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Total Face Irregularity Strength of type (α, β, γ) of Grid Graphs

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Abstract

In this article, all graphs $G = (V, E, F)$ under consideration are simple, finite, plane and undirected. The minimum integer k for which a vertex-edge labelled graph has distinct face weights is called the total face irregularity strength of the graph and is denoted by $tfs(G_n^m)$. We will describe new graph characteristics namely total (vertex, edge) face irregularity strength of generalized plane grid graphs G_n^m under $k-$labeling of type (α, β, γ) and investigate the exact value for the total face irregularity strength of some families of generalized plane grid graphs. Results are verified by the help of an example.
Keywords: Cartesian product of path graphs, Grid graphs, Total face labeling of type (α, β, γ), Total face irregularity strength.

1 Introduction

This complete research is purely based on simple, plane, finite and undirected graphs $G = (V, E, F)$ where $V(G)$ is the vertex set, $E(G)$ is the edge set and $F(G)$ is the face set of graph G. A mapping that maps graph elements into positive integers is called *graph labeling* and these positive integers are called *labels*. Suppose that k is a positive integer then *a k-labeling of type (α, β, γ)*, where $\alpha, \beta, \gamma \in \{0, 1\}$ is a mapping ϕ from the set of graph elements into the set of positive integers $\{1, 2, 3, \ldots, k\}$. A detail on graph $k-$labeling can be seen in [1]. If the domain of k-labeling of type (α, β, γ) is $V(G)$, $E(G)$, $F(G)$ or $V(G) \cup E(G)$ then we say this a *vertex k-labeling of type* $(1, 0, 0)$, *edge k-labeling of type* $(0, 1, 0)$, *face k-labeling of type* $(0, 0, 1)$ or *total k-labeling of type* $(1, 1, 0)$ respectively. The other possible cases are $V(G) \cup F(G)$, $E(G) \cup F(G)$ and $V(G) \cup E(G) \cup F(G)$ which are defined as *vertex-face k-labeling of type* $(1, 0, 1)$, *edge-face k-labeling of type* $(0, 1, 1)$ and *entire k-labeling of type* $(1, 1, 1)$ respectively. The trivial case $(\alpha, \beta, \gamma) = (0, 0, 0)$ is never allowed [18].
The weight of a vertex is the sum of labels of that vertex itself and the labels of its adjacent edges, *the weight of a graph edge* is the sum of label of that edge itself and the labels of its adjacent vertices, *the weight of a face* is the sum of labels of that face itself and the labels of its surrounding edges and vertices. In general, the weight of any face f of a plane graph G under k-labeling ϕ of type (α, β, γ) can be defined as,

$$Wt_{\phi_{(\alpha,\beta,\gamma)}}(f) = \alpha \sum_{v \sim f} \phi(v) + \beta \sum_{e \sim f} \phi(e) + \gamma\, \phi(f) \tag{1}$$

In this research, we have worked on total labeling in which we label both, vertices and edges, but the ultimate focus is always on *face weight* of the graph. A k-labeling ϕ of type (α, β, γ) of the plane graph G is called *face irregular k-labeling of type* (α, β, γ) of the plane graph G, if for every two different faces $f \neq g$ and $f, g \in G$, we have

$$Wt_{\phi_{(\alpha,\beta,\gamma)}}(f) \neq Wt_{\phi_{(\alpha,\beta,\gamma)}}(g)$$

In general, for a labelled graph G, the minimum integer k for which the graph G admits a face irregular $k-$labeling of type (α, β, γ) is called the *face irregularity strength of type* (α, β, γ) of the plane graph G and it is denoted by $\mathrm{fs}_{(\alpha,\beta,\gamma)}(G)$ and particularly, for a vertex-edge labelled graph, the minimum integer k for which the graph G admits a face irregular k-labeling of type (α, β, γ) is called the *total face irregularity strength of type* (α, β, γ) of the plane graph G and it is denoted by $\mathrm{tfs}_{(\alpha,\beta,\gamma)}(G)$. A deep survey on irregularity strength of graphs can be seen in [2, 3, 4, 6, 9, 10, 13, 14, 16].
Baca et al. [7], worked on total irregularity strength of graphs, calculated exact values and bounds for different families of graphs. A detailed concept of face irregular entire labeling as a modification of vertex irregular and edge irregular total labeling of plane graphs can be seen in [17]. Later, in 2020, Baca et al. investigated face irregular evaluations of plane graphs and obtained estimations on face irregularity strength of type (α, β, γ) for

ladder graphs [18]. This paper, total face irregularity strength of grid graphs under labeling ϕ of type (α,β,γ) is a modification of face irregular evaluations of plane graphs, referred as [18].
In this research, under consideration graphs are grid graphs G_n^m with n rows and m columns. Grid graphs are constructed by the graph cartesian products of path graphs, that is, $G_n^m = P_{n+1}\Box P_{m+1}$. This full article is based on those graphs where $\lfloor\frac{m+1}{3}\rfloor > m-2\lfloor\frac{m+1}{3}\rfloor$. We will prove the exact value for the total face irregularity strength under $k-$labeling ϕ of type (α,β,γ) of grid graphs.

Theorem 1. *[18] Let $G=(V,E,F)$ be a $2-$connected plane graph with n_i $i-$sided faces, $i\geq 3$. Let $\alpha,\beta,\gamma\in\{0,1\}$, $a=min\{i:n_i\neq 0\}$ and $b=max\{i:n_i\neq 0\}$, $n_b=1$ and $c=max\{i:n_i\neq 0, i\leq b\}$. Then the face irregularity strength of type α,β,γ of the plane graph G is*

$$fs_{(\alpha,\beta,\gamma)}(G)\geq\left\lceil\frac{(\alpha+\beta)\,a+\gamma+|F(G)|-2}{(\alpha+\beta)\,c+\gamma}\right\rceil$$

Proof. Proof can be seen in [18]. □

2 Main Results

We will obtain total face irregular labelings of type (α,β,γ) for $\alpha,\beta,\gamma\in\{0,1\}$ of grid graphs and the total face irregularity strength of generalized grid graphs under labeling ϕ of type (α,β,γ). The exact value of $tfs\,(G_n^m)$, that is calculated from grid graph G_n^m under a graph $k-$labeling of type $(1,1,0)$ is considered true if the differences in weights of the horizontal faces is 1 and the differences in weights of the vertical faces is m. Generalized grid graphs can be written as, $G_n^m = P_{n+1}\Box P_{m+1}$ and this research is focussed on those particular grid graphs G_n^m where $\lfloor\frac{m+1}{3}\rfloor > m-2\lfloor\frac{m+1}{3}\rfloor$.

The vertex set and the edge set of the grid graph can be defined as follows:

$$V(P_{n+1}\Box P_{m+1}) = \{v_i^j : i=1,2,\ldots,n+1\ ,\ j=1,2,\ldots,m+1\}$$
$$E(P_{n+1}\Box P_{m+1}) = \{v_i^j v_{i+1}^j : i=1,2,\ldots,n\ ,\ j=1,2,\ldots,m+1\}\cup\{v_i^j v_i^{j+1} : i=1,2,\ldots,n+1\ ,\ j=1,2,\ldots,m\}.$$

The vertices of the grid graph G_n^m for different intervals of i and j can be generalized as

$$f\left(v_i^j\right)=\begin{cases}1+\lfloor\frac{m+1}{3}\rfloor\lfloor\frac{i-1}{2}\rfloor, & i=1,2,3,\ldots,2\left\lceil\frac{k}{\lfloor\frac{m+1}{3}\rfloor}\right\rceil \text{ and } j=1,2,\ldots,m+1\\ k, & i=2\left\lceil\frac{k}{\lfloor\frac{m+1}{3}\rfloor}\right\rceil+1,\ldots,n+1 \text{ and } j=1,2,\ldots,m+1\end{cases}$$

The horizontal edges of the grid graph G_n^m can be generalized as

$$f\left(v_i^j v_i^{j+1}\right)=\begin{cases}1+\left(m-2\lfloor\frac{m+1}{3}\rfloor\right)\lfloor\frac{i-1}{2}\rfloor, & i=1,2,3,\ldots,2\left\lceil\frac{k}{m-2\lfloor\frac{m+1}{3}\rfloor}\right\rceil \text{ and } j=1,2,\ldots,m\\ k, & i=2\left\lceil\frac{k}{m-2\lfloor\frac{m+1}{3}\rfloor}\right\rceil+1,\ldots,n+1 \text{ and } j=1,2,\ldots,m\end{cases}$$

