New Insight into Quaternions and Their Matrices

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Abstract

The aim of this paper is to bring together quaternions and generalized complex numbers. Generalized quaternions with generalized complex number components are expressed and their algebraic structures are examined. Several matrix representations and computational results are introduced. As a crucial part, alternative approach for generalized quaternion matrix with elliptic number entries are developed.

Keywords: Generalized quaternion, generalized complex number, matrix representation.

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1 Introduction

Hamilton introduced the Hamiltonian quaternions for representing vectors in the space, [27,28]. The real quaternion is written as

\[ q = a_0 + a_1i + a_2j + a_3k, \]

where \( a_0, a_1, a_2, a_3 \in \mathbb{R} \) are components and \( i, j, k \not\in \mathbb{R} \) are versors, [26]. The set of real quaternions, as an extension of complex numbers, is an associative but non-commutative Clifford algebra and used in many fields in applied mathematics. The associative quaternions will be divided into two classes: in the first class there are the non-commutative quaternions (Hamiltonian, hyperbolic, split quaternions, generalized quaternions [1,17,18,25,31,37,43–45] etc.), in the second class there are the commutative quaternions (generalized Segre quaternions [13,14], dual quaternions [16,19,20,35,36] etc.).

The algebra of generalized quaternions as non-commutative system, denoted by \( \mathbb{Q}_{\alpha,\beta} \), includes a variety of well-known four-dimensional algebras as special cases. Conditions of the versors for them are given by:

\[ \begin{align*}
  i^2 &= -\alpha, \\
  j^2 &= -\beta, \\
  k^2 &= -\alpha\beta, \\
  ij &= -ji = k, \\
  jk &= -kj = \beta i, \\
  ki &= -ik = \alpha j,
\end{align*} \tag{1} \]

where \( \alpha, \beta \in \mathbb{R} \). For \( \alpha = \beta = 1 \) Hamiltonian quaternions, for \( \alpha = 1, \beta = -1 \) split quaternions, for \( \alpha = 1, \beta = 0 \) semi quaternions, for \( \alpha = -1, \beta = 0 \) split-semi quaternions, and for \( \alpha = \beta = 0 \) quasi quaternions are obtained.

Additionally, the general bidimensional hypercomplex systems (namely generalized complex numbers (GCN)) over the field of real numbers \( \mathbb{R} \) is given by the ring \( \mathbb{R}[X] / (h(X)) \approx \{ z = x_1 + x_2 I : I^2 = Iq + p, \ p, q, x_1, x_2 \in \mathbb{R}, I \not\in \mathbb{R} \} \), where \( h(X) = X^2 - qX - p \) is monic quadratic. By denoting this set with \( \mathbb{C}_{q,p} \) it is well known that the sign of \( \Delta = q^2 + 4p \) determines the properties of the general bidimensional systems. These systems are ring isomorphic with one of the following three types:

- for \( \Delta > 0 \) hyperbolic system; the canonical system is the system of hyperbolic (double, split complex, perplex) numbers \( \mathbb{H} \cong \mathbb{C}_{0,1} \) with \( p = 1, q = 0 \), [15,24,46]
• for \( \Delta < 0 \) elliptic system; the canonical system is the system of complex (ordinary) numbers \( \mathbb{C} \cong \mathbb{C}_{0,-1} \) with \( p = -1 \), \( q = 0 \). \[50\].

• for \( \Delta = 0 \) parabolic system; the canonical system is the system of dual numbers \( \mathbb{D} \cong \mathbb{C}_{0,0} \) with \( p = 0 \), \( q = 0 \). \[42,47\].

Regarding to the value \( D = z\bar{z} = (x_1 + x_2\ell)(x_1 - x_2\ell) = x_1^2 - px_2^2 + qx_1x_2 \), which is called the characteristic determinant, \( z \in \mathbb{C}_{n,p} \) are classified into three types, \[10\]. Hence \( z \in \mathbb{C}_{n,p} \) is called timelike, spacelike or null where \( D < 0 \), \( D > 0 \) and \( D = 0 \), respectively. Then all of the elements of the set \( \mathbb{C}_{0,-1} \) are spacelike. For \( q = 0 \), \( I^2 = p \in \mathbb{R} \), the generalized complex number system is denoted by \( \mathbb{C}_p \) and called \( p \)-complex plane \[30\].

In this paper, we aim to design generalized quaternions by taking the components as elements of \( \mathbb{C}_{q,p} \). Moreover, the algebraic structures and properties of these quaternions are investigated and several types of matrix representations are introduced. Also alternative approach for generalized quaternion matrix with elliptic number entries are developed as a further results.

2 Generalized Quaternions with \( \mathcal{GCN} \) Components

In this section, we present mathematical formulations of improved quaternions: generalized quaternions with \( \mathcal{GCN} \) and examine special matrix correspondences.

**Definition 1.** For \( \alpha, \beta \in \mathbb{R} \), the set of generalized quaternions with \( \mathcal{GCN} \) components are denoted by \( \tilde{Q}_{\alpha,\beta} \) and the elements of this set are defined as in the form:

\[
\tilde{q} = a_0 + a_1i + a_2j + a_3k,
\]

where \( a_0, a_1, a_2, a_3 \in \mathbb{C}_{q,p} \) and \( i, j, k \notin \mathbb{R} \) are versors that satisfy the properties in equation \[1\].

