} = \mathbf{k}, & \mathbf{jk} = -\mathbf{kj} = \beta\mathbf{i}, & \mathbf{ki} = -\mathbf{ik} = \alpha\mathbf{j}, \end{array} \tag{1}$$

where $\alpha, \beta \in \mathbb{R}$. For $\alpha = \beta = 1$ Hamiltonian quaternions, for $\alpha = 1, \beta = -1$ split quaternions, for $\alpha = 1, \beta = 0$ semi quaternions, for $\alpha = -1, \beta = 0$ split-semi quaternions, and for $\alpha = \beta = 0$ quasi quaternions are obtained.

Additionally, the general bidimensional hypercomplex systems (namely generalized complex numbers ($\mathcal{GCN}$)) over the field of real numbers $\mathbb{R}$ is given by the ring ([10–12, 30, 49]):

$$\frac{\mathbb{R}[X]}{\langle h(X)\rangle} \cong \left\{ z = x_1 + x_2 I : \ I^2 = I\mathfrak{q} + \mathfrak{p}, \ \ \mathfrak{p}, \mathfrak{q}, x_1, x_2 \in \mathbb{R}, I \notin \mathbb{R} \right\},$$

where $h(X) = X^2 - \mathfrak{q}X - \mathfrak{p}$ is monic quadratic. By denoting this set with $\mathbb{C}_{\mathfrak{q},\mathfrak{p}}$ it is well known that the sign of $\Delta = \mathfrak{q}^2 + 4\mathfrak{p}$ determines the properties of the general bidimensional systems. These systems are ring isomorphic with one of the following three types:

- for $\Delta > 0$ *hyperbolic system*; the canonical system is the system of hyperbolic (double, split complex, perplex) numbers $\mathbb{H} \cong \mathbb{C}_{0,1}$ with $\mathfrak{p} = 1$, $\mathfrak{q} = 0$, [15, 24, 46]

- for $\Delta < 0$ *elliptic system*; the canonical system is the system of complex (ordinary) numbers $\mathbb{C} \cong \mathbb{C}_{0,-1}$ with $\mathfrak{p} = -1$, $\mathfrak{q} = 0$, [50],
- for $\Delta = 0$ *parabolic system*; the canonical system is the system of dual numbers $\mathbb{D} \cong \mathbb{C}_{0,0}$ with $\mathfrak{p} = 0$, $\mathfrak{q} = 0$, [42, 47].

Regarding to the value $D = z\bar{z} = (x_1 + x_2 I)(x_1 - x_2 I) = {x_1}^2 - \mathfrak{p}{x_2}^2 + \mathfrak{q}x_1x_2$, which is called the characteristic determinant, $z \in \mathbb{C}_{\mathfrak{q},\mathfrak{p}}$ are classified into three types, [10]. Hence $z \in \mathbb{C}_{\mathfrak{q},\mathfrak{p}}$ is called timelike, spacelike or null where $D < 0$, $D > 0$ and $D = 0$, respectively. Then all of the elements of the set $\mathbb{C}_{0,-1}$ are spacelike. For $\mathfrak{q} = 0$, $I^2 = \mathfrak{p} \in \mathbb{R}$, the generalized complex number system is denoted by $\mathbb{C}_{\mathfrak{p}}$ and called $\mathfrak{p}$-complex plane [30].

In this paper, we aim to design generalized quaternions by taking the components as elements of $\mathbb{C}_{\mathfrak{q},\mathfrak{p}}$. Moreover, the algebraic structures and properties of these quaternions are investigated and several types of matrix representations are introduced. Also alternative approach for generalized quaternion matrix with elliptic number entries are developed as a further results.

2 Generalized Quaternions with $\mathcal{GCN}$ Components

In this section, we present mathematical formulations of improved quaternions: *generalized quaternions with* $\mathcal{GCN}$ and examine special matrix correspondences.

Definition 1. *For $\alpha, \beta \in \mathbb{R}$, the set of generalized quaternions with $\mathcal{GCN}$ components are denoted by $\widetilde{\mathcal{Q}}_{\alpha,\beta}$ and the elements of this set are defined as in the form:*

$$\widetilde{q} = a_0 + a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k},$$

where $a_0, a_1, a_2, a_3 \in \mathbb{C}_{\mathfrak{q},\mathfrak{p}}$ and $\mathbf{i}, \mathbf{j}, \mathbf{k} \notin \mathbb{R}$ are versors that satisfy the properties in equation (1).

Axiomatically, the generalized complex unit I commutes with the three quaternion versors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$, that is $\mathbf{i}I = I\mathbf{i}$, $\mathbf{j}I = I\mathbf{j}$ and $\mathbf{k}I = I\mathbf{k}$. It is obvious that for $\mathfrak{q} = 0, \mathfrak{p} = -1, \alpha = 1$, usual complex operator distinct from quaternion versor $\mathbf{i}$.

Throughout this section, $\widetilde{q} = a_0 + a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\widetilde{p} = b_0 + b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k} \in \widetilde{\mathcal{Q}}_{\alpha,\beta}$ are considered. Due to the generalized quaternions with $\mathcal{GCN}$ components are an extension of generalized quaternions, many properties of them are familiar. For any $q \in \widetilde{\mathcal{Q}}_{\alpha,\beta}$, $S_{\widetilde{q}} = a_0$ is the scalar part and $V_{\widetilde{q}} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ is the vector part. Equality of two improved quaternions is as follows: $\widetilde{p} = \widetilde{q} \Leftrightarrow S_{\widetilde{p}} = S_{\widetilde{q}}, V_{\widetilde{p}} = V_{\widetilde{q}}$. Addition (and hence subtraction) of $\widetilde{q}$ to another quaternion $\widetilde{p}$ acts in a componentwise way:

$$\begin{aligned}\widetilde{q} + \widetilde{p} &= (a_0 + b_0) + (a_1 + b_1)\,\mathbf{i} + (a_2 + b_2)\,\mathbf{j} + (a_3 + b_3)\,\mathbf{k} \\ &= S_{\widetilde{p}} + S_{\widetilde{q}} + V_{\widetilde{p}} + V_{\widetilde{q}}.\end{aligned}$$

The scalar multiplication of $\widetilde{q}$ with a scalar gives another improved quaternion

$$\begin{aligned}c\widetilde{q} &= ca_0 + ca_1\mathbf{i} + ca_2\mathbf{j} + ca_3\mathbf{k} \\ &= cS_{\widetilde{q}} + cV_{\widetilde{q}}, \quad c \in \mathbb{C}_{\mathfrak{q},\mathfrak{p}}.\end{aligned}$$

Multiplication of the two quaternions is carried out as follows:

$$\begin{aligned}\widetilde{q}\widetilde{p} &= (a_0b_0 - \alpha a_1b_1 - \beta a_2b_2 - \alpha\beta a_3b_3) + (a_0b_1 + a_1b_0 + \beta a_2b_3 - \beta a_3b_2)\,\mathbf{i} \\ &\quad + (a_0b_2 - \alpha a_1b_3 + a_2b_0 + \alpha a_3b_1)\,\mathbf{j} + (a_0b_3 + a_1b_2 - a_2b_1 + a_3b_0)\,\mathbf{k} \\ &= S_{\widetilde{q}}S_{\widetilde{p}} - \langle V_{\widetilde{q}}, V_{\widetilde{p}}\rangle_g + S_{\widetilde{q}}V_{\widetilde{p}} + S_{\widetilde{p}}V_{\widetilde{q}} + V_{\widetilde{p}} \times_g V_{\widetilde{q}},\end{aligned}$$