The generalized vertical edges of the grid graph G_n^m for different values of i and j are

$$f\left(v_i^j v_{i+1}^j\right)=\begin{cases}\lceil\frac{j}{2}\rceil, & i=1,2,\ldots,2\left\lceil\frac{k}{\lfloor\frac{m+1}{3}\rfloor}\right\rceil-1 \text{ and } j=1,2,\ldots,m-1\\ \lceil\frac{j}{2}\rceil+\left(\lfloor\frac{m+1}{3}\rfloor\left\lceil\frac{k}{\lfloor\frac{m+1}{3}\rfloor}\right\rceil+1-k\right) & \\ \left(i-2\left\lceil\frac{k}{\lfloor\frac{m+1}{3}\rfloor}\right\rceil+1\right), & i=2\left\lceil\frac{k}{\lfloor\frac{m+1}{3}\rfloor}\right\rceil, 2\left\lceil\frac{k}{\lfloor\frac{m+1}{3}\rfloor}\right\rceil+1 \text{ and } j=1,2,3,\ldots,m-1\\ \lceil\frac{j}{2}\rceil+2\left(\lfloor\frac{m+1}{3}\rfloor\left\lceil\frac{k}{\lfloor\frac{m+1}{3}\rfloor}\right\rceil\right)+2-2k & \\ +\left(\lfloor\frac{m+1}{3}\rfloor\right)\left(i-2\left\lceil\frac{k}{\lfloor\frac{m+1}{3}\rfloor}\right\rceil-1\right), & i=2\left\lceil\frac{k}{\lfloor\frac{m+1}{3}\rfloor}\right\rceil+2,\cdots,2\left\lceil\frac{k}{m-2\lfloor\frac{m+1}{3}\rfloor}\right\rceil-1 \text{ and}\\ & j=1,2,3,\ldots,m-1\end{cases}$$

$$f\left(v_i^j v_{i+1}^j\right)=\begin{cases}\left\lceil \frac{j}{2}\right\rceil+2\left(\left\lfloor \frac{m+1}{3}\right\rfloor\right)\left(\left\lceil \frac{k}{m-2\left\lfloor \frac{m+1}{3}\right\rfloor}\right\rceil-1\right)+2-2k \\ +\left\lfloor \frac{1}{2}\left(\left\lceil \frac{k}{m-2\left\lfloor \frac{m+1}{3}\right\rfloor}\right\rceil\left(m-2\left\lfloor \frac{m+1}{3}\right\rfloor\right)+2\left\lfloor \frac{m+1}{3}\right\rfloor+1-k\right)\right\rfloor \\ \left\lceil \frac{1}{2}\left(i-2\left\lceil \frac{k}{m-2\left\lfloor \frac{m+1}{3}\right\rfloor}\right\rceil+1\right)\right\rceil \\ +\left\lceil \frac{1}{2}\left(\left\lceil \frac{k}{m-2\left\lfloor \frac{m+1}{3}\right\rfloor}\right\rceil\left(m-2\left\lfloor \frac{m+1}{3}\right\rfloor\right)+2\left\lfloor \frac{m+1}{3}\right\rfloor+1-k\right)\right\rceil \\ \left\lfloor \frac{1}{2}\left(i-2\left\lceil \frac{k}{m-2\left\lfloor \frac{m+1}{3}\right\rfloor}\right\rceil+1\right)\right\rfloor, & i=2\left\lceil \frac{k}{m-2\left\lfloor \frac{m+1}{3}\right\rfloor}\right\rceil, 2\left\lceil \frac{k}{m-2\left\lfloor \frac{m+1}{3}\right\rfloor}\right\rceil+1 \\ & j=1,3,\ldots,m\,;\ m\equiv 1\,(2)\ \ or \\ & j=1,3,\ldots,m+1\,;\ m\equiv 0\,(2) \\ \left\lceil \frac{j}{2}\right\rceil+2\left(\left\lfloor \frac{m+1}{3}\right\rfloor\right)\left(\left\lceil \frac{k}{m-2\left\lfloor \frac{m+1}{3}\right\rfloor}\right\rceil-1\right)+2-2k \\ +\left\lceil \frac{1}{2}\left(\left\lceil \frac{k}{m-2\left\lfloor \frac{m+1}{3}\right\rfloor}\right\rceil\left(m-2\left\lfloor \frac{m+1}{3}\right\rfloor\right)+2\left\lfloor \frac{m+1}{3}\right\rfloor+1-k\right)\right\rceil \\ \left\lceil \frac{1}{2}\left(i-2\left\lceil \frac{k}{m-2\left\lfloor \frac{m+1}{3}\right\rfloor}\right\rceil+1\right)\right\rceil \\ +\left\lfloor \frac{1}{2}\left(\left\lceil \frac{k}{m-2\left\lfloor \frac{m+1}{3}\right\rfloor}\right\rceil\left(m-2\left\lfloor \frac{m+1}{3}\right\rfloor\right)+2\left\lfloor \frac{m+1}{3}\right\rfloor+1-k\right)\right\rfloor \\ \left\lfloor \frac{1}{2}\left(i-2\left\lceil \frac{k}{m-2\left\lfloor \frac{m+1}{3}\right\rfloor}\right\rceil+1\right)\right\rfloor, & i=2\left\lceil \frac{k}{m-2\left\lfloor \frac{m+1}{3}\right\rfloor}\right\rceil, 2\left\lceil \frac{k}{m-2\left\lfloor \frac{m+1}{3}\right\rfloor}\right\rceil+1 \\ & j=2,4,\ldots,m+1\,;\ m\equiv 1\,(2)\ \ or \\ & j=2,4,\ldots,m\,;\ m\equiv 0\,(2)\end{cases}$$

$$f\left(v_i^j v_{i+1}^j\right)=\begin{cases}\left\lceil \frac{j}{2}\right\rceil+3-3k+m\left\lceil \frac{k}{m-2\left\lfloor \frac{m+1}{3}\right\rfloor}\right\rceil \\ +\left\lfloor \frac{m}{2}\right\rfloor\left[\frac{1}{2}\left(i-2\left\lceil \frac{k}{m-2\left\lfloor \frac{m+1}{3}\right\rfloor}\right\rceil-1\right)\right. \\ \left.+\left\lceil \frac{m}{2}\right\rceil\ \frac{1}{2}\left(i-2\left\lceil \frac{k}{m-2\left\lfloor \frac{m+1}{3}\right\rfloor}\right\rceil-1\right)\right], & i=2\left\lceil \frac{k}{m-2\left\lfloor \frac{m+1}{3}\right\rfloor}\right\rceil+2,\cdots,n-1 \\ & j=1,3,\ldots,m\,;\ m\equiv 1\,(2)\ \ or \\ & j=1,3,\ldots,m+1\,;\ m\equiv 0\,(2) \\ \left\lceil \frac{j}{2}\right\rceil+3-3k+m\left\lceil \frac{k}{m-2\left\lfloor \frac{m+1}{3}\right\rfloor}\right\rceil \\ +\left\lceil \frac{m}{2}\right\rceil\left[\frac{1}{2}\left(i-2\left\lceil \frac{k}{m-2\left\lfloor \frac{m+1}{3}\right\rfloor}\right\rceil-1\right)\right. \\ \left.+\left\lfloor \frac{m}{2}\right\rfloor\ \frac{1}{2}\left(i-2\left\lceil \frac{k}{m-2\left\lfloor \frac{m+1}{3}\right\rfloor}\right\rceil-1\right)\right], & i=2\left\lceil \frac{k}{m-2\left\lfloor \frac{m+1}{3}\right\rfloor}\right\rceil+2,\cdots,n-1 \\ & j=2,4,\ldots,m+1\,;\ m\equiv 1\,(2)\ \ or \\ & j=2,4,\ldots,m\,;\ m\equiv 0\,(2)\end{cases}$$

The weights of the face f under $k-$labeling ϕ of type $(1,1,0)$ can be defined as

$$Wt_{\phi(1,1,0)}\left(f_i^j\right)=\phi\left(v_i^j\right)+\phi\left(v_i^{j+1}\right)+\phi\left(v_{i+1}^j\right)+\phi\left(v_{i+1}^{j+1}\right)+\phi\left(v_i^j v_i^{j+1}\right)+\phi\left(v_i^j v_{i+1}^j\right)+\phi\left(v_{i+1}^j v_{i+1}^{j+1}\right)$$
$$+\phi\left(v_i^{j+1} v_{i+1}^{j+1}\right)$$

The differences in weights of the horizontal faces can be defined as

$$\begin{aligned}Wt_{\phi(1,1,0)}\left(f_i^{j+1}\right)-Wt_{\phi(1,1,0)}\left(f_i^j\right)&=\phi\left(v_i^{j+1}\right)+\phi\left(v_i^{j+2}\right)+\phi\left(v_{i+1}^{j+1}\right)+\phi\left(v_{i+1}^{j+2}\right)+\phi\left(v_i^{j+1}v_i^{j+2}\right)\\&+\phi\left(v_i^{j+1}v_{i+1}^{j+1}\right)+\phi\left(v_{i+1}^{j+1}v_{i+1}^{j+2}\right)+\phi\left(v_i^{j+2}v_{i+1}^{j+2}\right)-\phi\left(v_i^j\right)\\&-\phi\left(v_i^{j+1}\right)-\phi\left(v_{i+1}^j\right)-\phi\left(v_{i+1}^{j+1}\right)-\phi\left(v_i^j v_i^{j+1}\right)\\&-\phi\left(v_i^j v_{i+1}^j\right)-\phi\left(v_{i+1}^j v_{i+1}^{j+1}\right)-\phi\left(v_i^{j+1}v_{i+1}^{j+1}\right)\end{aligned}$$

$$= \phi\left(v_i^{j+2}\right) + \phi\left(v_{i+1}^{j+2}\right) + \phi\left(v_i^{j+1}v_i^{j+2}\right) + \phi\left(v_{i+1}^{j+1}v_{i+1}^{j+2}\right)$$
$$+ \phi\left(v_i^{j+2}v_{i+1}^{j+2}\right) - \phi\left(v_i^{j}\right) - \phi\left(v_{i+1}^{j}\right) - \phi\left(v_i^{j}v_i^{j+1}\right)$$
$$- \phi\left(v_i^{j}v_{i+1}^{j}\right) - \phi\left(v_{i+1}^{j}v_{i+1}^{j+1}\right)$$

Theorem 2. *If G_n^m is a grid graph which is generalized for positive integers, $n \geq 2$ rows and $m \geq 2$ columns, with $\left\lfloor \frac{m+1}{3} \right\rfloor > m - 2\left\lfloor \frac{m+1}{3} \right\rfloor$ under $k-$labeling ϕ of type $(1,1,0)$ then the differences in weights of horizontal faces is 1.*