Axiomatically, the generalized complex unit \( I \) commutes with the three quaternion versors \( i \) and \( k \), that is \( II = I, Ij = Ij \) and \( kI = Ik \). It is obvious that for \( q = 0 \), \( p = -1 \), \( \alpha = 1 \), usual complex operator distinct from quaternion versor \( i \).

Throughout this section, \( \tilde{q} = a_0 + a_1i + a_2j + a_3k \) and \( \tilde{p} = b_0 + b_1i + b_2j + b_3k \in \tilde{Q}_{\alpha,\beta} \) are considered. Due to the generalized quaternions with \( \mathcal{GCN} \) components are an extension of generalized quaternions, many properties of them are familiar. For any \( q \in \tilde{Q}_{\alpha,\beta} \), \( S_q = a_0 \) is the scalar part and \( V_q = a_1i + a_2j + a_3k \) is the vector part. Equality of two improved quaternions is as follows: \( \tilde{p} = \tilde{q} \iff S_p = S_q, V_p = V_q \). Addition (and hence subtraction) of \( \tilde{q} \) to another quaternion \( \tilde{p} \) acts in a componentwise way:

\[
\tilde{q} + \tilde{p} = (a_0 + b_0) + (a_1 + b_1)i + (a_2 + b_2)j + (a_3 + b_3)k
= S_p + S_q + V_p + V_q.
\]

The scalar multiplication of \( \tilde{q} \) with a scalar gives another improved quaternion

\[
\tilde{c\tilde{q}} = ca_0 + ca_1i + ca_2j + ca_3k
= cS_q + cV_q, \quad c \in \mathbb{C}_{q,p}.
\]

Multiplication of the two quaternions is carried out as follows:

\[
\tilde{q}\tilde{p} = (a_0b_0 - a_0a_1b_1 - a_0a_2b_2 - a_0a_3b_3 + a_1b_1 + a_1b_0 + a_1a_2b_3 - a_1a_3b_2)i
+ (a_0b_2 - a_0a_1b_3 + a_0a_2b_0 + a_0a_3b_1)j + (a_0b_3 + a_1b_0 - a_1a_2b_1 - a_1a_3b_2)k
= S_qS_p - (V_q,V_p)g + S_qV_p + S_pV_q + V_p \times_g V_q,
\]

(see generalized inner and cross products in \[31\]). The conjugate of \( \tilde{q} \) is the quaternion \( \bar{\tilde{q}} = a_0 - a_1i - a_2j - a_3k = S_q - V_q \). The norm of \( \tilde{q} \) is

\[
||\tilde{q}||^2 = |\tilde{q}\bar{\tilde{q}}| = |\tilde{q}\tilde{q}| = |a_0^2 + a_1^2 + a_2^2 + a_3^2|,
\]

and the inverse is

\[
(\tilde{q})^{-1} = \frac{\tilde{q}}{||\tilde{q}||} \text{ for } ||\tilde{q}||^2 \neq 0.
\]

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Proposition 1. \( \tilde{Q}_{\alpha, \beta} \) is a 4-dimensional module over \( \mathbb{C}_{q, p} \) with base \( \{1, i, j, k\} \) and is an 8-dimensional vector space over \( \mathbb{R} \) with base \( \{1, I, i, I, j, I, k, I, k\} \).

Proposition 2. For any \( \tilde{q}, \tilde{p} \in \tilde{Q}_{\alpha, \beta} \) and \( c_1, c_2 \in \mathbb{R} \), the conjugate and norm hold following properties:

i. \( \tilde{x} \tilde{q} = \tilde{q} \),

ii. \( \overline{c_1 \tilde{p} + c_2 \tilde{q}} = c_1 \tilde{p} + c_2 \tilde{q} \),

iii. \( \overline{\tilde{p} \tilde{q}} = \tilde{q} \tilde{p} \),

iv. \( \|c_1 \tilde{q}\| = |c_1| \|\tilde{q}\| \),

v. \( \|\tilde{q} \tilde{p}\| = \|\tilde{q}\| \|\tilde{p}\| \),

vi. \( \|\tilde{q}\|_I = \|\tilde{q}\|_I \).

Definition 2. For any \( \tilde{q}, \tilde{p} \in \tilde{Q}_{\alpha, \beta} \), the scalar and vector products on \( \tilde{Q}_{\alpha, \beta} \) are, respectively, defined by:

\[
\langle \tilde{q}, \tilde{p} \rangle_g = S_q S_p + (V_q, V_p)_g = a_0 b_0 + a_1 b_1 + \beta a_2 b_2 + \alpha \beta a_3 b_3 = S_q \tilde{p},
\]

\[
\tilde{q} \times \tilde{p} = S_q V_p + S_p V_q - V_q \times V_p = V_q \tilde{p},
\]

where \( \alpha, \beta \in \mathbb{R}^* \) and \( \langle \cdot, \cdot \rangle_g \) is a generalized scalar product.