(see generalized inner and cross products in [31]). The conjugate of $\widetilde{q}$ is the quaternion $\overline{\widetilde{q}} = a_0 - a_1\mathbf{i} - a_2\mathbf{j} - a_3\mathbf{k} = S_{\widetilde{q}} - V_{\widetilde{q}}$. The norm of $\widetilde{q}$ is

$$\|\widetilde{q}\|^2 = \left|\widetilde{q}\,\overline{\widetilde{q}}\right| = \left|\overline{\widetilde{q}}\,\widetilde{q}\right| = \left|a_0^2 + \alpha a_1^2 + \beta a_2^2 + \alpha\beta a_3^2\right|,$$

and the inverse is

$$(\widetilde{q})^{-1} = \frac{\overline{\widetilde{q}}}{\|\widetilde{q}\|^2} \quad \text{for} \quad \|\widetilde{q}\|^2 \neq 0.$$

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Proposition 1. *$\widetilde{\mathcal{Q}}_{\alpha,\beta}$ is a 4-dimensional module over $\mathbb{C}_{\mathfrak{q},\mathfrak{p}}$ with base $\{1,\mathbf{i},\mathbf{j},\mathbf{k}\}$ and is an 8-dimensional vector space over $\mathbb{R}$ with base $\{1,I,\mathbf{i},I\mathbf{i},\mathbf{j},I\mathbf{j},\mathbf{k},I\mathbf{k}\}$.*

Proposition 2. *For any $\widetilde{q},\widetilde{p}\in\widetilde{\mathcal{Q}}_{\alpha,\beta}$ and $c_1,c_2\in\mathbb{R}$, the conjugate and norm hold following properties:*

i. $\overline{\overline{\widetilde{q}}}=\widetilde{q}$,

ii. $\overline{c_1\widetilde{p}+c_2\widetilde{q}}=c_1\overline{\widetilde{p}}+c_2\overline{\widetilde{q}}$,

iii. $\overline{\widetilde{p}\,\widetilde{q}}=\overline{\widetilde{q}}\,\overline{\widetilde{p}}$,

iv. $\|c_1\widetilde{q}\|=|c_1|\|\widetilde{q}\|$,

v. $\|\widetilde{q}\,\widetilde{p}\|=\|\widetilde{q}\|\|\widetilde{p}\|$,

vi. $\left\|\frac{\widetilde{q}}{\widetilde{p}}\right\|=\frac{\|\widetilde{q}\|}{\|\widetilde{p}\|}$.

Definition 2. *For any $\widetilde{q},\widetilde{p}\in\widetilde{\mathcal{Q}}_{\alpha,\beta}$, the scalar and vector products on $\widetilde{\mathcal{Q}}_{\alpha,\beta}$ are, respectively, defined by:*

$$\begin{aligned}\langle\widetilde{q},\widetilde{p}\rangle_g &= S_{\widetilde{q}}S_{\widetilde{p}}+\langle V_{\widetilde{q}},V_{\widetilde{p}}\rangle_g=a_0b_0+\alpha a_1b_1+\beta a_2b_2+\alpha\beta a_3b_3=S_{\widetilde{q}\,\overline{\widetilde{p}}},\\ \widetilde{q}\times_g\widetilde{p} &= S_{\widetilde{q}}V_{\overline{\widetilde{p}}}+S_{\overline{\widetilde{p}}}V_{\widetilde{q}}-V_{\widetilde{q}}\times_g V_{\widetilde{p}}=V_{\widetilde{q}\,\overline{\widetilde{p}}},\end{aligned}$$

where $\alpha,\beta\in\mathbb{R}^+$ and $\langle,\rangle_g$ is a generalized scalar product.

Remark 1. *As an another perspective to $\widetilde{q}\in\widetilde{\mathcal{Q}}_{\alpha,\beta}$, the following can be calculated:*

$$\begin{aligned}\widetilde{q} &= a_0+a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k}\\ &= (x_{01}+x_{02}I)+(x_{11}+x_{12}I)\,\mathbf{i}+(x_{21}+x_{22}I)\,\mathbf{j}+(x_{31}+x_{32}I)\,\mathbf{k}\\ &= q_0+q_1I,\end{aligned}\tag{2}$$

where $a_i=x_{i1}+x_{i2}I\in\mathbb{C}_{\mathfrak{q},\mathfrak{p}}$, $q_{j-1}=x_{0j}+x_{1j}\mathbf{i}+x_{2j}\mathbf{j}+x_{3j}\mathbf{k}\in Q_{\alpha\beta}$ for $0\leq i\leq 3$, $1\leq j\leq 2$. So, for $\widetilde{q}=q_0+q_1I$ and $\widetilde{p}=p_0+p_1I\in\widetilde{\mathcal{Q}}_{\alpha,\beta}$, the generalized scalar product is also defined by as follows:

$$\langle\widetilde{q},\widetilde{p}\rangle_g = S_{q_0\overline{p}_0}+\mathfrak{p}S_{q_1\overline{p}_1}+\left(S_{q_0\overline{p}_1}+S_{q_1\overline{p}_0}+\mathfrak{q}S_{q_1\overline{p}_1}\right)I.$$

It is worthy to note that $\widetilde{\mathcal{Q}}_{\alpha,\beta}$ is a 2-dimensional module over $\mathcal{Q}_{\alpha,\beta}$ (skew-field) with base $\{1,I\}$. For $\widetilde{q}=q_0+q_1I$ and $\widetilde{p}=p_0+p_1I$, if $\widetilde{p}=\widetilde{q}$, then $p_0=q_0, p_1=q_1$. Also $\widetilde{p}+\widetilde{q}=(p_0+q_0)+(p_1+q_1)\,I$,

$$\widetilde{q}\,\widetilde{p}=(q_0p_0+\mathfrak{p}q_1p_1)+(q_0p_1+q_1p_0+\mathfrak{q}q_1p_1)\,I,$$

and $c\widetilde{q}=cq_0+cq_1I$, $c\in\mathbb{R}$. The conjugate and anti conjugate are $\widetilde{q}^{\dagger_1}=q_0+\mathfrak{q}q_1-q_1I$ and $\widetilde{q}^{\dagger_2}=q_1-q_0I$, respectively. Moduli is $\|\widetilde{q}\|^2_{\dagger_1}=\widetilde{q}\,\widetilde{q}^{\dagger_1}$ and the inverse is $(\widetilde{q})^{-1}=\frac{\widetilde{q}^{\dagger_1}}{\widetilde{q}\,\widetilde{q}^{\dagger_1}}$ for $\widetilde{q}\widetilde{q}^{\dagger_1}\neq 0$. Additionally, the followings hold for $\widetilde{q},\widetilde{p}\in\widetilde{\mathcal{Q}}_{\alpha,\beta}$ and $c_1,c_2\in\mathbb{R}$:

i. $\left(\widetilde{q}^{\dagger_1}\right)^{\dagger_1}=\widetilde{q}$,

ii. $\left(\widetilde{q}^{\dagger_2}\right)^{\dagger_2}=-\widetilde{q}$,

iii. $(c_1\widetilde{q}\pm c_2\widetilde{p})^{\dagger_1}=c_1\widetilde{q}^{\dagger_1}\pm c_2\widetilde{p}^{\dagger_1}$,