Proof. To prove this theorem, it will be sufficient to show that among all the intervals of labeling, we have $Wt_{\phi(1,1,0)}\left(f_i^{j+1}\right) - Wt_{\phi(1,1,0)}\left(f_i^{j}\right) = 1$. In order to do this, we will proceed as follows:

$$\text{For } \ i = 1,2,3,\ldots,2\left\lceil \frac{k}{\left\lfloor \frac{m+1}{3} \right\rfloor} \right\rceil - 1 \text{ and } j = 1,2,3,\ldots,m-1$$
$$Wt_{\phi(1,1,0)}\left(f_i^{j+1}\right) - Wt_{\phi(1,1,0)}\left(f_i^{j}\right) =$$
$$= \phi\left(v_i^{j+2}\right) + \phi\left(v_{i+1}^{j+2}\right) + \phi\left(v_i^{j+1}v_i^{j+2}\right) + \phi\left(v_{i+1}^{j+1}v_{i+1}^{j+2}\right) + \phi\left(v_i^{j+2}v_{i+1}^{j+2}\right)$$
$$- \phi\left(v_i^{j}\right) - \phi\left(v_{i+1}^{j}\right) - \phi\left(v_i^{j}v_i^{j+1}\right) - \phi\left(v_i^{j}v_{i+1}^{j}\right) - \phi\left(v_{i+1}^{j}v_{i+1}^{j+1}\right)$$
$$= \phi\left(v_i^{j+2}v_{i+1}^{j+2}\right) - \phi\left(v_i^{j}v_{i+1}^{j}\right)$$
$$= \left\lceil \frac{j+2}{2} \right\rceil - \left\lceil \frac{j}{2} \right\rceil$$
$$= 1$$

$$\text{For } \ i = 2\left\lceil \frac{k}{\left\lfloor \frac{m+1}{3} \right\rfloor} \right\rceil, 2\left\lceil \frac{k}{\left\lfloor \frac{m+1}{3} \right\rfloor} \right\rceil + 1 \text{ and } j = 1,2,3,\ldots,m-1$$
$$Wt_{\phi(1,1,0)}\left(f_i^{j+1}\right) - Wt_{\phi(1,1,0)}\left(f_i^{j}\right) =$$
$$= \phi\left(v_i^{j+2}\right) + \phi\left(v_{i+1}^{j+2}\right) + \phi\left(v_i^{j+1}v_i^{j+2}\right) + \phi\left(v_{i+1}^{j+1}v_{i+1}^{j+2}\right) + \phi\left(v_i^{j+2}v_{i+1}^{j+2}\right)$$
$$- \phi\left(v_i^{j}\right) - \phi\left(v_{i+1}^{j}\right) - \phi\left(v_i^{j}v_i^{j+1}\right) - \phi\left(v_i^{j}v_{i+1}^{j}\right) - \phi\left(v_{i+1}^{j}v_{i+1}^{j+1}\right)$$
$$= \left\lceil \frac{j+2}{2} \right\rceil + \left(\left\lfloor \frac{m+1}{3} \right\rfloor \left\lceil \frac{k}{\left\lfloor \frac{m+1}{3} \right\rfloor} \right\rceil + 1 - k\right)\left(i - 2\left\lceil \frac{k}{\left\lfloor \frac{m+1}{3} \right\rfloor} \right\rceil + 1\right)$$
$$- \left\lceil \frac{j}{2} \right\rceil - \left(\left\lfloor \frac{m+1}{3} \right\rfloor \left\lceil \frac{k}{\left\lfloor \frac{m+1}{3} \right\rfloor} \right\rceil + 1 - k\right)\left(i - 2\left\lceil \frac{k}{\left\lfloor \frac{m+1}{3} \right\rfloor} \right\rceil + 1\right)$$
$$= \left\lceil \frac{j+2}{2} \right\rceil - \left\lceil \frac{j}{2} \right\rceil$$
$$= 1$$

$$\text{For } \ i = 2\left\lceil \frac{k}{\left\lfloor \frac{m+1}{3} \right\rfloor} \right\rceil + 2, \cdots, 2\left\lceil \frac{k}{m - 2\left\lfloor \frac{m+1}{3} \right\rfloor} \right\rceil - 1 \text{ and } j = 1,2,3,\ldots,m-1$$
$$Wt_{\phi(1,1,0)}\left(f_i^{j+1}\right) - Wt_{\phi(1,1,0)}\left(f_i^{j}\right) =$$
$$= \phi\left(v_i^{j+2}\right) + \phi\left(v_{i+1}^{j+2}\right) + \phi\left(v_i^{j+1}v_i^{j+2}\right) + \phi\left(v_{i+1}^{j+1}v_{i+1}^{j+2}\right) + \phi\left(v_i^{j+2}v_{i+1}^{j+2}\right)$$
$$- \phi\left(v_i^{j}\right) - \phi\left(v_{i+1}^{j}\right) - \phi\left(v_i^{j}v_i^{j+1}\right) - \phi\left(v_i^{j}v_{i+1}^{j}\right) - \phi\left(v_{i+1}^{j}v_{i+1}^{j+1}\right)$$
$$= \left\lceil \frac{j+2}{2} \right\rceil + 2\left(\left\lfloor \frac{m+1}{3} \right\rfloor \left\lceil \frac{k}{\left\lfloor \frac{m+1}{3} \right\rfloor} \right\rceil\right) + 2 - 2k + \left(\left\lfloor \frac{m+1}{3} \right\rfloor\right)\left(i - 2\left\lceil \frac{k}{\left\lfloor \frac{m+1}{3} \right\rfloor} \right\rceil - 1\right)$$

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$$
\begin{aligned}
&-\left\lceil \frac{j}{2}\right\rceil - 2\left(\left\lfloor \frac{m+1}{3}\right\rfloor \left\lceil \frac{k}{\left\lfloor \frac{m+1}{3}\right\rfloor}\right\rceil\right) - 2 + 2k + \left(\left\lfloor \frac{m+1}{3}\right\rfloor\right)\left(i - 2\left\lceil \frac{k}{\left\lfloor \frac{m+1}{3}\right\rfloor}\right\rceil - 1\right)\\
&= \left\lceil \frac{j+2}{2}\right\rceil - \left\lceil \frac{j}{2}\right\rceil\\
&= 1
\end{aligned}
$$

For $i = 2\left\lceil \frac{k}{m-2\left\lfloor \frac{m+1}{3}\right\rfloor}\right\rceil, 2\left\lceil \frac{k}{m-2\left\lfloor \frac{m+1}{3}\right\rfloor}\right\rceil + 1$

and $j = 1, 3, \ldots, m\ ;\ m \equiv 1\,(2)$ or $j = 1, 3, \ldots, m+1\ ;\ m \equiv 0\,(2)$

$$
\begin{aligned}
&Wt_{\phi(1,1,0)}\left(f_i^{j+1}\right) - Wt_{\phi(1,1,0)}\left(f_i^{j}\right) =\\
&= \phi\left(v_i^{j+2}\right) + \phi\left(v_{i+1}^{j+2}\right) + \phi\left(v_i^{j+1} v_i^{j+2}\right) + \phi\left(v_{i+1}^{j+1} v_{i+1}^{j+2}\right) + \phi\left(v_i^{j+2} v_{i+1}^{j+2}\right)\\
&- \phi\left(v_i^{j}\right) - \phi\left(v_{i+1}^{j}\right) - \phi\left(v_i^{j} v_i^{j+1}\right) - \phi\left(v_i^{j} v_{i+1}^{j}\right) - \phi\left(v_{i+1}^{j} v_{i+1}^{j+1}\right)\\
&= \left\lceil \frac{j+2}{2}\right\rceil + 2\left\lfloor \frac{m+1}{3}\right\rfloor \left(\left\lceil \frac{k}{m-2\left\lfloor \frac{m+1}{3}\right\rfloor}\right\rceil - 1\right) + 2 - 2k\\
&+ \left\lfloor \frac{1}{2}\left(\left\lceil \frac{k}{m-2\left\lfloor \frac{m+1}{3}\right\rfloor}\right\rceil \left(m - 2\left\lfloor \frac{m+1}{3}\right\rfloor\right) + 2\left\lfloor \frac{m+1}{3}\right\rfloor + 1 - k\right)\right\rfloor \left\lceil \frac{1}{2}\left(i - 2\left\lceil \frac{k}{m-2\left\lfloor \frac{m+1}{3}\right\rfloor}\right\rceil + 1\right)\right\rceil\\
&+ \left\lceil \frac{1}{2}\left(\left\lceil \frac{k}{m-2\left\lfloor \frac{m+1}{3}\right\rfloor}\right\rceil \left(m - 2\left\lfloor \frac{m+1}{3}\right\rfloor\right) + 2\left\lfloor \frac{m+1}{3}\right\rfloor + 1 - k\right)\right\rceil \left\lfloor \frac{1}{2}\left(i - 2\left\lceil \frac{k}{m-2\left\lfloor \frac{m+1}{3}\right\rfloor}\right\rceil + 1\right)\right\rfloor\\
&- \left\lceil \frac{j}{2}\right\rceil - 2\left(\left\lfloor \frac{m+1}{3}\right\rfloor\right)\left(\left\lceil \frac{k}{m-2\left\lfloor \frac{m+1}{3}\right\rfloor}\right\rceil - 1\right) - 2 + 2k\\
&- \left\lfloor \frac{1}{2}\left(\left\lceil \frac{k}{m-2\left\lfloor \frac{m+1}{3}\right\rfloor}\right\rceil \left(m - 2\left\lfloor \frac{m+1}{3}\right\rfloor\right) + 2\left\lfloor \frac{m+1}{3}\right\rfloor + 1 - k\right)\right\rfloor \left\lceil \frac{1}{2}\left(i - 2\left\lceil \frac{k}{m-2\left\lfloor \frac{m+1}{3}\right\rfloor}\right\rceil + 1\right)\right\rceil\\
&- \left\lceil \frac{1}{2}\left(\left\lceil \frac{k}{m-2\left\lfloor \frac{m+1}{3}\right\rfloor}\right\rceil \left(m - 2\left\lfloor \frac{m+1}{3}\right\rfloor\right) + 2\left\lfloor \frac{m+1}{3}\right\rfloor + 1 - k\right)\right\rceil \left\lfloor \frac{1}{2}\left(i - 2\left\lceil \frac{k}{m-2\left\lfloor \frac{m+1}{3}\right\rfloor}\right\rceil + 1\right)\right\rfloor\\
&= \left\lceil \frac{j+2}{2}\right\rceil - \left\lceil \frac{j}{2}\right\rceil\\
&= 1
\end{aligned}
$$