Remark 1. As an another perspective to \( \tilde{q} \in \tilde{Q}_{\alpha, \beta} \), the following can be calculated:

\[
\tilde{q} = a_0 + a_1 i + a_2 j + a_3 k = (x_{01} + x_{02} I) + (x_{11} + x_{12} I) i + (x_{21} + x_{22} I) j + (x_{31} + x_{32} I) k = q_0 + q_1 I,
\]

where \( a_i = x_{ii} + x_{i2} I \in \mathbb{C}_{q, p}, q_{i-1} = x_{0j} + x_{1j} i + x_{2j} j + x_{3j} k \in Q_{\alpha, \beta} \) for \( 0 \leq i \leq 3, 1 \leq j \leq 2 \). So, for \( \tilde{q} = q_0 + q_1 I \) and \( \tilde{p} = p_0 + p_1 I \in \tilde{Q}_{\alpha, \beta} \), the generalized scalar product is also defined by as follows:

\[
\langle \tilde{q}, \tilde{p} \rangle_g = S_{q_0} S_{p_0} + p S_{q_1} S_{p_1} + (S_{q_0} S_{p_1} + S_{q_1} S_{p_0} + q S_{q_1} S_{p_1}) I.
\]

It is worthy to note that \( \tilde{Q}_{\alpha, \beta} \) is a 2-dimensional module over \( Q_{\alpha, \beta} \) (skew-field) with base \( \{1, I\} \). For \( \tilde{q} = q_0 + q_1 I \) and \( \tilde{p} = p_0 + p_1 I, \) if \( \tilde{p} = \tilde{q}, \) then \( p_0 = q_0, p_1 = q_1. \) Also \( \tilde{p} + \tilde{q} = (p_0 + q_0) + (p_1 + q_1) I, \)

\[
\tilde{p} \tilde{q} = (q_0 p_0 + q_1 p_1) + (q_0 p_1 + q_1 p_0 + q_0 q_1) I,
\]

and \( c \tilde{q} = c q_0 + c q_1 I, c \in \mathbb{R}. \) The conjugate and anti conjugate are \( \tilde{q}^{*1} = q_0 + q_1 I \) and \( \tilde{q}^{*2} = q_1 - q_0 I \), respectively. Moduli is \( \|\tilde{q}\|^2_1 = \tilde{q} \tilde{q}^{*1} \) and the inverse is \( (\tilde{q})^{-1} = \frac{\tilde{q}^{*1}}{\|\tilde{q}\|^2_1} = \frac{\tilde{q}^{*1}}{q_0^2 + q_1^2} \neq 0. \) Additionally, the followings hold for \( \tilde{q}, \tilde{p} \in \tilde{Q}_{\alpha, \beta} \) and \( c_1, c_2 \in \mathbb{R}: \)

i. \( (\tilde{q})^{*1} = \tilde{q}, \)

ii. \( (\tilde{q})^{*2} = -\tilde{q}, \)

iii. \( (c_1 \tilde{q} \pm c_2 \tilde{p})^{*1} = c_1 \tilde{q}^{*1} \pm c_2 \tilde{p}^{*1}, \)

iv. \( (c_1 \tilde{q} \pm c_2 \tilde{p})^{*2} = c_1 \tilde{q}^{*2} \pm c_2 \tilde{p}^{*2}, \)

v. \( \tilde{q} + \tilde{q}^{*1} = 2 q_0 + q q_1, \)

vi. \( (\tilde{q} \tilde{p})^{*1} = \tilde{q}^{*1} \tilde{p}^{*1}, \)

vii. \( \|c_1 \tilde{q}\|^{*1}_1 = |c_1| \|\tilde{q}\|^{*1}_1, \)

viii. \( \|\tilde{q} \tilde{p}\|^{*1}_1 = \|\tilde{q}\|^{*1}_1 \|\tilde{p}\|^{*1}_1, \)

ix. \( \left\|\frac{\tilde{q}}{\tilde{p}}\right\|^{*1}_1 = \frac{|\tilde{q}|^{*1}_1}{|\tilde{p}|^{*1}_1}. \)
2.1 Matrix Correspondences

For \( \tilde{q} \in \tilde{Q}_{\alpha,\beta} \), \( L : \tilde{Q}_{\alpha,\beta} \rightarrow \mathcal{R} \), 
\( L(\tilde{q}) = \begin{bmatrix} a_0 + a_3k & a_1i + a_2j \\ a_1i + a_2j & a_0 + a_3k \end{bmatrix} \)

is a linear transformation, where \( \mathcal{R} := \{ A_{\tilde{q}} \in M_2(\tilde{Q}_{\alpha,\beta}) : A_{\tilde{q}} = \begin{bmatrix} a_0 + a_3k & a_1i + a_2j \\ a_1i + a_2j & a_0 + a_3k \end{bmatrix} \} \)

is a subset of \( M_2(\tilde{Q}_{\alpha,\beta}) \). So there exists a correspondence between \( \tilde{Q}_{\alpha,\beta} \) and \( \mathcal{R} \) via the map \( L \). Hence, 2 \times 2 quaternionic matrix representation of \( \tilde{q} \) is \( A_{\tilde{q}} \).