iv. $(c_1\widetilde{q}\pm c_2\widetilde{p})^{\dagger_2}=c_1\widetilde{q}^{\dagger_2}\pm c_2\widetilde{p}^{\dagger_2}$,

v. $\widetilde{q}+\widetilde{q}^{\dagger_1}=2q_0+\mathfrak{q}q_1$,

vi. $(\widetilde{q}\,\widetilde{p})^{\dagger_1}=\widetilde{q}^{\dagger_1}\,\widetilde{p}^{\dagger_1}$,

vii. $\|c_1\widetilde{q}\|_{\dagger_1}=|c_1|\,\|\widetilde{q}\|_{\dagger_1}$,

viii. $\|\widetilde{q}\,\widetilde{p}\|_{\dagger_1}=\|\widetilde{q}\|_{\dagger_1}\,\|\widetilde{p}\|_{\dagger_1}$,

ix. $\left\|\frac{\widetilde{q}}{\widetilde{p}}\right\|_{\dagger_1}=\frac{\|\widetilde{q}\|_{\dagger_1}}{\|\widetilde{p}\|_{\dagger_1}}$.

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2.1 Matrix Correspondences

For $\widetilde{q} \in \widetilde{\mathcal{Q}}_{\alpha,\beta}$, $\mathcal{L} : \widetilde{\mathcal{Q}}_{\alpha,\beta} \to \mathcal{R}$, $\mathcal{L}(\widetilde{q}) = \begin{bmatrix} a_0 + a_3\mathbf{k} & a_1\mathbf{i} + a_2\mathbf{j} \\ a_1\mathbf{i} + a_2\mathbf{j} & a_0 + a_3\mathbf{k} \end{bmatrix}$ is a linear transformation, where

$$\mathcal{R} := \left\{ \mathcal{A}_{\widetilde{q}} \in \mathbb{M}_2(\widetilde{\mathcal{Q}}_{\alpha,\beta}) \ : \ \mathcal{A}_{\widetilde{q}} = \begin{bmatrix} a_0 + a_3\mathbf{k} & a_1\mathbf{i} + a_2\mathbf{j} \\ a_1\mathbf{i} + a_2\mathbf{j} & a_0 + a_3\mathbf{k} \end{bmatrix} \right\}$$

is a subset of $\mathbb{M}_2(\widetilde{\mathcal{Q}}_{\alpha,\beta})$. So there exists a correspondence between $\widetilde{\mathcal{Q}}_{\alpha,\beta}$ and $\mathcal{R}$ via the map $\mathcal{L}$. Hence, 2×2 quaternionic matrix representation of $\widetilde{q}$ is $\mathcal{A}_{\widetilde{q}}$.

Theorem 1. *Every generalized quaternion with $\mathcal{GCN}$ components can be represented by a 2×2 quaternionic matrix. $\widetilde{\mathcal{Q}}_{\alpha,\beta}$ is the subset of $\mathbb{M}_2(\widetilde{\mathcal{Q}}_{\alpha,\beta})$.*

Corollary 1. *$\mathcal{L}$ can be determined as the following representation:*

$$\mathcal{L}(a_0 + a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}) = a_0 I_2 + a_1\mathsf{I} + a_2\mathsf{J} + a_3\mathsf{K}, \tag{3}$$

where

$$\mathsf{I} = \begin{bmatrix} 0 & \mathbf{i} \\ \mathbf{i} & 0 \end{bmatrix}, \mathsf{J} = \begin{bmatrix} 0 & \mathbf{j} \\ \mathbf{j} & 0 \end{bmatrix}, \mathsf{K} = \begin{bmatrix} \mathbf{k} & 0 \\ 0 & \mathbf{k} \end{bmatrix}.$$

Thus

$$\begin{array}{ccc} \mathsf{I}^2 = -\alpha I_2, & \mathsf{J}^2 = -\beta I_2, & \mathsf{K}^2 = -\alpha\beta I_2 \\ \mathsf{IJ} = -\mathsf{JI} = \mathsf{K}, & \mathsf{JK} = -\mathsf{KJ} = -\beta\mathsf{I}, & \mathsf{KI} = -\mathsf{IK} = \alpha\mathsf{J}. \end{array}$$

Theorem 2. *For $\widetilde{q}, \widetilde{p} \in \widetilde{\mathcal{Q}}_{\alpha,\beta}$ and $\lambda \in \mathbb{R}$, then the following identities hold:*

i. $\widetilde{q} = \widetilde{p} \Leftrightarrow \mathcal{A}_{\widetilde{q}} = \mathcal{A}_{\widetilde{p}}$,

ii. $\mathcal{A}_{\widetilde{q}+\widetilde{p}} = \mathcal{A}_{\widetilde{q}} + \mathcal{A}_{\widetilde{p}}$,

iii. $\mathcal{A}_{\widetilde{q}\widetilde{p}} = \mathcal{A}_{\widetilde{q}}\mathcal{A}_{\widetilde{p}}$,

iv. $\mathcal{A}_{\lambda\widetilde{q}} = \lambda\mathcal{A}_{\widetilde{q}}$.

For $\widetilde{q} \in \widetilde{\mathcal{Q}}_{\alpha,\beta}$, denote $\mathcal{K}$ as a subset of $\mathbb{M}(\mathbb{C}_{\mathfrak{q},\mathfrak{p}})$ given by:

$$\mathcal{K} := \left\{ \mathcal{B}^l_{\widetilde{q}} \in \mathbb{M}_4(\mathbb{C}_{\mathfrak{q},\mathfrak{p}}) \ : \ \mathcal{B}^l_{\widetilde{q}} = \begin{bmatrix} a_0 & -\alpha a_1 & -\beta a_2 & -\alpha\beta a_3 \\ a_1 & a_0 & -\beta a_3 & \beta a_2 \\ a_2 & \alpha a_3 & a_0 & -\alpha a_1 \\ a_3 & -a_2 & a_1 & a_0 \end{bmatrix} \right\}$$

and define the map $\mathcal{N} : \widetilde{\mathcal{Q}}_{\alpha,\beta} \to \mathcal{K}$, $\mathcal{N}(\widetilde{q}) = \begin{bmatrix} a_0 & -\alpha a_1 & -\beta a_2 & -\alpha\beta a_3 \\ a_1 & a_0 & -\beta a_3 & \beta a_2 \\ a_2 & \alpha a_3 & a_0 & -\alpha a_1 \\ a_3 & -a_2 & a_1 & a_0 \end{bmatrix}$.

There exists a correspondence between $\widetilde{\mathcal{Q}}_{\alpha,\beta}$ and $\mathcal{K}$ via the map $\mathcal{N}$. $\mathcal{B}^l_{\widetilde{q}}$ is the 4×4 left generalized complex matrix representation of $\widetilde{q}$ according to the standard basis $\{1, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$.

4×4 right generalized complex matrix representation of $\widetilde{q}$ can be calculated similarly[1]. Throughout this paper $\mathcal{B}^l_{\widetilde{q}}$ will be considered.