For $i = 2\left\lceil \frac{k}{m-2\left\lfloor \frac{m+1}{3}\right\rfloor}\right\rceil, 2\left\lceil \frac{k}{m-2\left\lfloor \frac{m+1}{3}\right\rfloor}\right\rceil + 1$

and $j = 2, 4, \ldots, m+1\ ;\ m \equiv 1\,(2)$ or $j = 2, 4, \ldots, m\ ;\ m \equiv 0\,(2)$

$$
\begin{aligned}
&Wt_{\phi(1,1,0)}\left(f_i^{j+1}\right) - Wt_{\phi(1,1,0)}\left(f_i^{j}\right) =\\
&= \phi\left(v_i^{j+2}\right) + \phi\left(v_{i+1}^{j+2}\right) + \phi\left(v_i^{j+1} v_i^{j+2}\right) + \phi\left(v_{i+1}^{j+1} v_{i+1}^{j+2}\right) + \phi\left(v_i^{j+2} v_{i+1}^{j+2}\right)\\
&- \phi\left(v_i^{j}\right) - \phi\left(v_{i+1}^{j}\right) - \phi\left(v_i^{j} v_i^{j+1}\right) - \phi\left(v_i^{j} v_{i+1}^{j}\right) - \phi\left(v_{i+1}^{j} v_{i+1}^{j+1}\right)\\
&= \left\lceil \frac{j+2}{2}\right\rceil + 2\left(\left\lfloor \frac{m+1}{3}\right\rfloor\right)\left(\left\lceil \frac{k}{m-2\left\lfloor \frac{m+1}{3}\right\rfloor}\right\rceil - 1\right) + 2 - 2k\\
&+ \left\lceil \frac{1}{2}\left(\left\lceil \frac{k}{m-2\left\lfloor \frac{m+1}{3}\right\rfloor}\right\rceil \left(m - 2\left\lfloor \frac{m+1}{3}\right\rfloor\right) + 2\left\lfloor \frac{m+1}{3}\right\rfloor + 1 - k\right)\right\rceil \left\lceil \frac{1}{2}\left(i - 2\left\lceil \frac{k}{m-2\left\lfloor \frac{m+1}{3}\right\rfloor}\right\rceil + 1\right)\right\rceil\\
&+ \left\lfloor \frac{1}{2}\left(\left\lceil \frac{k}{m-2\left\lfloor \frac{m+1}{3}\right\rfloor}\right\rceil \left(m - 2\left\lfloor \frac{m+1}{3}\right\rfloor\right) + 2\left\lfloor \frac{m+1}{3}\right\rfloor + 1 - k\right)\right\rfloor \left\lfloor \frac{1}{2}\left(i - 2\left\lceil \frac{k}{m-2\left\lfloor \frac{m+1}{3}\right\rfloor}\right\rceil + 1\right)\right\rfloor\\
&- \left\lceil \frac{j}{2}\right\rceil - 2\left(\left\lfloor \frac{m+1}{3}\right\rfloor\right)\left(\left\lceil \frac{k}{m-2\left\lfloor \frac{m+1}{3}\right\rfloor}\right\rceil - 1\right) - 2 + 2k
\end{aligned}
$$

$$
\begin{aligned}
&-\left[\frac{1}{2}\left(\left\lceil\frac{k}{m-2\left\lfloor\frac{m+1}{3}\right\rfloor}\right\rceil\left(m-2\left\lfloor\frac{m+1}{3}\right\rfloor\right)+2\left\lfloor\frac{m+1}{3}\right\rfloor+1-k\right)\right]\left[\frac{1}{2}\left(i-2\left\lceil\frac{k}{m-2\left\lfloor\frac{m+1}{3}\right\rfloor}\right\rceil+1\right)\right] \\
&-\left\lfloor\frac{1}{2}\left(\left\lceil\frac{k}{m-2\left\lfloor\frac{m+1}{3}\right\rfloor}\right\rceil\left(m-2\left\lfloor\frac{m+1}{3}\right\rfloor\right)+2\left\lfloor\frac{m+1}{3}\right\rfloor+1-k\right)\right\rfloor\left\lfloor\frac{1}{2}\left(i-2\left\lceil\frac{k}{m-2\left\lfloor\frac{m+1}{3}\right\rfloor}\right\rceil+1\right)\right\rfloor \\
&=\left\lceil\frac{j+2}{2}\right\rceil-\left\lceil\frac{j}{2}\right\rceil \\
&=1
\end{aligned}
$$

$$
\begin{aligned}
&\text{For } \ i=2\left\lceil\frac{k}{m-2\left\lfloor\frac{m+1}{3}\right\rfloor}\right\rceil+2,\ldots,n-1 \\
&\text{and } \ j=1,3,\ldots,m\ ;\ m\equiv 1\,(2) \ \text{ or } j=1,3,\ldots,m+1\ ;\ m\equiv 0\,(2) \\
&Wt_{\phi(1,1,0)}\left(f_i^{j+1}\right)-Wt_{\phi(1,1,0)}\left(f_i^{j}\right)= \\
&=\phi\left(v_i^{j+2}\right)+\phi\left(v_{i+1}^{j+2}\right)+\phi\left(v_i^{j+1}v_i^{j+2}\right)+\phi\left(v_{i+1}^{j+1}v_{i+1}^{j+2}\right)+\phi\left(v_i^{j+2}v_{i+1}^{j+2}\right) \\
&-\phi\left(v_i^{j}\right)-\phi\left(v_{i+1}^{j}\right)-\phi\left(v_i^{j}v_i^{j+1}\right)-\phi\left(v_i^{j}v_{i+1}^{j}\right)-\phi\left(v_{i+1}^{j}v_{i+1}^{j+1}\right) \\
&=\left\lceil\frac{j+2}{2}\right\rceil+3-3k+m\left\lceil\frac{k}{m-2\left\lfloor\frac{m+1}{3}\right\rfloor}\right\rceil+\left\lfloor\frac{m}{2}\right\rfloor\left\lceil\frac{1}{2}\left(i-2\left\lceil\frac{k}{m-2\left\lfloor\frac{m+1}{3}\right\rfloor}\right\rceil-1\right)\right\rceil \\
&+\left\lceil\frac{m}{2}\right\rceil\left\lfloor\frac{1}{2}\left(i-2\left\lceil\frac{k}{m-2\left\lfloor\frac{m+1}{3}\right\rfloor}\right\rceil-1\right)\right\rfloor-\left\lceil\frac{j}{2}\right\rceil-3+3k-m\left\lceil\frac{k}{m-2\left\lfloor\frac{m+1}{3}\right\rfloor}\right\rceil \\
&-\left\lfloor\frac{m}{2}\right\rfloor\left\lceil\frac{1}{2}\left(i-2\left\lceil\frac{k}{m-2\left\lfloor\frac{m+1}{3}\right\rfloor}\right\rceil-1\right)\right\rceil-\left\lceil\frac{m}{2}\right\rceil\left\lfloor\frac{1}{2}\left(i-2\left\lceil\frac{k}{m-2\left\lfloor\frac{m+1}{3}\right\rfloor}\right\rceil-1\right)\right\rfloor \\
&=\left\lceil\frac{j+2}{2}\right\rceil-\left\lceil\frac{j}{2}\right\rceil \\
&=1
\end{aligned}
$$

$$
\begin{aligned}
&\text{For } \ i=2\left\lceil\frac{k}{m-2\left\lfloor\frac{m+1}{3}\right\rfloor}\right\rceil+2,\ldots,n-1 \\
&\text{and } \ j=2,4,\ldots,m+1\ ;\ m\equiv 1\,(2) \ \text{ or } j=2,4,\ldots,m\ ;\ m\equiv 0\,(2) \\
&Wt_{\phi(1,1,0)}\left(f_i^{j+1}\right)-Wt_{\phi(1,1,0)}\left(f_i^{j}\right)= \\
&=\phi\left(v_i^{j+2}\right)+\phi\left(v_{i+1}^{j+2}\right)+\phi\left(v_i^{j+1}v_i^{j+2}\right)+\phi\left(v_{i+1}^{j+1}v_{i+1}^{j+2}\right)+\phi\left(v_i^{j+2}v_{i+1}^{j+2}\right) \\
&-\phi\left(v_i^{j}\right)-\phi\left(v_{i+1}^{j}\right)-\phi\left(v_i^{j}v_i^{j+1}\right)-\phi\left(v_i^{j}v_{i+1}^{j}\right)-\phi\left(v_{i+1}^{j}v_{i+1}^{j+1}\right) \\
&=\left\lceil\frac{j+2}{2}\right\rceil+3-3k+m\left\lceil\frac{k}{m-2\left\lfloor\frac{m+1}{3}\right\rfloor}\right\rceil+\left\lceil\frac{m}{2}\right\rceil\left\lceil\frac{1}{2}\left(i-2\left\lceil\frac{k}{m-2\left\lfloor\frac{m+1}{3}\right\rfloor}\right\rceil-1\right)\right\rceil \\
&+\left\lfloor\frac{m}{2}\right\rfloor\left\lfloor\frac{1}{2}\left(i-2\left\lceil\frac{k}{m-2\left\lfloor\frac{m+1}{3}\right\rfloor}\right\rceil-1\right)\right\rfloor-\left\lceil\frac{j}{2}\right\rceil-3+3k-m\left\lceil\frac{k}{m-2\left\lfloor\frac{m+1}{3}\right\rfloor}\right\rceil \\
&-\left\lceil\frac{m}{2}\right\rceil\left\lceil\frac{1}{2}\left(i-2\left\lceil\frac{k}{m-2\left\lfloor\frac{m+1}{3}\right\rfloor}\right\rceil-1\right)\right\rceil-\left\lfloor\frac{m}{2}\right\rfloor\left\lfloor\frac{1}{2}\left(i-2\left\lceil\frac{k}{m-2\left\lfloor\frac{m+1}{3}\right\rfloor}\right\rceil-1\right)\right\rfloor \\
&=\left\lceil\frac{j+2}{2}\right\rceil-\left\lceil\frac{j}{2}\right\rceil \\
&=1
\end{aligned}
$$