**Theorem 1.** Every generalized quaternion with GCN components can be represented by a 2 \times 2 quaternionic matrix. \( \tilde{Q}_{\alpha,\beta} \) is the subset of \( M_2(\tilde{Q}_{\alpha,\beta}) \).

**Corollary 1.** \( L \) can be determined as the following representation:

\[
L(a_0 + a_1i + a_2j + a_3k) = a_0I_2 + a_1i + a_2J + a_3K,
\]

where

\[
l = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}, \quad j = \begin{bmatrix} 0 & j \\ j & 0 \end{bmatrix}, \quad K = \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}.
\]

Thus

\[
|l|^2 = -\alpha I_2, \quad |j|^2 = -\beta I_2, \quad |K|^2 = -\alpha\beta I_2,
\]

\[
lJ = -Jl, \quad JK = -KJ = -\beta l, \quad KL = -lK = \alpha J.
\]

**Theorem 2.** For \( \tilde{q}, \tilde{p} \in \tilde{Q}_{\alpha,\beta} \) and \( \lambda \in \mathbb{R} \), then the following identities hold:

i. \( \tilde{q} = \tilde{p} \Leftrightarrow A_{\tilde{q}} = A_{\tilde{p}} \)

ii. \( A_{\tilde{q} + \tilde{p}} = A_{\tilde{q}} + A_{\tilde{p}} \)

iii. \( A_{\tilde{q}\tilde{p}} = A_{\tilde{q}}A_{\tilde{p}} \)

iv. \( A_{\lambda\tilde{q}} = \lambda A_{\tilde{q}} \)

For \( \tilde{q} \in \tilde{Q}_{\alpha,\beta} \), denote \( \mathcal{K} \) as a subset of \( M(C_{q,p}) \) given by:

\[
\mathcal{K} := \{ B_\tilde{q}^l \in M_4(C_{q,p}) : B_\tilde{q}^l = \begin{bmatrix} a_0 & -\alpha a_1 & -\beta a_2 & -\alpha \beta a_3 \\ a_1 & a_0 & -\beta a_3 & \beta a_2 \\ a_2 & a_3 & a_0 & -\alpha a_1 \\ a_3 & -a_2 & a_1 & a_0 \end{bmatrix} \}
\]

and define the map \( \mathcal{N} : \tilde{Q}_{\alpha,\beta} \rightarrow \mathcal{K}, \mathcal{N}(\tilde{q}) = \begin{bmatrix} a_0 & -\alpha a_1 & -\beta a_2 & -\alpha \beta a_3 \\ a_1 & a_0 & -\beta a_3 & \beta a_2 \\ a_2 & a_3 & a_0 & -\alpha a_1 \\ a_3 & -a_2 & a_1 & a_0 \end{bmatrix} \).

There exists a correspondence between \( \tilde{Q}_{\alpha,\beta} \) and \( \mathcal{K} \) via the map \( \mathcal{N} \). \( B_\tilde{q}^l \) is the 4 \times 4 left generalized complex matrix representation of \( \tilde{q} \) according to the standard basis \{1, i, j, k\}.

4 \times 4 right generalized complex matrix representation of \( \tilde{q} \) can be calculated similarly. Throughout this paper \( B_\tilde{q}^l \) will be considered.

**Theorem 3.** Every generalized quaternion with GCN components can be represented by a 4 \times 4 generalized complex matrix. \( \tilde{Q}_{\alpha,\beta} \) is the subset of \( M_4(C_{q,p}) \).

\[4 \times 4\] right generalized complex matrix representation of \( \tilde{q} \) is:

\[
B_\tilde{q}^l = \begin{bmatrix} a_0 & -\alpha a_1 & -\beta a_2 & -\alpha \beta a_3 \\ a_1 & a_0 & -\beta a_3 & \beta a_2 \\ a_2 & a_3 & a_0 & -\alpha a_1 \\ a_3 & -a_2 & a_1 & a_0 \end{bmatrix}.
\]
Corollary 2. Considering the basis \{1,i,j,k\}, the column matrix representation of \(\tilde{p} \in \tilde{Q}_{a,\beta}\) is given by:

\[
\tilde{p} = \begin{bmatrix} b_0 & b_1 & b_2 & b_3 \end{bmatrix}^T.
\]

By using \(B_q^l\), the multiplication of \(\tilde{q}, \tilde{p} \in \tilde{Q}_{a,\beta}\) can also be written by: \(\tilde{q}\tilde{p} = B_q^l \tilde{p}\).

Theorem 4. Let \(\tilde{q} \in \tilde{Q}_{a,\beta}\). \(B_q^l\) can be determined as:

\[
B_q^l = a_0I_4 + a_1I + a_2J + a_3K,
\]

where

\[
I = \begin{bmatrix} 0 & -\alpha & 0 & 0 \\ 0 & 0 & -\alpha & 0 \\ 0 & 0 & 0 & -\alpha \\ 1 & 0 & 0 & 0 \end{bmatrix},
J = \begin{bmatrix} 0 & 0 & -\beta & 0 \\ 0 & 0 & 0 & \beta \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix},
K = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha \\ 0 & 0 & 0 & \alpha \beta \\ 0 & 0 & \alpha & \alpha \beta \end{bmatrix}.
\]

Undoubtedly, \(I, J, K\) satisfy the generalized quaternion versors conditions in equation (1).