Theorem 3. *Every generalized quaternion with $\mathcal{GCN}$ components can be represented by a 4×4 generalized complex matrix. $\widetilde{\mathcal{Q}}_{\alpha,\beta}$ is the subset of $\mathbb{M}_4(\mathbb{C}_{\mathfrak{q},\mathfrak{p}})$.*

[1] 4×4 right generalized complex matrix representation of $\widetilde{q}$ is:

$$\mathcal{B}^r_{\widetilde{q}} = \begin{bmatrix} a_0 & -\alpha a_1 & -\beta a_2 & -\alpha\beta a_3 \\ a_1 & a_0 & \beta a_3 & -\beta a_2 \\ a_2 & -\alpha a_3 & a_0 & \alpha a_1 \\ a_3 & a_2 & -a_1 & a_0 \end{bmatrix}.$$

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Corollary 2. *Considering the basis* $\{1, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$, *the column matrix representation of* $\widetilde{p} \in \widetilde{\mathcal{Q}}_{\alpha,\beta}$ *is given by:*

$$\widetilde{p} = \left[\begin{array}{cccc} b_0 & b_1 & b_2 & b_3 \end{array} \right]^T .$$

By using $\mathcal{B}^l_{\widetilde{q}}$, *the multiplication of* $\widetilde{q}, \widetilde{p} \in \widetilde{\mathcal{Q}}_{\alpha,\beta}$ *can also be written by:* $\widetilde{q}\widetilde{p} = \mathcal{B}^l_{\widetilde{q}}\widetilde{p}$.

Theorem 4. *Let* $\widetilde{q} \in \widetilde{\mathcal{Q}}_{\alpha,\beta}$. $\mathcal{B}^l_{\widetilde{q}}$ *can be determined as:*

$$\mathcal{B}^l_{\widetilde{q}} = a_0 I_4 + a_1\mathbf{I} + a_2\mathbf{J} + a_3\mathbf{K},$$

where

$$\mathbf{I} = \left[\begin{array}{cccc} 0 & -\alpha & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\alpha \\ 0 & 0 & 1 & 0 \end{array} \right], \mathbf{J} = \left[\begin{array}{cccc} 0 & 0 & -\beta & 0 \\ 0 & 0 & 0 & \beta \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{array} \right], \mathbf{K} = \left[\begin{array}{cccc} 0 & 0 & 0 & -\alpha\beta \\ 0 & 0 & -\beta & 0 \\ 0 & \alpha & 0 & 0 \\ 1 & 0 & 0 & 0 \end{array} \right].$$

Undoubtedly, $\mathbf{I}, \mathbf{J}, \mathbf{K}$ *satisfy the generalized quaternion versors conditions in equation* (1).

Theorem 5. *Let* $\widetilde{q} \in \widetilde{\mathcal{Q}}_{\alpha,\beta}$. *Then, we have*

$$\sigma\mathcal{B}^l_{\widetilde{q}}\sigma = \mathcal{B}^l_{\widetilde{q}^*},$$

where $\sigma = \operatorname{diag}(1, 1, -1, -1)$ *and* $\widetilde{q}^* = a_0 + a_1\mathbf{i} - a_2\mathbf{j} - a_3\mathbf{k} \in \widetilde{\mathcal{Q}}_{\alpha,\beta}$.

Theorem 6. *Let* $\widetilde{q}, \widetilde{p} \in \widetilde{\mathcal{Q}}_{\alpha,\beta}$ *and* $\lambda \in \mathbb{R}$, *the following properties are satisfied:*

i. $\widetilde{q} = \widetilde{p} \Leftrightarrow \mathcal{B}^l_{\widetilde{q}} = \mathcal{B}^l_{\widetilde{p}}$,

ii. $\mathcal{B}^l_{\widetilde{q}+\widetilde{p}} = \mathcal{B}^l_{\widetilde{q}} + \mathcal{B}^l_{\widetilde{p}}$,

iii. $\mathcal{B}^l_{\lambda\widetilde{q}} = \lambda(\mathcal{B}_{\widetilde{q}})$,

iv. $\mathcal{B}^l_{\widetilde{q}\widetilde{p}} = \mathcal{B}^l_{\widetilde{q}}\mathcal{B}^l_{\widetilde{p}}$,

v. $\det(\mathcal{B}^l_{\widetilde{q}}) = \left(a_0^2 + \alpha a_1^2 + \beta a_2^2 + \alpha\beta a_3^2\right)^2 = \|\widetilde{q}\|^4$,

vi. $\operatorname{tr}(\mathcal{B}^l_{\widetilde{q}}) = 4S_{\widetilde{q}}$.

Theorem 7. *Let* $\widetilde{q} \in \widetilde{\mathcal{Q}}_{\alpha,\beta}$ *and* $\widetilde{q}^{-1}$ *be inverse of* $\widetilde{q}$. *Then,*

$$\mathcal{B}^l_{\widetilde{q}^{-1}} = \frac{1}{\sqrt{\det(\mathcal{B}^l_{\widetilde{q}})}}\mathcal{B}^l_{\widetilde{q}}.$$

For $\widetilde{q} = q_0 + q_1 I \in \widetilde{\mathcal{Q}}_{\alpha,\beta}$, denote $\mathcal{T}$ as a subset of $\mathbb{M}_2(Q_{\alpha\beta})$ given by:

$$\mathcal{T} := \left\{ \mathcal{D}_{\widetilde{q}} \in \mathbb{M}_2(Q_{\alpha\beta}) \, : \, \mathcal{D}_{\widetilde{q}} = \left[\begin{array}{cc} q_0 & \mathfrak{p}q_1 \\ q_1 & q_0 + \mathfrak{q}q_1 \end{array} \right] \right\},$$

and define the map $\mathcal{M} : \widetilde{\mathcal{Q}}_{\alpha,\beta} \to \mathcal{T}$, $\mathcal{M}(\widetilde{q}) = \left[\begin{array}{cc} q_0 & \mathfrak{p}q_1 \\ q_1 & q_0 + \mathfrak{q}q_1 \end{array} \right]$.

It can be concluded that there exists a correspondence between $\widetilde{\mathcal{Q}}_{\alpha,\beta}$ and $\mathcal{T}$ via the map $\mathcal{M}$. Hence, 2×2 generalized complex matrix representation of $\widetilde{q}$ with respect to the standard basis $\{1, I\}$ is the matrix $\mathcal{D}_{\widetilde{q}}$.

Theorem 8. *Every* $\mathcal{GCN}$ *with generalized quaternion components can be represented by a* 2×2 *generalized quaternion matrix.* $\widetilde{\mathcal{Q}}_{\alpha,\beta}$ *is the subset of* $\mathbb{M}_2(Q_{\alpha\beta})$.

By using $\mathcal{D}_{\widetilde{q}}$ and $\widetilde{p} = \left[\begin{array}{cc} p_0 & p_1 \end{array} \right]^T$, we have: $\widetilde{q}\widetilde{p} = \mathcal{D}_{\widetilde{q}}\widetilde{p}$. Moreover, $\mathcal{D}_{\widetilde{q}}$ is also in the form $\mathcal{D}_{\widetilde{q}} = q_0 I_2 + q_1 \mathsf{I}$, where $\mathsf{I} = \left[\begin{array}{cc} 0 & \mathfrak{p} \\ 1 & \mathfrak{q} \end{array} \right]$ is the representation of I. It should be noted that there are many ways to choose I, for instance: $\mathsf{I} = \left[\begin{array}{cc} \mathfrak{q} & 1 \\ \mathfrak{p} & 0 \end{array} \right]$ (see in [39]).