$\square$

The weights of the face f under $k-$labeling ϕ of type $(1,1,0)$ can be defined as

$$Wt_{\phi(1,1,0)}\left(f_i^{j}\right)=\phi\left(v_i^{j}\right)+\phi\left(v_i^{j+1}\right)+\phi\left(v_{i+1}^{j}\right)+\phi\left(v_{i+1}^{j+1}\right)+\phi\left(v_i^{j}v_i^{j+1}\right)+\phi\left(v_i^{j}v_{i+1}^{j}\right)$$

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$$+\phi\left(v_{i+1}^{j}v_{i+1}^{j+1}\right)+\phi\left(v_{i}^{j+1}v_{i+1}^{j+1}\right)$$

The vertical difference in weights can be defined as

$$\begin{aligned}Wt_{\phi(1,1,0)}\left(f_{i+1}^{j}\right)-Wt_{\phi(1,1,0)}\left(f_{i}^{j}\right)&=\phi\left(v_{i+1}^{j}\right)+\phi\left(v_{i+1}^{j+1}\right)+\phi\left(v_{i+2}^{j}\right)+\phi\left(v_{i+2}^{j+1}\right)+\phi\left(v_{i+1}^{j}v_{i+1}^{j+1}\right)\\&+\phi\left(v_{i+1}^{j}v_{i+2}^{j}\right)+\phi\left(v_{i+2}^{j}v_{i+2}^{j+1}\right)+\phi\left(v_{i+1}^{j+1}v_{i+2}^{j+1}\right)-\phi\left(v_{i}^{j}\right)\\&-\phi\left(v_{i}^{j+1}\right)-\phi\left(v_{i+1}^{j}\right)-\phi\left(v_{i+1}^{j+1}\right)-\phi\left(v_{i}^{j}v_{i}^{j+1}\right)\\&-\phi\left(v_{i}^{j}v_{i+1}^{j}\right)-\phi\left(v_{i+1}^{j}v_{i+1}^{j+1}\right)-\phi\left(v_{i}^{j+1}v_{i+1}^{j+1}\right)\\&=\phi\left(v_{i+2}^{j}\right)+\phi\left(v_{i+2}^{j+1}\right)+\phi\left(v_{i+1}^{j}v_{i+2}^{j}\right)+\phi\left(v_{i+2}^{j}v_{i+2}^{j+1}\right)+\phi\left(v_{i+1}^{j+1}v_{i+2}^{j+1}\right)\\&-\phi\left(v_{i}^{j}\right)-\phi\left(v_{i}^{j+1}\right)-\phi\left(v_{i}^{j}v_{i}^{j+1}\right)-\phi\left(v_{i}^{j}v_{i+1}^{j}\right)-\phi\left(v_{i}^{j+1}v_{i+1}^{j+1}\right)\end{aligned}$$

Theorem 3. *If G_n^m is a grid graph which is generalized for positive integers, $n\geq 2$ rows and $m\geq 2$ columns, with $\left\lfloor\frac{m+1}{3}\right\rfloor > m-2\left\lfloor\frac{m+1}{3}\right\rfloor$ under $k-$labeling ϕ of type $(1,1,0)$ then the differences in weights of vertical faces is m.*

Proof. To prove this theorem, it will be sufficient to show that among all the intervals of labeling, we have $Wt_{\phi(1,1,0)}\left(f_{i+1}^{j}\right)-Wt_{\phi(1,1,0)}\left(f_{i}^{j}\right)=m$. In order to do this, we will proceed as follows:

For $i=1,2,\ldots,2\left\lceil\frac{k}{\left\lfloor\frac{m+1}{3}\right\rfloor}\right\rceil-2\,;\ j=1,2,\ldots,m$

$$\begin{aligned}&Wt_{\phi(1,1,0)}\left(f_{i+1}^{j}\right)-Wt_{\phi(1,1,0)}\left(f_{i}^{j}\right)=\\&=\phi\left(v_{i+2}^{j}\right)+\phi\left(v_{i+2}^{j+1}\right)+\phi\left(v_{i+1}^{j}v_{i+2}^{j}\right)+\phi\left(v_{i+2}^{j}v_{i+2}^{j+1}\right)+\phi\left(v_{i+1}^{j+1}v_{i+2}^{j+1}\right)-\phi\left(v_{i}^{j}\right)-\phi\left(v_{i}^{j+1}\right)-\phi\left(v_{i}^{j}v_{i}^{j+1}\right)\\&-\phi\left(v_{i}^{j}v_{i+1}^{j}\right)-\phi\left(v_{i}^{j+1}v_{i+1}^{j+1}\right)\\&=1+\left\lfloor\frac{m+1}{3}\right\rfloor\left\lfloor\frac{i+1}{2}\right\rfloor+1+\left\lfloor\frac{m+1}{3}\right\rfloor\left\lfloor\frac{i+1}{2}\right\rfloor+\left\lceil\frac{j}{2}\right\rceil+1+\left(m-2\left\lfloor\frac{m+1}{3}\right\rfloor\right)\left\lfloor\frac{i+1}{2}\right\rfloor+\left\lceil\frac{j+1}{2}\right\rceil\\&-1-\left\lfloor\frac{m+1}{3}\right\rfloor\left\lfloor\frac{i-1}{2}\right\rfloor-1-\left\lfloor\frac{m+1}{3}\right\rfloor\left\lfloor\frac{i-1}{2}\right\rfloor-1-\left(m-2\left\lfloor\frac{m+1}{3}\right\rfloor\right)\left\lfloor\frac{i-1}{2}\right\rfloor-\left\lceil\frac{j}{2}\right\rceil-\left\lceil\frac{j+1}{2}\right\rceil\\&=2\left\lfloor\frac{m+1}{3}\right\rfloor\left(\left\lfloor\frac{i+1}{2}\right\rfloor-\left\lfloor\frac{i-1}{2}\right\rfloor\right)+\left(m-2\left\lfloor\frac{m+1}{3}\right\rfloor\right)\left(\left\lfloor\frac{i+1}{2}\right\rfloor-\left\lfloor\frac{i-1}{2}\right\rfloor\right)\\&=2\left\lfloor\frac{m+1}{3}\right\rfloor+m-2\left\lfloor\frac{m+1}{3}\right\rfloor\\&=m\end{aligned}$$

For $i=2\left\lceil\frac{k}{\left\lfloor\frac{m+1}{3}\right\rfloor}\right\rceil-1\,;\ j=1,2,\ldots,m$

$$\begin{aligned}&Wt_{\phi(1,1,0)}\left(f_{i+1}^{j}\right)-Wt_{\phi(1,1,0)}\left(f_{i}^{j}\right)=\\&=\phi\left(v_{i+2}^{j}\right)+\phi\left(v_{i+2}^{j+1}\right)+\phi\left(v_{i+1}^{j}v_{i+2}^{j}\right)+\phi\left(v_{i+2}^{j}v_{i+2}^{j+1}\right)+\phi\left(v_{i+1}^{j+1}v_{i+2}^{j+1}\right)-\phi\left(v_{i}^{j}\right)-\phi\left(v_{i}^{j+1}\right)-\phi\left(v_{i}^{j}v_{i}^{j+1}\right)\\&-\phi\left(v_{i}^{j}v_{i+1}^{j}\right)-\phi\left(v_{i}^{j+1}v_{i+1}^{j+1}\right)\\&=k+k+\left\lceil\frac{j}{2}\right\rceil+\left(\left\lfloor\frac{m+1}{3}\right\rfloor\left\lceil\frac{k}{\left\lfloor\frac{m+1}{3}\right\rfloor}\right\rceil+1-k\right)\left(i-2\left\lceil\frac{k}{\left\lfloor\frac{m+1}{3}\right\rfloor}\right\rceil+2\right)+1+\left(m-2\left\lfloor\frac{m+1}{3}\right\rfloor\right)\left\lfloor\frac{i+1}{2}\right\rfloor\\&+\left\lceil\frac{j+1}{2}\right\rceil+\left(\left\lfloor\frac{m+1}{3}\right\rfloor\left\lceil\frac{k}{\left\lfloor\frac{m+1}{3}\right\rfloor}\right\rceil+1-k\right)\left(i-2\left\lceil\frac{k}{\left\lfloor\frac{m+1}{3}\right\rfloor}\right\rceil+2\right)-1-\left\lfloor\frac{m+1}{3}\right\rfloor\left\lfloor\frac{i-1}{2}\right\rfloor-1\\&-\left\lfloor\frac{m+1}{3}\right\rfloor\left\lfloor\frac{i-1}{2}\right\rfloor-1-\left(m-2\left\lfloor\frac{m+1}{3}\right\rfloor\right)\left\lfloor\frac{i-1}{2}\right\rfloor-\left\lceil\frac{j}{2}\right\rceil-\left\lceil\frac{j+1}{2}\right\rceil\end{aligned}$$