Theorem 5. Let \(\tilde{q} \in \tilde{Q}_{a,\beta}\). Then, we have

\[
\sigma B_q^l \sigma = B_{\overline{q}}^l,
\]

where \(\sigma = \text{diag}(1,1,-1,-1)\) and \(\overline{q} = a_0 + a_1i - a_2j - a_3k \in \tilde{Q}_{a,\beta}\).

Theorem 6. Let \(\tilde{q}, \tilde{p} \in \tilde{Q}_{a,\beta}\) and \(\lambda \in \mathbb{R}\), the following properties are satisfied:

i. \(\tilde{q} = \tilde{p} \iff B_q^l = B_p^l\),
ii. \(B_{q+p}^l = B_q^l + B_p^l\),
iii. \(B_{\lambda\tilde{q}}^l = \lambda(B_q^l)\),
iv. \(B_{\overline{q}}^l = B_q^l B_{\overline{p}}^l\),

v. \(\det(B_q^l) = (\alpha^2 + \alpha \alpha_1^2 + \beta a_2^2 + \alpha \beta \alpha_3)^2 = ||\tilde{q}||^4\),
vi. \(\text{tr}(B_q^l) = 4S_q\).

Theorem 7. Let \(\tilde{q} \in \tilde{Q}_{a,\beta}\) and \(\tilde{q}^{-1}\) be inverse of \(\tilde{q}\). Then,

\[
B_{\tilde{q}^{-1}}^l = \frac{1}{\sqrt{\det(B_{\overline{q}}^l)}} B_{\overline{q}}^l
\]

For\(\tilde{q} = q_0 + q_1I \in \tilde{Q}_{a,\beta}\), denote \(T\) as a subset of \(M_2(Q_{a,\beta})\) given by:

\[
T := \left\{ D_q \in M_2(Q_{a,\beta}) : D_q = \begin{bmatrix} q_0 & pq_1 \\ q_1 & q_0 + q_1 \end{bmatrix} \right\},
\]

and define the map \(M : \tilde{Q}_{a,\beta} \rightarrow T\), \(M(\tilde{q}) = \begin{bmatrix} q_0 & pq_1 \\ q_1 & q_0 + q_1 \end{bmatrix}\).

It can be concluded that there exists a correspondence between \(\tilde{Q}_{a,\beta}\) and \(T\) via the map \(M\). Hence, \(2 \times 2\) generalized complex matrix representation of \(\tilde{q}\) with respect to the standard basis \(\{1,I\}\) is the matrix \(D_q\).

Theorem 8. Every GCN with generalized quaternion components can be represented by a \(2 \times 2\) generalized quaternion matrix. \(\tilde{Q}_{a,\beta}\) is the subset of \(M_2(Q_{a,\beta})\).

By using \(D_q\) and \(\tilde{p} = [p_0 \ p_1]^T\), we have: \(\tilde{q}\tilde{p} = D_q\tilde{p}\). Moreover, \(D_q\) is also in the form \(D_q = q_0I_2 + q_1I\), where \(I = \begin{bmatrix} 0 & p \\ 1 & q \end{bmatrix}\) is the representation of \(I\). It should be noted that there are many ways to choose \(I\), for instance: \(I = \begin{bmatrix} q & 1 \\ p & 0 \end{bmatrix}\) (see in [39]).
Theorem 9. For any \( \tilde{q} = q_0 + q_1 I \) and \( \tilde{p} = p_0 + p_1 I \in \tilde{Q}_{\alpha, \beta} \) and \( \lambda \in \mathbb{R} \), the following properties are satisfied:

i. \( \tilde{p} = \tilde{p} \iff D_{\tilde{q}} = D_{\tilde{p}} \),

ii. \( D_{\tilde{q} + \tilde{p}} = D_{\tilde{q}} + D_{\tilde{p}} \),

iii. \( D_{\lambda \tilde{q}} = \lambda D_{\tilde{q}} \),

iv. \( D_{\tilde{q} \tilde{p}} = D_{\tilde{q}} D_{\tilde{p}} \).

v. \( \det(D_{\tilde{q}}) = \tilde{q}_0^2 + q_1 q_0 - p_1^2 \), where the notation \( \det \) represents the determinant of the quaternion matrix.

Definition 3. Let \( \tilde{q} = q_0 + q_1 I \in \tilde{Q}_{\alpha, \beta} \). The vector representation of \( \tilde{q} \) is defined as

\[
\tilde{\mathbf{q}} = \begin{bmatrix} \tilde{q}_0 & \tilde{q}_1 \end{bmatrix}^T = \begin{bmatrix} \tilde{q}_0 \\ \tilde{q}_1 \end{bmatrix},
\]

where \( q_{j-1} = x_{0j} + x_{1j} i + x_{2j} j + x_{3j} k \in Q_{\alpha, \beta} \) and

\[
\tilde{q}_{j-1} = (x_{0j}, x_{1j}, x_{2j}, x_{3j}) = [x_{0j}, x_{1j}, x_{2j}, x_{3j}]^T
\]

are vectors (matrices) for \( 1 \leq j \leq 2 \).