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Theorem 9. *For any $\widetilde{q} = q_0 + q_1 I$ and $\widetilde{p} = p_0 + p_1 I \in \widetilde{\mathcal{Q}}_{\alpha,\beta}$ and $\lambda \in \mathbb{R}$, the following properties are satisfied:*

i. $\widetilde{q} = \widetilde{p} \Leftrightarrow \mathcal{D}_{\widetilde{q}} = \mathcal{D}_{\widetilde{p}}$,

ii. $\mathcal{D}_{\widetilde{q}+\widetilde{p}} = \mathcal{D}^l_{\widetilde{q}} + \mathcal{D}_{\widetilde{p}}$,

iii. $\mathcal{D}_{\lambda\widetilde{q}} = \lambda(\mathcal{D}_{\widetilde{q}})$,

iv. $\mathcal{D}_{\widetilde{q}\widetilde{p}} = \mathcal{D}_{\widetilde{q}}\mathcal{D}_{\widetilde{p}}$,

v. $\det(\mathcal{D}_{\widetilde{q}}) = q_0^2 + \mathfrak{q}q_1q_0 - \mathfrak{p}q_1^2$, *where the notation* det *represents the determinant of the quaternion matrix.* [2]

Definition 3. *Let $\widetilde{q} = q_0 + q_1 I \in \widetilde{\mathcal{Q}}_{\alpha,\beta}$. The vector representation of $\widetilde{q}$ is defined as*

$$\vec{\widetilde{q}} = \begin{bmatrix} \vec{q}_0 & \vec{q}_1 \end{bmatrix}^T = \begin{bmatrix} \vec{q}_0 \\ \vec{q}_1 \end{bmatrix},$$

where $q_{j-1} = x_{0j} + x_{1j}\mathbf{i} + x_{2j}\mathbf{j} + x_{3j}\mathbf{k} \in \mathcal{Q}_{\alpha,\beta}$ and

$$\vec{q}_{j-1} = (x_{0j}, x_{1j}, x_{2j}, x_{3j}) = [x_{0j}\ x_{1j}\ x_{2j}\ x_{3j}]^T$$

are vectors (matrices) for $1 \leq j \leq 2$.

Theorem 10. *Let $\widetilde{q} = q_0 + q_1 I \in \widetilde{\mathcal{Q}}_{\alpha,\beta}$. Then*

i. $\mathcal{X}\vec{\widetilde{q}} = \vec{\widetilde{q}}^{\dagger_1}$, *where* $\mathcal{X} = \begin{bmatrix} I_4 & \mathfrak{q}I_4 \\ 0 & -I_4 \end{bmatrix} \in \mathbb{M}_8(\mathbb{R})$.

ii. $\mathcal{Y}\vec{\widetilde{q}} = \vec{\widetilde{q}}^{\dagger_2}$, *where* $\mathcal{Y} = \begin{bmatrix} 0 & I_4 \\ -I_4 & 0 \end{bmatrix} \in \mathbb{M}_8(\mathbb{R})$.

By applying the isomorphism $\Gamma(x_{i1} + x_{i2}I) = \begin{bmatrix} x_{i1} & \mathfrak{p}x_{i2} \\ x_{i2} & x_{i1} + \mathfrak{q}x_{i2} \end{bmatrix}$ to $\mathcal{B}^l_{\widetilde{q}}$, where $x_{i1} + x_{i2}I \in \mathbb{C}_{\mathfrak{q},\mathfrak{p}}$, for $0 \leq i \leq 3$, the left real matrix representation of $\widetilde{q}$ (see in (2)) with respect to the base $\{1, I, \mathbf{i}, I\mathbf{i}, \mathbf{j}, I\mathbf{j}, \mathbf{k}, I\mathbf{k}\}$ is given by:

$$\mathcal{C}^l_{\widetilde{q}} = \begin{bmatrix} x_{01} & \mathfrak{p}x_{02} & -\alpha x_{11} & -\alpha\mathfrak{p}x_{12} & -\beta x_{21} & -\beta\mathfrak{p}x_{22} & -\alpha\beta x_{31} & -\alpha\beta\mathfrak{p}x_{32} \\ x_{02} & x_{01}+\mathfrak{q}x_{02} & -\alpha x_{12} & -\alpha\left(x_{11}+\mathfrak{q}x_{12}\right) & -\beta x_{22} & -\beta\left(x_{21}+\mathfrak{q}x_{22}\right) & -\alpha\beta x_{32} & -\alpha\beta\left(x_{31}+\mathfrak{q}x_{32}\right) \\ x_{11} & \mathfrak{p}x_{12} & x_{01} & \mathfrak{p}x_{02} & \beta x_{31} & \beta\mathfrak{p}x_{32} & -\beta x_{21} & -\mathfrak{p}\beta x_{22} \\ x_{12} & x_{11}+\mathfrak{q}x_{12} & x_{02} & x_{01}+\mathfrak{q}x_{02} & \beta x_{32} & \beta\left(x_{31}+\mathfrak{q}x_{32}\right) & -\beta x_{22} & -\beta\left(x_{21}+\mathfrak{q}x_{22}\right) \\ x_{21} & \mathfrak{p}x_{22} & \alpha x_{31} & \mathfrak{p}\alpha x_{32} & x_{01} & \mathfrak{p}x_{02} & -\alpha x_{11} & -\mathfrak{p}\alpha x_{12} \\ x_{22} & x_{21}+\mathfrak{q}x_{22} & \alpha x_{32} & \alpha\left(x_{31}+\mathfrak{q}x_{32}\right) & x_{02} & x_{01}+\mathfrak{q}x_{02} & -\alpha x_{12} & -\alpha\left(x_{11}+\mathfrak{q}x_{12}\right) \\ x_{31} & \mathfrak{p}x_{32} & -x_{21} & -\mathfrak{p}x_{22} & x_{11} & \mathfrak{p}x_{12} & x_{01} & \mathfrak{p}x_{02} \\ x_{32} & x_{31}+\mathfrak{q}x_{32} & -x_{22} & -x_{21}-\mathfrak{q}x_{22} & x_{12} & x_{11}+\mathfrak{q}x_{12} & x_{02} & x_{01}+\mathfrak{q}x_{02} \end{bmatrix}.$$

So, $\widetilde{\mathcal{Q}}_{\alpha,\beta}$ is the subset of $\mathbb{M}_8(\mathbb{R})$.