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$$= 2k + 2\left(\left\lfloor\frac{m+1}{3}\right\rfloor\left\lceil\frac{k}{\lfloor\frac{m+1}{3}\rfloor}\right\rceil + 1 - k\right)\left(\left(2\left\lceil\frac{k}{\lfloor\frac{m+1}{3}\rfloor}\right\rceil - 1\right) - 2\left\lceil\frac{k}{\lfloor\frac{m+1}{3}\rfloor}\right\rceil + 2\right)$$
$$+ \left(m - 2\left\lfloor\frac{m+1}{3}\right\rfloor\right)\left(\left\lfloor\frac{i+1}{2}\right\rfloor + \left\lfloor\frac{i-1}{2}\right\rfloor\right) - 2\left\lfloor\frac{m+1}{3}\right\rfloor\left\lfloor\frac{i-1}{2}\right\rfloor - 2$$
$$= 2k + 2\left\lfloor\frac{m+1}{3}\right\rfloor\left\lceil\frac{k}{\lfloor\frac{m+1}{3}\rfloor}\right\rceil + 2 - 2k + m - 2\left\lfloor\frac{m+1}{3}\right\rfloor - 2\left\lfloor\frac{m+1}{3}\right\rfloor\left\lfloor\frac{i-1}{2}\right\rfloor - 2$$
$$= m + 2\left\lfloor\frac{m+1}{3}\right\rfloor\left(\left\lceil\frac{k}{\lfloor\frac{m+1}{3}\rfloor}\right\rceil - 1 - \left\lfloor\left\lceil\frac{k}{\lfloor\frac{m+1}{3}\rfloor}\right\rceil - 1\right\rfloor\right)$$
$$= m$$

For $i = 2\left\lceil\frac{k}{\lfloor\frac{m+1}{3}\rfloor}\right\rceil$; $j = 1, 2, \ldots, m$

$$Wt_{\phi(1,1,0)}\left(f^j_{i+1}\right) - Wt_{\phi(1,1,0)}\left(f^j_i\right) =$$
$$= \phi\left(v^j_{i+2}\right) + \phi\left(v^{j+1}_{i+2}\right) + \phi\left(v^j_{i+1}v^j_{i+2}\right) + \phi\left(v^j_{i+2}v^{j+1}_{i+2}\right) + \phi\left(v^{j+1}_{i+1}v^{j+1}_{i+2}\right) - \phi\left(v^j_i\right) - \phi\left(v^{j+1}_i\right) - \phi\left(v^j_i v^{j+1}_i\right)$$
$$- \phi\left(v^j_i v^j_{i+1}\right) - \phi\left(v^{j+1}_i v^{j+1}_{i+1}\right)$$
$$= k + k + \left\lceil\frac{j}{2}\right\rceil + \left(\left\lfloor\frac{m+1}{3}\right\rfloor\left\lceil\frac{k}{\lfloor\frac{m+1}{3}\rfloor}\right\rceil + 1 - k\right)\left(i - 2\left\lceil\frac{k}{\lfloor\frac{m+1}{3}\rfloor}\right\rceil + 2\right) + 1 + \left(m - 2\left\lfloor\frac{m+1}{3}\right\rfloor\right)\left\lfloor\frac{i+1}{2}\right\rfloor$$
$$+ \left\lceil\frac{j+1}{2}\right\rceil + \left(\left\lfloor\frac{m+1}{3}\right\rfloor\left\lceil\frac{k}{\lfloor\frac{m+1}{3}\rfloor}\right\rceil + 1 - k\right)\left(i - 2\left\lceil\frac{k}{\lfloor\frac{m+1}{3}\rfloor}\right\rceil + 2\right) - 1 - \left\lfloor\frac{m+1}{3}\right\rfloor\left\lfloor\frac{i-1}{2}\right\rfloor - 1$$
$$- \left\lfloor\frac{m+1}{3}\right\rfloor\left\lfloor\frac{i-1}{2}\right\rfloor - 1 - \left(m - 2\left\lfloor\frac{m+1}{3}\right\rfloor\right)\left\lfloor\frac{i-1}{2}\right\rfloor - \left\lceil\frac{j}{2}\right\rceil$$
$$- \left(\left\lfloor\frac{m+1}{3}\right\rfloor\left\lceil\frac{k}{\lfloor\frac{m+1}{3}\rfloor}\right\rceil + 1 - k\right)\left(i - 2\left\lceil\frac{k}{\lfloor\frac{m+1}{3}\rfloor}\right\rceil + 1\right) - \left\lceil\frac{j+1}{2}\right\rceil$$
$$- \left(\left\lfloor\frac{m+1}{3}\right\rfloor\left\lceil\frac{k}{\lfloor\frac{m+1}{3}\rfloor}\right\rceil + 1 - k\right)\left(i - 2\left\lceil\frac{k}{\lfloor\frac{m+1}{3}\rfloor}\right\rceil + 1\right)$$
$$= 2k + 2\left(\left\lfloor\frac{m+1}{3}\right\rfloor\left\lceil\frac{k}{\lfloor\frac{m+1}{3}\rfloor}\right\rceil + 1 - k\right)\left(i - 2\left\lceil\frac{k}{\lfloor\frac{m+1}{3}\rfloor}\right\rceil + 2 - i + 2\left\lceil\frac{k}{\lfloor\frac{m+1}{3}\rfloor}\right\rceil - 1\right)$$
$$+ \left(m - 2\left\lfloor\frac{m+1}{3}\right\rfloor\right)\left(\left\lfloor\frac{i+1}{2}\right\rfloor + \left\lfloor\frac{i-1}{2}\right\rfloor\right) - 2\left\lfloor\frac{m+1}{3}\right\rfloor\left\lfloor\frac{i-1}{2}\right\rfloor - 2$$
$$= 2k + 2\left\lfloor\frac{m+1}{3}\right\rfloor\left\lceil\frac{k}{\lfloor\frac{m+1}{3}\rfloor}\right\rceil + 2 - 2k + m - 2\left\lfloor\frac{m+1}{3}\right\rfloor - 2\left\lfloor\frac{m+1}{3}\right\rfloor\left\lfloor\frac{i-1}{2}\right\rfloor - 2$$
$$= m + 2\left\lfloor\frac{m+1}{3}\right\rfloor\left(\left\lceil\frac{k}{\lfloor\frac{m+1}{3}\rfloor}\right\rceil - 1 - \left\lfloor\left\lceil\frac{k}{\lfloor\frac{m+1}{3}\rfloor}\right\rceil - \frac{1}{2}\right\rfloor\right)$$
$$= m$$

For $i = 2\left\lceil\frac{k}{\lfloor\frac{m+1}{3}\rfloor}\right\rceil + 1$; $j = 1, 2, \ldots, m$

$$Wt_{\phi(1,1,0)}\left(f^j_{i+1}\right) - Wt_{\phi(1,1,0)}\left(f^j_i\right) =$$
$$= \phi\left(v^j_{i+2}\right) + \phi\left(v^{j+1}_{i+2}\right) + \phi\left(v^j_{i+1}v^j_{i+2}\right) + \phi\left(v^j_{i+2}v^{j+1}_{i+2}\right) + \phi\left(v^{j+1}_{i+1}v^{j+1}_{i+2}\right) - \phi\left(v^j_i\right) - \phi\left(v^{j+1}_i\right) - \phi\left(v^j_i v^{j+1}_i\right)$$
$$- \phi\left(v^j_i v^j_{i+1}\right) - \phi\left(v^{j+1}_i v^{j+1}_{i+1}\right)$$
$$= k + k + \left\lceil\frac{j}{2}\right\rceil + 2\left(\left\lfloor\frac{m+1}{3}\right\rfloor\left\lceil\frac{k}{\lfloor\frac{m+1}{3}\rfloor}\right\rceil\right) + 2 - 2k + \left(\left\lfloor\frac{m+1}{3}\right\rfloor\right)\left(i - 2\left\lceil\frac{k}{\lfloor\frac{m+1}{3}\rfloor}\right\rceil\right) + 1$$