Theorem 10. Let \( \tilde{q} = q_0 + q_1 I \in \tilde{Q}_{\alpha, \beta} \). Then

i. \( \mathcal{X}_{\tilde{q}} = \tilde{\mathbf{q}}^T \), where \( \mathcal{X} = \begin{bmatrix} I_4 & qI_4 \\ 0 & -I_4 \end{bmatrix} \in M_8(\mathbb{R}) \).

ii. \( \mathcal{Y}_{\tilde{q}} = \tilde{\mathbf{q}}^T \), where \( \mathcal{Y} = \begin{bmatrix} 0 & I_4 \\ -I_4 & 0 \end{bmatrix} \in M_8(\mathbb{R}) \).

By applying the isomorphism \( \Gamma(x_{i1} + x_{i2} I) = \begin{bmatrix} x_{i1} & px_{i2} \\ x_{i2} & x_{i1} + qx_{i2} \end{bmatrix} \) to \( B_{\tilde{q}}^T \), where \( x_{i1} + x_{i2} I \in \mathbb{C}_{p, q} \), for \( 0 \leq i \leq 3 \), the left real matrix representation of \( \tilde{q} \) (see in (3)) with respect to the base \( \{I, I, i, I, j, I, k, I, k\} \) is given by:

\[
C_{\tilde{q}}^T = \begin{bmatrix} x_{01} & px_{02} & -\alpha x_{11} & -\alpha px_{12} & -\beta x_{21} & -\beta px_{22} & -\alpha \beta x_{31} & -\alpha \beta px_{32} \\ x_{02} & x_{01} + qx_{02} & -\alpha x_{12} & -\alpha x_{11} + qx_{12} & -\beta x_{12} & -\beta x_{21} + qx_{22} & -\alpha \beta x_{12} & -\alpha \beta x_{11} + qx_{12} \\ x_{11} & px_{12} & 0 & px_{02} & 0 & px_{22} & 0 & px_{32} \\ x_{12} & x_{11} + qx_{12} & x_{01} + qx_{02} & x_{01} + qx_{02} & x_{02} + qx_{02} & x_{01} + qx_{02} & x_{01} + qx_{02} & x_{01} + qx_{02} \\ x_{21} & px_{22} & \alpha x_{31} & px_{31} & 0 & px_{32} & 0 & px_{32} \\ x_{22} & x_{21} + qx_{22} & \alpha x_{32} & px_{31} & 0 & px_{32} & 0 & px_{32} \\ x_{31} & px_{32} & -x_{21} & -x_{21} - qx_{22} & x_{12} & x_{11} & x_{11} + qx_{12} & x_{01} + qx_{02} \\ x_{32} & x_{31} + qx_{32} & -x_{22} & -x_{21} - qx_{22} & x_{12} & x_{11} & x_{11} + qx_{12} & x_{01} + qx_{02} \end{bmatrix}.
\]

So, \( \tilde{Q}_{\alpha, \beta} \) is the subset of \( M_8(\mathbb{R}) \).

Example 1. Take \( \tilde{q} \in \tilde{Q}_{2,1} \) with \( \mathcal{GCN} \) components for \( p = -1 \) and \( q = 1 \):

\[
\tilde{q} = 1 + (-1 + I) i + I j + (1 + 2I) k.
\]

Then,

\[
A_{\tilde{q}} = \begin{bmatrix} 1 + I k \\ -1 + I k \end{bmatrix},
\]

\[
B_{\tilde{q}}^T = \begin{bmatrix} 1 & -2 & -I & -2(1 + 2I) \\ 1 & 1 & -2I & I \\ -2I & 2(1 + 2I) & 1 & -2(1 + I) \\ 1 + 2I & -I & -1 + I & 1 \end{bmatrix};
\]

\footnote{The determinant of an arbitrary \( 2 \times 2 \) quaternion matrix is defined by \( \det \left( \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) = da - cb. \) [1]}
\[
C_q = \begin{bmatrix}
1 & 0 & -2 & 0 & 0 & 1 & -2 & 4 \\
0 & 1 & 0 & -2 & -1 & -1 & -4 & -6 \\
-1 & -1 & 1 & 0 & -1 & 2 & 0 & -1 \\
1 & 0 & 0 & 1 & -2 & -3 & 1 & 1 \\
0 & -1 & 2 & -4 & 1 & 0 & 2 & 2 \\
1 & 1 & 4 & 6 & 0 & 1 & -2 & 0 \\
1 & -2 & 0 & 1 & -1 & -1 & 1 & 0 \\
2 & 3 & -1 & -1 & 1 & 0 & 0 & 1
\end{bmatrix},
\]

\[
D_q = \begin{bmatrix}
1 - i + k & -i - j - 2k \\
i + j + 2k & 1 + j + 3k
\end{bmatrix},
\]

and

\[
B_{q^{-1}} = \frac{1}{\sqrt{-237 + 51T}} \begin{bmatrix}
1 & -2 & I & 2(1 + 2I) \\
-1 + I & 1 & 1 + 2I & -I \\
-I & -2(1 + 2I) & 1 & -2(-1 + I) \\
-1 - 2I & I & -1 + I & 1
\end{bmatrix}.
\]