Example 1. *Take $\widetilde{q} \in \widetilde{\mathcal{Q}}_{2,1}$ with $\mathcal{GCN}$ components for $\mathfrak{p} = -1$ and $\mathfrak{q} = 1$:*

$$\widetilde{q} = 1 + (-1+I)\,\mathbf{i} + I\mathbf{j} + (1+2I)\,\mathbf{k}.$$

Then,

$$\mathcal{A}_{\widetilde{q}} = \begin{bmatrix} 1 + I\mathbf{k} & (-1+I)\,\mathbf{i} + I\mathbf{j} \\ (-1+I)\,\mathbf{i} + I\mathbf{j} & 1 + (1+2I)\mathbf{k} \end{bmatrix},$$

$$\mathcal{B}^l_{\widetilde{q}} = \begin{bmatrix} 1 & -2 & -I & -2(1+2I) \\ -1+I & 1 & -1-2I & I \\ I & 2(1+2I) & 1 & -2(-1+I) \\ 1+2I & -I & -1+I & 1 \end{bmatrix},$$

[2]The determinant of an arbitrary 2×2 quaternion matrix is defined by $\det\left(\begin{bmatrix} \mathsf{a} & \mathsf{b} \\ \mathsf{c} & \mathsf{d} \end{bmatrix}\right) = \mathsf{da} - \mathsf{cb}$, [51].

$$\mathcal{C}^l_{\widetilde{q}} = \begin{bmatrix} 1 & 0 & -2 & 0 & 0 & 1 & -2 & 4 \\ 0 & 1 & 0 & -2 & -1 & -1 & -4 & -6 \\ -1 & -1 & 1 & 0 & -1 & 2 & 0 & -1 \\ 1 & 0 & 0 & 1 & -2 & -3 & 1 & 1 \\ 0 & -1 & 2 & -4 & 1 & 0 & 2 & 2 \\ 1 & 1 & 4 & 6 & 0 & 1 & -2 & 0 \\ 1 & -2 & 0 & 1 & -1 & -1 & 1 & 0 \\ 2 & 3 & -1 & -1 & 1 & 0 & 0 & 1 \end{bmatrix},$$

$$\mathcal{D}_{\widetilde{q}} = \begin{bmatrix} 1-\mathbf{i}+\mathbf{k} & -\mathbf{i}-\mathbf{j}-2\mathbf{k} \\ \mathbf{i}+\mathbf{j}+2\mathbf{k} & 1+\mathbf{j}+3\mathbf{k} \end{bmatrix},$$

and

$$\mathcal{B}^l_{\widetilde{q}^{-1}} = \frac{1}{\sqrt{-237+51I}} \begin{bmatrix} 1 & -2 & I & 2(1+2I) \\ -1+I & 1 & 1+2I & -I \\ -I & -2(1+2I) & 1 & -2(-1+I) \\ -1-2I & I & -1+I & 1 \end{bmatrix}.$$

Also, the vector representation of $\widetilde{q}^{\dagger}$ *is computed by:*

$$\begin{aligned} \vec{\widetilde{q}^{\dagger_1}} = \mathcal{X}\vec{\widetilde{q}} &= \begin{bmatrix} I_4 & I_4 \\ 0 & -I_4 \end{bmatrix} \begin{bmatrix} \begin{bmatrix} 1 & -1 & 0 & 1 \end{bmatrix}^T \\ \begin{bmatrix} 0 & 1 & 1 & 2 \end{bmatrix}^T \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 1 & 3 & 0 & -1 & -1 & -2 \end{bmatrix}^T. \end{aligned}$$

3 Further Result: An Alternative Matrix Approach

The questions about numbers, hypercomplex numbers and quaternions included questions about their matrices. Inspired by matrix forms in the study [23], we give an answer for the question of the alternative representation of generalized quaternion matrix with elliptic number entries (see elliptic biquaternions in [40]). So this matrix is in the form:

$$\widetilde{Q} = A_0 I_2 + A_1\mathcal{I} + A_2\mathcal{J} + A_3\mathcal{K},$$

where $A_0, A_1, A_2, A_3 \in \mathbb{C}_{\mathfrak{p}}$ are elliptic numbers for $\mathfrak{p} < 0$. The base elements can be defined as follows:
Case 1: For $\alpha, \beta \in \mathbb{R}^+$

$$\mathcal{I} = \begin{bmatrix} \sqrt{\frac{\alpha}{|\mathfrak{p}|}}I & 0 \\ 0 & -\sqrt{\frac{\alpha}{|\mathfrak{p}|}}I \end{bmatrix}, \mathcal{J} = \begin{bmatrix} 0 & \sqrt{\beta} \\ -\sqrt{\beta} & 0 \end{bmatrix}, \mathcal{K} = \begin{bmatrix} 0 & \sqrt{\frac{\alpha\beta}{|\mathfrak{p}|}}\ I \\ \sqrt{\frac{\alpha\beta}{|\mathfrak{p}|}}\ I & 0 \end{bmatrix},$$

Case 2: For $\alpha \in \mathbb{R}^+$, $\beta \in \mathbb{R}^-$

$$\mathcal{I} = \begin{bmatrix} \sqrt{\frac{\alpha}{|\mathfrak{p}|}}I & 0 \\ 0 & -\sqrt{\frac{\alpha}{|\mathfrak{p}|}}I \end{bmatrix}, \mathcal{J} = \begin{bmatrix} 0 & \sqrt{-\beta} \\ \sqrt{-\beta} & 0 \end{bmatrix}, \mathcal{K} = \begin{bmatrix} 0 & \sqrt{\frac{-\alpha\beta}{|\mathfrak{p}|}}\ I \\ -\sqrt{\frac{-\alpha\beta}{|\mathfrak{p}|}}\ I & 0 \end{bmatrix},$$

Case 3: For $\alpha \in \mathbb{R}^-$, $\beta \in \mathbb{R}^+$

$$\mathcal{I} = \begin{bmatrix} 0 & \sqrt{-\alpha} \\ \sqrt{-\alpha} & 0 \end{bmatrix}, \mathcal{J} = \begin{bmatrix} -\sqrt{\frac{\beta}{|\mathfrak{p}|}}I & 0 \\ 0 & \sqrt{\frac{\beta}{|\mathfrak{p}|}}I \end{bmatrix}, \mathcal{K} = \begin{bmatrix} 0 & \sqrt{\frac{-\alpha\beta}{|\mathfrak{p}|}}\ I \\ -\sqrt{\frac{-\alpha\beta}{|\mathfrak{p}|}}\ I & 0 \end{bmatrix},$$

Case 4: For $\alpha, \beta \in \mathbb{R}^-$

$$\mathcal{I} = \begin{bmatrix} 0 & \sqrt{\frac{-\alpha}{|\mathfrak{p}|}}I \\ -\sqrt{\frac{-\alpha}{|\mathfrak{p}|}}I & 0 \end{bmatrix}, \mathcal{J} = \begin{bmatrix} 0 & \sqrt{-\beta} \\ \sqrt{-\beta} & 0 \end{bmatrix}, \mathcal{K} = \begin{bmatrix} \sqrt{\frac{\alpha\beta}{|\mathfrak{p}|}}\ I & 0 \\ 0 & -\sqrt{\frac{\alpha\beta}{|\mathfrak{p}|}}\ I \end{bmatrix}.$$

These elements satisfy the following conditions:

$$\begin{array}{ll} \mathcal{I}^2 = -\alpha I_2 & \mathcal{I}\mathcal{J} = -\mathcal{J}\mathcal{I} = \mathcal{K} \\ \mathcal{J}^2 = -\beta I_2 & \mathcal{J}\mathcal{K} = -\mathcal{K}\mathcal{J} = \beta\mathcal{I} \\ \mathcal{K}^2 = -\alpha\beta I_2 & \mathcal{K}\mathcal{I} = -\mathcal{I}\mathcal{K} = \alpha\mathcal{J}. \end{array}$$

Taking into account Case 1, $\widetilde{Q}$ is rewritten as

$$\widetilde{Q} = \begin{bmatrix} A_0 + \sqrt{\frac{\alpha}{|\mathfrak{p}|}} I A_1 & \sqrt{\beta} A_2 + \sqrt{\frac{\alpha\beta}{|\mathfrak{p}|}}\, I A_3 \\ -\sqrt{\beta} A_2 + \sqrt{\frac{\alpha\beta}{|\mathfrak{p}|}}\, I A_3 & A_0 - \sqrt{\frac{\alpha}{|\mathfrak{p}|}} I A_1 \end{bmatrix}.$$

One can see this matrix in Tian's paper [48] related to biquaternions (complexified quaternion) for $\alpha = \beta = 1$ and $\mathfrak{p} = -1$.