$$+\left(m-2\left\lfloor\frac{m+1}{3}\right\rfloor\right)\left\lfloor\frac{i+1}{2}\right\rfloor+\left\lceil\frac{j+1}{2}\right\rceil+2\left(\left\lfloor\frac{m+1}{3}\right\rfloor\left\lceil\frac{k}{\lfloor\frac{m+1}{3}\rfloor}\right\rceil\right)+2-2k$$
$$+\left(\left\lfloor\frac{m+1}{3}\right\rfloor\right)\left(i-2\left\lceil\frac{k}{\lfloor\frac{m+1}{3}\rfloor}\right\rceil\right)-k-k-1-\left(m-2\left\lfloor\frac{m+1}{3}\right\rfloor\right)\left\lfloor\frac{i-1}{2}\right\rfloor$$
$$-\left\lceil\frac{j}{2}\right\rceil-\left(\left\lfloor\frac{m+1}{3}\right\rfloor\left\lceil\frac{k}{\lfloor\frac{m+1}{3}\rfloor}\right\rceil+1-k\right)\left(i-2\left\lceil\frac{k}{\lfloor\frac{m+1}{3}\rfloor}\right\rceil+1\right)$$
$$-\left\lceil\frac{j+1}{2}\right\rceil-\left(\left\lfloor\frac{m+1}{3}\right\rfloor\left\lceil\frac{k}{\lfloor\frac{m+1}{3}\rfloor}\right\rceil+1-k\right)\left(i-2\left\lceil\frac{k}{\lfloor\frac{m+1}{3}\rfloor}\right\rceil+1\right)$$
$$=4\left(\left\lfloor\frac{m+1}{3}\right\rfloor\left\lceil\frac{k}{\lfloor\frac{m+1}{3}\rfloor}\right\rceil\right)+4+2\left(\left\lfloor\frac{m+1}{3}\right\rfloor\right)\left(i-2\left\lceil\frac{k}{\lfloor\frac{m+1}{3}\rfloor}\right\rceil\right)+\left(m-2\left\lfloor\frac{m+1}{3}\right\rfloor\right)\left(\left\lfloor\frac{i+1}{2}\right\rfloor+\left\lfloor\frac{i-1}{2}\right\rfloor\right)$$
$$-2k-2k-2\left(\left\lfloor\frac{m+1}{3}\right\rfloor\left\lceil\frac{k}{\lfloor\frac{m+1}{3}\rfloor}\right\rceil+1-k\right)\left(i-2\left\lceil\frac{k}{\lfloor\frac{m+1}{3}\rfloor}\right\rceil+1\right)$$
$$=4\left(\left\lfloor\frac{m+1}{3}\right\rfloor\left\lceil\frac{k}{\lfloor\frac{m+1}{3}\rfloor}\right\rceil\right)+4+2\left(\left\lfloor\frac{m+1}{3}\right\rfloor\right)\left(2\left\lceil\frac{k}{\lfloor\frac{m+1}{3}\rfloor}\right\rceil+1-2\left\lceil\frac{k}{\lfloor\frac{m+1}{3}\rfloor}\right\rceil\right)+\left(m-2\left\lfloor\frac{m+1}{3}\right\rfloor\right)$$
$$-4k-2\left(\left\lfloor\frac{m+1}{3}\right\rfloor\left\lceil\frac{k}{\lfloor\frac{m+1}{3}\rfloor}\right\rceil+1-k\right)\left(2\left\lceil\frac{k}{\lfloor\frac{m+1}{3}\rfloor}\right\rceil+1-2\left\lceil\frac{k}{\lfloor\frac{m+1}{3}\rfloor}\right\rceil+1\right)$$
$$=4\left(\left\lfloor\frac{m+1}{3}\right\rfloor\left\lceil\frac{k}{\lfloor\frac{m+1}{3}\rfloor}\right\rceil\right)+4+2\left(\left\lfloor\frac{m+1}{3}\right\rfloor\right)+\left(m-2\left\lfloor\frac{m+1}{3}\right\rfloor\right)-4k-4\left(\left\lfloor\frac{m+1}{3}\right\rfloor\left\lceil\frac{k}{\lfloor\frac{m+1}{3}\rfloor}\right\rceil+1-k\right)$$
$$=4\left\lfloor\frac{m+1}{3}\right\rfloor\left\lceil\frac{k}{\lfloor\frac{m+1}{3}\rfloor}\right\rceil+4+2\left\lfloor\frac{m+1}{3}\right\rfloor+m-2\left\lfloor\frac{m+1}{3}\right\rfloor-4k-4\left\lfloor\frac{m+1}{3}\right\rfloor\left\lceil\frac{k}{\lfloor\frac{m+1}{3}\rfloor}\right\rceil-4+4k$$
$$=m$$

For $i=2\left\lceil\frac{k}{\lfloor\frac{m+1}{3}\rfloor}\right\rceil+2,\cdots,2\left\lceil\frac{k}{m-2\lfloor\frac{m+1}{3}\rfloor}\right\rceil-2$ and $j=1,2,\ldots,m$

$$Wt_{\phi(1,1,0)}\left(f_{i+1}^j\right)-Wt_{\phi(1,1,0)}\left(f_i^j\right)=$$
$$=\phi\left(v_{i+2}^j\right)+\phi\left(v_{i+2}^{j+1}\right)+\phi\left(v_{i+1}^jv_{i+2}^j\right)+\phi\left(v_{i+2}^jv_{i+2}^{j+1}\right)+\phi\left(v_{i+1}^{j+1}v_{i+2}^{j+1}\right)-\phi\left(v_i^j\right)-\phi\left(v_i^{j+1}\right)-\phi\left(v_i^jv_i^{j+1}\right)$$
$$-\phi\left(v_i^jv_{i+1}^j\right)-\phi\left(v_i^{j+1}v_{i+1}^{j+1}\right)$$
$$=\left\lceil\frac{j}{2}\right\rceil+\left\lceil\frac{j+1}{2}\right\rceil+4\left(\left\lfloor\frac{m+1}{3}\right\rfloor\left\lceil\frac{k}{\lfloor\frac{m+1}{3}\rfloor}\right\rceil\right)+4-4k+2\left(\left\lfloor\frac{m+1}{3}\right\rfloor\right)\left(i-2\left\lceil\frac{k}{\lfloor\frac{m+1}{3}\rfloor}\right\rceil\right)+1$$
$$+\left(m-2\left\lfloor\frac{m+1}{3}\right\rfloor\right)\left\lfloor\frac{i+1}{2}\right\rfloor-1-\left(m-2\left\lfloor\frac{m+1}{3}\right\rfloor\right)\left\lfloor\frac{i-1}{2}\right\rfloor-\left\lceil\frac{j}{2}\right\rceil-\left\lceil\frac{j+1}{2}\right\rceil$$
$$-4\left(\left\lfloor\frac{m+1}{3}\right\rfloor\left\lceil\frac{k}{\lfloor\frac{m+1}{3}\rfloor}\right\rceil\right)-4+4k-2\left(\left\lfloor\frac{m+1}{3}\right\rfloor\right)\left(i-2\left\lceil\frac{k}{\lfloor\frac{m+1}{3}\rfloor}\right\rceil-1\right)$$
$$=2\left(\left\lfloor\frac{m+1}{3}\right\rfloor\right)\left(i-2\left\lceil\frac{k}{\lfloor\frac{m+1}{3}\rfloor}\right\rceil-i+2\left\lceil\frac{k}{\lfloor\frac{m+1}{3}\rfloor}\right\rceil+1\right)+\left(m-2\left\lfloor\frac{m+1}{3}\right\rfloor\right)\left(\left\lfloor\frac{i+1}{2}\right\rfloor-\left\lfloor\frac{i-1}{2}\right\rfloor\right)$$
$$=2\left\lfloor\frac{m+1}{3}\right\rfloor+m-2\left\lfloor\frac{m+1}{3}\right\rfloor$$
$$=m$$

For $i=2\left\lceil\frac{k}{m-2\lfloor\frac{m+1}{3}\rfloor}\right\rceil-1$ and $j=1,2,\ldots,m$