Also, the vector representation of \(\tilde{q}^1\) is computed by:

\[
\tilde{q}^1 = \mathcal{K}\tilde{q} = \begin{bmatrix}
I_4 & I_4 \\
0 & -I_4
\end{bmatrix} \begin{bmatrix}
1 & -1 & 0 & 1 \\
0 & 1 & 1 & 2
\end{bmatrix}^T = \begin{bmatrix}
1 & 0 & 1 & 3 \\
0 & -1 & -1 & -2
\end{bmatrix}^T.
\]

3 Further Result: An Alternative Matrix Approach

The questions about numbers, hypercomplex numbers and quaternions included questions about their matrices. Inspired by matrix forms in the study [23], we give an answer for the question of the alternative representation of generalized quaternion matrix with elliptic number entries (see elliptic biquaternions in [40]). So this matrix is in the form:

\[
\tilde{Q} = A_0 I_2 + A_1 I + A_2 \mathcal{J} + A_3 \mathcal{K},
\]

where \(A_0, A_1, A_2, A_3 \in \mathbb{C}_p\) are elliptic numbers for \(p < 0\). The base elements can be defined as follows:

**Case 1:** For \(\alpha, \beta \in \mathbb{R}^+\)

\[
I = \begin{bmatrix}
\sqrt{\frac{\alpha}{|\alpha|}} I & 0 \\
0 & -\sqrt{\frac{\alpha}{|\alpha|}} I
\end{bmatrix}, \mathcal{J} = \begin{bmatrix}
0 & \sqrt{\beta} \\
-\sqrt{\beta} & 0
\end{bmatrix}, \mathcal{K} = \begin{bmatrix}
0 & \sqrt{\frac{\alpha}{|\alpha|}} I \\
\sqrt{\frac{\alpha}{|\alpha|}} I & 0
\end{bmatrix},
\]

**Case 2:** For \(\alpha \in \mathbb{R}^+, \beta \in \mathbb{R}^-\)

\[
I = \begin{bmatrix}
\sqrt{\frac{\alpha}{|\alpha|}} I & 0 \\
0 & -\sqrt{\frac{\alpha}{|\alpha|}} I
\end{bmatrix}, \mathcal{J} = \begin{bmatrix}
0 & \sqrt{-\beta} \\
\sqrt{-\beta} & 0
\end{bmatrix}, \mathcal{K} = \begin{bmatrix}
0 & \sqrt{-\frac{\alpha}{|\alpha|}} I \\
-\sqrt{-\frac{\alpha}{|\alpha|}} I & 0
\end{bmatrix},
\]

**Case 3:** For \(\alpha \in \mathbb{R}^-, \beta \in \mathbb{R}^+\)

\[
I = \begin{bmatrix}
0 & \sqrt{-\alpha} \\
\sqrt{-\alpha} & 0
\end{bmatrix}, \mathcal{J} = \begin{bmatrix}
-\sqrt{\frac{\beta}{|\beta|}} I & 0 \\
0 & \sqrt{\frac{\beta}{|\beta|}} I
\end{bmatrix}, \mathcal{K} = \begin{bmatrix}
0 & \sqrt{-\frac{\alpha}{|\alpha|}} I \\
-\sqrt{-\frac{\alpha}{|\alpha|}} I & 0
\end{bmatrix},
\]

**Case 4:** For \(\alpha, \beta \in \mathbb{R}^-\)

\[
I = \begin{bmatrix}
0 & \sqrt{-\alpha} I \\
-\sqrt{-\alpha} I & 0
\end{bmatrix}, \mathcal{J} = \begin{bmatrix}
0 & \sqrt{-\beta} \\
\sqrt{-\beta} & 0
\end{bmatrix}, \mathcal{K} = \begin{bmatrix}
\sqrt{\frac{\alpha}{|\alpha|}} I & 0 \\
0 & -\sqrt{\frac{\alpha}{|\alpha|}} I
\end{bmatrix}.
\]
These elements satisfy the following conditions:
\[
\begin{align*}
T^2 &= -\alpha I_2 & TJ &= -JT = K, \\
J^2 &= -\beta I_2 & JK &= -KJ = \beta I, \\
K^2 &= -\alpha\beta I_2 & KT &= -TK = \alpha J.
\end{align*}
\]

Taking into account Case 1, \( \tilde{Q} \) is rewritten as
\[
\tilde{Q} = \begin{bmatrix}
A_0 + \sqrt{\frac{\alpha}{|p|}} IA_1 & \sqrt{\beta} A_2 + \sqrt{\frac{\alpha\beta}{|p|}} IA_3 \\
-\sqrt{\beta} A_2 + \sqrt{\frac{\alpha\beta}{|p|}} IA_3 & A_0 - \sqrt{\frac{\alpha}{|p|}} IA_1
\end{bmatrix}
\]

One can see this matrix in Tian’s paper \cite{48} related to biquaternions (complexified quaternion) for \( \alpha = \beta = 1 \) and \( p = -1 \).