The conjugate (same as the adjoint), transpose, the elliptic conjugate, the total conjugate and determinant $\widetilde{Q}$ can be given as follows:

$$\begin{aligned} \overline{\widetilde{Q}} &= A_0 I_2 - A_1\mathcal{I} - A_2\mathcal{J} - A_3\mathcal{K} = \operatorname{Adj}\widetilde{Q}, \\ \widetilde{Q}^T &= A_0 I_2 + A_1\mathcal{I} - A_2\mathcal{J} + A_3\mathcal{K}, \\ \widetilde{Q}^{\mathbb{C}_\mathfrak{p}} &= A_0 I_2 - A_1\mathcal{I} + A_2\mathcal{J} - A_3\mathcal{K} = \overline{\widetilde{Q}}^T, \\ \overline{\widetilde{Q}}^{\mathbb{C}_\mathfrak{p}} &= A_0 I_2 + A_1\mathcal{I} - A_2\mathcal{J} + A_3\mathcal{K} = \overline{\left(\widetilde{Q}^{\mathbb{C}_\mathfrak{p}}\right)}, \end{aligned}$$

and

$$\begin{aligned} \det\widetilde{Q} &= A_0^2 + \alpha A_1^2 + \beta A_2^2 + \alpha\beta A_3^2 \\ &= A_0^2 + A_1^2 \det\mathcal{I} + A_2^2 \det\mathcal{J} + A_3^2 \det\mathcal{K}. \end{aligned}$$

For $\det\widetilde{Q} \neq 0$ then the inverse of $\widetilde{Q}$ is defined by:

$$\widetilde{Q}^{-1} = \frac{1}{\det\widetilde{Q}}\overline{\widetilde{Q}} = \frac{1}{A_0^2 + \alpha A_1^2 + \beta A_2^2 + \alpha\beta A_3^2}(A_0 I_2 - A_1\mathcal{I} - A_2\mathcal{J} - A_3\mathcal{K}).$$

Similar calculations can be given for other cases. Additionally, the relationships between the above operations and some properties of generalized quaternion matrices with elliptic number entries can be easily proved so we omit them. For $A_0, A_1, A_2, A_3 \in \mathbb{C}_{-1}$, we refer to [2] under the condition that $\alpha = \beta = 1$ and $\alpha = 1, \beta = -1$.

4 Concluding Remarks

Our paper is motivated by the question: What happens if the components of quaternions become $\mathcal{GCN}$? Based on this question, we develop the theory of generalized quaternions (non-commutative system) with $\mathcal{GCN}$ components for $\mathfrak{p}, \mathfrak{q} \in \mathbb{R}$. Also, we investigate the algebraic structures and properties by considering them as a $\mathcal{GCN}$ and as a quaternion. With specific values of α and β, we obtained different types of quaternions with $\mathcal{GCN}$ components in Section 3. Additionally, we established matrix representations and gave a numerical example. In Section 3, an alternative approach for generalized quaternion matrix with elliptic number entries are developed.

The crucial part of this paper is that one can reduce the calculations to mentioned types of quaternions with hyperbolic, elliptic and parabolic number components considering $\Delta = \mathfrak{q}^2 + 4\mathfrak{p}$ (see in Table 1). As a natural consequence of this situation, taking into account special conditions, definition of special quaternions which mentioned in the papers [4, 6–9, 22, 29, 33, 34, 38] are generalized via Definition 1, the papers [3, 21, 32, 41] are generalized from the viewpoint of definition, algebraic properties, relations and matrix representations of quaternions and finally different matrix forms in the papers [2, 40, 48] are generalized in Section 3. All of these situations can be examined in Table 2. For instance, all of the obtained calculations are in agreement with complex quaternions for $\alpha = \beta = 1, \mathfrak{q} = 0, \mathfrak{p} = -1$.

With this unified method, we believe that these results give rise to ease of calculation via mathematical concordance and in the future studies, we intend to investigate another commutative and non-commutative quaternions created with $\mathcal{GCN}$ components in this manner. Now, the necessary and sufficient condition for similarity, co-similarity and semi-similarity for elements of set of the generalized quaternions with $\mathcal{GCN}$ components for $\mathfrak{p}, \mathfrak{q} \in \mathbb{R}$ is an open problem for researchers.

Table 1: Basic classification regarding components

$\Delta = \mathfrak{q}^2 + 4\mathfrak{p}$	Type of components	References
$\Delta < 0$	elliptic	biquaternion [40, 41] (for $\mathfrak{q} = 0$)
$\Delta = 0$	parabolic	[7] (for $\mathfrak{q} = 0$)
$\Delta > 0$	hyperbolic	

Table 2: Classification considering components with regard to the value of $\mathfrak{p}, \mathfrak{q}, \alpha$ and β

Condition	α	β	Type of components	Type of quaternion	References
$\mathfrak{q} = 0, \mathfrak{p} = -1$	1	1	complex	Hamiltonian	biquaternion [16, 29, 48]
	1	-1	complex	split	[21]
	1	0	complex	semi	[4, 9]
	-1	0	complex	split semi	
	0	0	complex	quasi	
$\mathfrak{q} = 0, \mathfrak{p} = 0$	1	1	dual	Hamiltonian	[33, 38]
	1	-1	dual	split	[34]
	1	0	dual	semi	[8]
	-1	0	dual	split semi	[32]
	0	0	dual	quasi	
$\mathfrak{q} = 0, \mathfrak{p} = 1$	1	1	hyperbolic	Hamiltonian	split biquaternion [6]
	1	-1	hyperbolic	split	[22]
	1	0	hyperbolic	semi	
	-1	0	hyperbolic	split semi	[3]
	0	0	hyperbolic	quasi	

References

[1] Y. Alagöz, K. H. Oral and S. Yüce, *Split quaternion matrices*, Miskolc Mathematical Notes, **13**(2), 223-232, 2012.

[2] Y. Alagöz and G. Özyurt, *Some properties of complex quaternion and complex split quaternion matrices*, Miskolc Mathematical Notes, **20**(1), 45-58, 2019.

[3] Y. Alagöz and G. Özyurt, *Real and Hyperbolic Matrices of Split Semi Quaternions*, Adv. Appl. Clifford Algebras, **29**, 29-53, 2019.

[4] Y. Alagöz and G. Özyurt, *Linear Equations Systems of Real and Complex Semi-Quaternions*, Iranian Journal of Science and Technology, Transactions A: Science, **44**(5), 1483-1493, 2020.

[5] D. Alfsmann, *On families of 2N-dimensional hypercomplex algebras suitable for digital signal processing*, in Proc. European Signal Processing Conf. (EUSIPCO), Florence, Italy, 2006.

[6] G. Arizmendi and M. A. P. D. la Rosa, *Some extensions of quaternions and symmetries of simply connected space forms.* arXiv preprint arXiv:1906.11370, 2019.