The conjugate (same as the adjoint), transpose, the elliptic conjugate, the total conjugate and determinant \( \tilde{Q} \) can be given as follows:
\[
\begin{align*}
\tilde{Q} &= A_0 I_2 - A_1 I - A_2 J - A_3 K = \text{Adj} \tilde{Q}, \\
\tilde{Q}^T &= A_0 I_2 + A_1 I - A_2 J + A_3 K, \\
\tilde{Q}^C & = A_0 I_2 - A_1 I + A_2 J - A_3 K = \tilde{Q}^T, \\
\tilde{Q}^p & = A_0 I_2 + A_1 I - A_2 J + A_3 K = (\tilde{Q}^C)^T,
\end{align*}
\]

and
\[
\det \tilde{Q} = A_0^2 + \alpha A_1^2 + \beta A_2^2 + \alpha\beta A_3^2 \\
= A_0^2 + A_1^2 \det I + A_2^2 \det J + A_3^2 \det K.
\]

For \( \det \tilde{Q} \neq 0 \) then the inverse of \( \tilde{Q} \) is defined by:
\[
\tilde{Q}^{-1} = \frac{1}{\det \tilde{Q}} = \frac{1}{A_0^2 + \alpha A_1^2 + \beta A_2^2 + \alpha\beta A_3^2} (A_0 I_2 - A_1 I - A_2 J - A_3 K).
\]

Similar calculations can be given for other cases. Additionally, the relationships between the above operations and some properties of generalized quaternion matrices with elliptic number entries can be easily proved so we omit them. For \( A_0, A_1, A_2, A_3 \in \mathbb{C}^{-1} \), we refer to \cite{2} under the condition that \( \alpha = \beta = 1 \) and \( \alpha = 1, \beta = -1 \).

4 Concluding Remarks

Our paper is motivated by the question: What happens if the components of quaternions become \( \mathcal{GCN} \)? Based on this question, we develop the theory of generalized quaternions (non-commutative system) with \( \mathcal{GCN} \) components for \( p, q \in \mathbb{R} \). Also, we investigate the algebraic structures and properties by considering them as a \( \mathcal{GCN} \) and as a quaternion. With specific values of \( \alpha \) and \( \beta \), we obtained different types of quaternions with \( \mathcal{GCN} \) components in Section 3. Additionally, we established matrix representations and gave a numerical example. In Section 3, an alternative approach for generalized quaternion matrix with elliptic number entries are developed.

The crucial part of this paper is that one can reduce the calculations to mentioned types of quaternions with hyperbolic, elliptic and parabolic number components considering \( \Delta = q^2 + 4p \) (see in Table \ref{tab:1}). As a natural consequence of this situation, taking into account special conditions, definition of special quaternions which mentioned in the papers \cite{1,6,9,22,29,33,34,38} are generalized via Definition \ref{def:1} the papers \cite{3,21,32,41} are generalized from the viewpoint of definition, algebraic properties, relations and matrix representations of quaternions and finally different matrix forms in the papers \cite{2,40,48} are generalized in Section 3. All of these situations can be examined in Table \ref{tab:2}. For instance, all of the obtained calculations are in agreement with complex quaternions for \( \alpha = \beta = 1, q = 0, p = -1 \).

With this unified method, we believe that these results give rise to ease of calculation via mathematical concordance and in the future studies, we intend to investigate another commutative and non-commutative quaternions created with \( \mathcal{GCN} \) components in this manner. Now, the necessary and sufficient condition for similarity, co-similarity and semi-similarity for elements of set of the generalized quaternions with \( \mathcal{GCN} \) components for \( p, q \in \mathbb{R} \) is an open problem for researchers.
Table 1: Basic classification regarding components

<table>
<thead>
<tr>
<th>$\Delta = q^2 + 4p$</th>
<th>Type of components</th>
<th>References</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Delta &lt; 0$</td>
<td>elliptic</td>
<td>biquaternion [40,41] (for $q = 0$)</td>
</tr>
<tr>
<td>$\Delta = 0$</td>
<td>parabolic</td>
<td>[7] (for $q = 0$)</td>
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<tr>
<td>$\Delta &gt; 0$</td>
<td>hyperbolic</td>
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</tr>
</tbody>
</table>

Table 2: Classification considering components with regard to the value of $p, q, \alpha$ and $\beta$

<table>
<thead>
<tr>
<th>Condition $q = 0, p = -1$</th>
<th>$\alpha$</th>
<th>$\beta$</th>
<th>Type of components</th>
<th>Type of quaternion</th>
<th>References</th>
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<tr>
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<td>Hamiltonian</td>
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<tr>
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<td>-1</td>
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<td>split</td>
<td>[21]</td>
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<tr>
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<td>semi</td>
<td>[4,9]</td>
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<td>[33,38]</td>
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<td>split semi</td>
<td>[32]</td>
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<td>0</td>
<td>dual</td>
<td>quasi</td>
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</table>

<table>
<thead>
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<th>Condition $q = 0, p = 1$</th>
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<th>$\beta$</th>
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References