[7] E. Ata and Y. Yıldırım, *A different polar representation for generalized and generalized dual quaternions*, Advances in Applied Clifford Algebras, **28**(4), 1-20, 2018.duallerin hepsi

[8] M. Bekar, *Dual-Quasi Elliptic Planar Motion*, Mathematical Sciences and Applications E-Notes, **4**(1), 136-143, 2016.

[9] M. Bekar and Y. Yaylı, *A Study on Complexified Semi-Quaternions*, International Conference on Operators in Morrey-type Spaces and Applications (OMTSA), 115, 2017.

[10] F. Catoni, D. Boccaletti, R. Cannata, V. Catoni, E. Nichelatti and P. Zampetti, *The mathematics of Minkowski space-time and an introduction to commutative hypercomplex numbers*, Birkhauser Verlag, Basel, 2008.

[11] F. Catoni, R. Cannata, V. Catoni and P. Zampetti, *Two-dimensional hypercomplex numbers and related trigonometries and geometries*, Adv. Appl. Clifford Algebr., **14**, 47–68, 2004.

[12] F. Catoni, R. Cannata, V. Catoni and P. Zampetti, *N-dimensional geometries generated by hypercomplex numbers*, Adv. Appl. Clifford Algebr., **15**, **1**, 1–25, 2005.

[13] F. Catoni, R. Cannata, V. Catoni and P. Zampetti, *An introduction to commutative quaternions*, Adv. Appl. Clifford Algebr., **16**, **1**, 1–28, 2006.

[14] F. Catoni, R. Cannata and P. Zampetti, *An Introduction to Constant Curvature Spaces in the Commutative (Segre) Quaternion Geometry*, Adv. Appl. Clifford Algebr., **16** , **1**, 85–101, 2006.

[15] W. K. Clifford, Mathematical papers (ed. R. Tucker), Chelsea Pub. Co., Bronx, NY, 1968.

[16] W. K. Clifford, *Preliminary sketch of bi-quaternions*, Proc. Lond. Math. Soc., **s1–4(1)**, 381–395, 1873.

[17] J. Cockle, *On a New Imaginary in Algebra*, Philosophical Magazine, London-Dublin-Edinburgh, **34**(3), 37-47, 1849.

[18] L. E. Dickson, *On the Theory of Numbers and Generalized Quaternions*, American Journal of Mathematics **46**(1), 1-16, 1924.

[19] J. D. Jr. Edmonds, Relativistic Reality: A Modern View, World Scientific, Singapore, 1997.

[20] Z. Ercan and S. Yüce, *On properties of the dual quaternions*, European Journal of Pure and Applied Mathematics, **4**(2), 142–146, 2011.

[21] M. Erdoğdu and M. Özdemir, *On complex split quaternion matrices*, Adv. Appl. Clifford Algebr., **23**(3), 625-638, 2013.

[22] M. Erdoğdu and M. Özdemir, *Matrices over hyperbolic split quaternions*, Filomat, **30**(4), 913-920, 2016.

[23] C. Flaut and V. Shpakivskyi, *An efficient method for solving equations in generalized quaternion and octonion algebras*, Adv. Appl. Clifford Algebr., **25**(2), 337-350, 2015.

[24] P. Fjelstad, *Extending special relativity via the Perplex numbers*, American Journal of Physics, **54**, 416–422, 1986.

[25] L. W. Griffiths, *Generalized Quaternion Algebras and the Theory of Numbers*, American Journal of Mathematics 50(2), 303-314, 1928.

[26] W. R. Hamilton, *On Quaternions; or on a New System of Imaginaries in Algebra*, The London, Edinburgh and Dublin Philosophical Magazine and Journal of Science (3rd Series), xxv-xxxvi, 1844–1850.

[27] W. R. Hamilton Elements of Quaternions, Chelsea Pub. Com., New York, 1969.

[28] W. R. Hamilton, Lectures on Quaternions. Hodges and Smith, Dublin, 1853.

[29] W. R. Hamilton, On the geometrical interpretation of some results obtained by calculation with biquaternions, In Halberstam and Ingram and first published in Proceedings of the Royal Irish Academy, 1853.

[30] A. A. Harkin and J. B. Harkin, *Geometry of generalized complex numbers*, Math. Mag., **77**, 118-129, 2004.

[31] M. Jafari and Y. Yaylı, *Generalized quaternions and their algebratic properties*, Commun. Fac. Sci. Ank. Series A1, **64**, 15–27, 2015.

[32] M. Jafari, *A Note on the Representation and Definition of Dual Split Semi-Quaternions Algebra*, Global Journal of Science Frontier Research: F Mathematics and Decision Sciences, **17**(8F), 2017.

[33] A. P. Kotelnikov, *Screw calculus and some applications to geometry and mechanics*, Annal. Imp. Univ. Kazan, 1895

[34] L. Kula and Y. Yaylı, *Dual split quaternions and screw motion in Minkowski 3-space*, Iranian Journal of Science and Technology (Sciences), **30**(3), 245-258, 2006.

[35] V. Majernik, *Quaternion formulation of the Galilean space-time transformation*, Acta Phy. Slovaca., **56**, 9–14, 2006.

[36] V. Majernik and M. Nagy, *Quaternionic form of Maxwell's equations with sources*, Lett. Nuovo Cimento., **16**, 165–169, 1976.

[37] A. Mamagani and M. Jafari, *On properties of generalized quaternion algebra*, Journal of Novel Applied Sciences, **2**, 683–689, 2013.

[38] A. McAulay, *Octonions: a development of Clifford's Biquaternions*, 1898.

[39] F. Messelmi, *Generalized numbers and their holomorphic functions*, Int. J. Open Problems Complex Analysis, **7**, 35–47, 2015.

[40] K. E. Özen and M. Tosun, *On the matrix algebra of elliptic biquaternions*, Mathematical Methods in the Applied Sciences, **43**(6), 2984-2998, 2020.

[41] K. E. Özen and M. Tosun, *Elliptic matrix representations of elliptic biquaternions and their applications*, International Electronic Journal of Geometry, **11**(2), 96-103, 2018.

[42] E. Pennestrì and R. Stefanelli, *Linear algebra and numerical algorithms using dual numbers*, Multibody Syst. Dyn., **18**, 323–344, 2007.

[43] H. Pottman and J. Wallner, Computational line geometry, Springer-Verlag, Berlin Heidelberg New York, 2000.

[44] B. A. Rosenfeld, Geometry of Lie groups, Kluwer academic publishers, Dordrecht, 1997.

[45] D. Savin, C. Flaut and C. Ciobanu, *Some properties of the symbol algebras*, Carpathian J. Math., arxiv:0906.2715V1, 2009.

[46] G. Sobczyk, *The hyperbolic number plane*, College Math. J., **26**, 268–280, 1995.

[47] E. Study, Geometrie der Dynamen, Leipzig, 1903.

[48] Y. Tian, *Biquaternions and their complex matrix representations*, Beiträge zur Algebra und Geometrie/Contributions to Algebra and Geometry, **54**(2), 575-592, 2013.

[49] S. Veldsman, *Generalized complex numbers over near-fields*, Quaest. Math., **42**, 181–200, 2019.

[50] I. M. Yaglom, Complex numbers in geometry, Academic Press, New York, 1968.

[51] F. Zhang, *Quaternions and Matrices of Quaternions*, Linear Algebra and its Applications, **251**, 21-57, 1997.