$$Wt_{\phi(1,1,0)}\left(f_{i+1}^j\right)-Wt_{\phi(1,1,0)}\left(f_i^j\right)=$$

$$= \phi\left(v_{i+2}^{j}\right) + \phi\left(v_{i+2}^{j+1}\right) + \phi\left(v_{i+1}^{j}v_{i+2}^{j}\right) + \phi\left(v_{i+2}^{j}v_{i+2}^{j+1}\right) + \phi\left(v_{i+1}^{j+1}v_{i+2}^{j+1}\right) - \phi\left(v_{i}^{j}\right) - \phi\left(v_{i}^{j+1}\right) - \phi\left(v_{i}^{j}v_{i}^{j+1}\right)$$
$$- \phi\left(v_{i}^{j}v_{i+1}^{j}\right) - \phi\left(v_{i}^{j+1}v_{i+1}^{j+1}\right)$$
$$= \left\lceil \frac{j}{2} \right\rceil + \left\lceil \frac{j+1}{2} \right\rceil + 4\left(\left\lfloor \frac{m+1}{3} \right\rfloor\right)\left(\left\lceil \frac{k}{m-2\left\lfloor \frac{m+1}{3} \right\rfloor} \right\rceil - 1\right) + 4 - 4k + \left\lceil \frac{1}{2}\left(i+1-2\left\lceil \frac{k}{m-2\left\lfloor \frac{m+1}{3} \right\rfloor} \right\rceil + 1\right) \right\rceil$$
$$\left(\left\lfloor \frac{\left(\left\lceil \frac{k}{m-2\left\lfloor \frac{m+1}{3} \right\rfloor} \right\rceil\left(m-2\left\lfloor \frac{m+1}{3} \right\rfloor\right) + 2\left\lfloor \frac{m+1}{3} \right\rfloor + 1 - k\right)}{2} \right\rfloor + \left\lceil \frac{\left(\left\lceil \frac{k}{m-2\left\lfloor \frac{m+1}{3} \right\rfloor} \right\rceil\left(m-2\left\lfloor \frac{m+1}{3} \right\rfloor\right) + 2\left\lfloor \frac{m+1}{3} \right\rfloor + 1 - k\right)}{2} \right\rceil\right)$$
$$+ \left\lfloor \frac{1}{2}\left(i+1-2\left\lceil \frac{k}{m-2\left\lfloor \frac{m+1}{3} \right\rfloor} \right\rceil + 1\right) \right\rfloor$$
$$\left(\left\lceil \frac{\left(\left\lceil \frac{k}{m-2\left\lfloor \frac{m+1}{3} \right\rfloor} \right\rceil\left(m-2\left\lfloor \frac{m+1}{3} \right\rfloor\right) + 2\left\lfloor \frac{m+1}{3} \right\rfloor + 1 - k\right)}{2} \right\rceil + \left\lfloor \frac{\left(\left\lceil \frac{k}{m-2\left\lfloor \frac{m+1}{3} \right\rfloor} \right\rceil\left(m-2\left\lfloor \frac{m+1}{3} \right\rfloor\right) + 2\left\lfloor \frac{m+1}{3} \right\rfloor + 1 - k\right)}{2} \right\rfloor\right)$$
$$+ k - 1 - \left(m - 2\left\lfloor \frac{m+1}{3} \right\rfloor\right)\left\lfloor \frac{i-1}{2} \right\rfloor - \left\lceil \frac{j}{2} \right\rceil - \left\lceil \frac{j+1}{2} \right\rceil - 4\left(\left\lfloor \frac{m+1}{3} \right\rfloor\left\lceil \frac{k}{\left\lfloor \frac{m+1}{3} \right\rfloor} \right\rceil\right) - 4 + 4k$$
$$- 2\left(\left\lfloor \frac{m+1}{3} \right\rfloor\right)\left(i - 2\left\lceil \frac{k}{\left\lfloor \frac{m+1}{3} \right\rfloor} \right\rceil - 1\right)$$
$$= 4\left(\left\lfloor \frac{m+1}{3} \right\rfloor\right)\left(\left\lceil \frac{k}{m-2\left\lfloor \frac{m+1}{3} \right\rfloor} \right\rceil - 1\right) + (1)\left(\left\lceil \frac{k}{m-2\left\lfloor \frac{m+1}{3} \right\rfloor} \right\rceil\left(m - 2\left\lfloor \frac{m+1}{3} \right\rfloor\right) + 2\left\lfloor \frac{m+1}{3} \right\rfloor + 1 - k\right) + 0$$
$$+ k - 1 - \left(m - 2\left\lfloor \frac{m+1}{3} \right\rfloor\right)\left\lfloor \frac{i-1}{2} \right\rfloor - 4\left(\left\lfloor \frac{m+1}{3} \right\rfloor\left\lceil \frac{k}{\left\lfloor \frac{m+1}{3} \right\rfloor} \right\rceil\right) - 2\left(\left\lfloor \frac{m+1}{3} \right\rfloor\right)\left(i - 2\left\lceil \frac{k}{\left\lfloor \frac{m+1}{3} \right\rfloor} \right\rceil - 1\right)$$
$$= 4\left(\left\lfloor \frac{m+1}{3} \right\rfloor\right)\left(\left\lceil \frac{k}{m-2\left\lfloor \frac{m+1}{3} \right\rfloor} \right\rceil - 1\right) + \left\lceil \frac{k}{m-2\left\lfloor \frac{m+1}{3} \right\rfloor} \right\rceil\left(m - 2\left\lfloor \frac{m+1}{3} \right\rfloor\right) + 2\left\lfloor \frac{m+1}{3} \right\rfloor + 1 - k + k$$
$$- 1 - \left(m - 2\left\lfloor \frac{m+1}{3} \right\rfloor\right)\left\lfloor \frac{i-1}{2} \right\rfloor - 4\left(\left\lfloor \frac{m+1}{3} \right\rfloor\left\lceil \frac{k}{\left\lfloor \frac{m+1}{3} \right\rfloor} \right\rceil\right) - 2\left(\left\lfloor \frac{m+1}{3} \right\rfloor\right)\left(i - 2\left\lceil \frac{k}{\left\lfloor \frac{m+1}{3} \right\rfloor} \right\rceil - 1\right)$$
$$= 2\left(\left\lfloor \frac{m+1}{3} \right\rfloor\right)\left(2\left\lceil \frac{k}{m-2\left\lfloor \frac{m+1}{3} \right\rfloor} \right\rceil - 2 + 1 - 2\left\lceil \frac{k}{\left\lfloor \frac{m+1}{3} \right\rfloor} \right\rceil - i + 2\left\lceil \frac{k}{\left\lfloor \frac{m+1}{3} \right\rfloor} \right\rceil + 1\right)$$
$$+ \left(m - 2\left\lfloor \frac{m+1}{3} \right\rfloor\right)\left(\left\lceil \frac{k}{m-2\left\lfloor \frac{m+1}{3} \right\rfloor} \right\rceil - \left\lfloor \frac{i-1}{2} \right\rfloor\right)$$
$$= m$$

For $i = 2\left\lceil \frac{k}{m-2\left\lfloor \frac{m+1}{3} \right\rfloor} \right\rceil$ and $j = 1, 2, \ldots, m$

$$Wt_{\phi(1,1,0)}\left(f_{i+1}^{j}\right) - Wt_{\phi(1,1,0)}\left(f_{i}^{j}\right) =$$
$$= \phi\left(v_{i+2}^{j}\right) + \phi\left(v_{i+2}^{j+1}\right) + \phi\left(v_{i+1}^{j}v_{i+2}^{j}\right) + \phi\left(v_{i+2}^{j}v_{i+2}^{j+1}\right) + \phi\left(v_{i+1}^{j+1}v_{i+2}^{j+1}\right) - \phi\left(v_{i}^{j}\right) - \phi\left(v_{i}^{j+1}\right)$$
$$- \phi\left(v_{i}^{j}v_{i}^{j+1}\right) - \phi\left(v_{i}^{j}v_{i+1}^{j}\right) - \phi\left(v_{i}^{j+1}v_{i+1}^{j+1}\right)$$
$$= \left\lceil \frac{j}{2} \right\rceil + \left\lceil \frac{j+1}{2} \right\rceil + 4\left(\left\lfloor \frac{m+1}{3} \right\rfloor\right)\left(\left\lceil \frac{k}{m-2\left\lfloor \frac{m+1}{3} \right\rfloor} \right\rceil - 1\right) + 4 - 4k + \left\lceil \frac{1}{2}\left(i+1-2\left\lceil \frac{k}{m-2\left\lfloor \frac{m+1}{3} \right\rfloor} \right\rceil + 1\right) \right\rceil$$
$$\left(\left\lfloor \frac{\left(\left\lceil \frac{k}{m-2\left\lfloor \frac{m+1}{3} \right\rfloor} \right\rceil\left(m-2\left\lfloor \frac{m+1}{3} \right\rfloor\right) + 2\left\lfloor \frac{m+1}{3} \right\rfloor + 1 - k\right)}{2} \right\rfloor + \left\lceil \frac{\left(\left\lceil \frac{k}{m-2\left\lfloor \frac{m+1}{3} \right\rfloor} \right\rceil\left(m-2\left\lfloor \frac{m+1}{3} \right\rfloor\right) + 2\left\lfloor \frac{m+1}{3} \right\rfloor + 1 - k\right)}{2} \right\rceil\right)$$

$$+\left\lfloor\frac{1}{2}\left(i+1-2\left\lceil\frac{k}{m-2\left\lfloor\frac{m+1}{3}\right\rfloor}\right\rceil+1\right)\right\rfloor$$
$$\left(\left\lceil\frac{\left(\left\lceil\frac{k}{m-2\left\lfloor\frac{m+1}{3}\right\rfloor}\right\rceil\left(m-2\left\lfloor\frac{m+1}{3}\right\rfloor\right)+2\left\lfloor\frac{m+1}{3}\right\rfloor+1-k\right)}{2}\right\rceil+\left\lfloor\frac{\left(\left\lceil\frac{k}{m-2\left\lfloor\frac{m+1}{3}\right\rfloor}\right\rceil\left(m-2\left\lfloor\frac{m+1}{3}\right\rfloor\right)+2\left\lfloor\frac{m+1}{3}\right\rfloor+1-k\right)}{2}\right\rfloor\right)$$
$$+k-1-\left(m-2\left\lfloor\frac{m+1}{3}\right\rfloor\right)\left\lfloor\frac{i-1}{2}\right\rfloor-\left\lceil\frac{j}{2}\right\rceil-\left\lceil\frac{j+1}{2}\right\rceil-4\left(\left\lfloor\frac{m+1}{3}\right\rfloor\right)\left(\left\lceil\frac{k}{m-2\left\lfloor\frac{m+1}{3}\right\rfloor}\right\rceil-1\right)-4+4k$$
$$-\left\lfloor\frac{1}{2}\left(i-2\left\lceil\frac{k}{m-2\left\lfloor\frac{m+1}{3}\right\rfloor}\right\rceil+1\right)\right\rfloor$$
$$\left(\left\lfloor\frac{\left(\left\lceil\frac{k}{m-2\left\lfloor\frac{m+1}{3}\right\rfloor}\right\rceil\left(m-2\left\lfloor\frac{m+1}{3}\right\rfloor\right)+2\left\lfloor\frac{m+1}{3}\right\rfloor+1-k\right)}{2}\right\rfloor+\left\lceil\frac{\left(\left\lceil\frac{k}{m-2\left\lfloor\frac{m+1}{3}\right\rfloor}\right\rceil\left(m-2\left\lfloor\frac{m+1}{3}\right\rfloor\right)+2\left\lfloor\frac{m+1}{3}\right\rfloor+1-k\right)}{2}\right\rceil\right)$$
$$-\left\lfloor\frac{1}{2}\left(i-2\left\lceil\frac{k}{m-2\left\lfloor\frac{m+1}{3}\right\rfloor}\right\rceil+1\right)\right\rfloor$$
$$\left(\left\lceil\frac{\left(\left\lceil\frac{k}{m-2\left\lfloor\frac{m+1}{3}\right\rfloor}\right\rceil\left(m-2\left\lfloor\frac{m+1}{3}\right\rfloor\right)+2\left\lfloor\frac{m+1}{3}\right\rfloor+1-k\right)}{2}\right\rceil+\left\lfloor\frac{\left(\left\lceil\frac{k}{m-2\left\lfloor\frac{m+1}{3}\right\rfloor}\right\rceil\left(m-2\left\lfloor\frac{m+1}{3}\right\rfloor\right)+2\left\lfloor\frac{m+1}{3}\right\rfloor+1-k\right)}{2}\right\rfloor\right)$$
$$=4\left(\left\lfloor\frac{m+1}{3}\right\rfloor\right)\left(\left\lceil\frac{k}{m-2\left\lfloor\frac{m+1}{3}\right\rfloor}\right\rceil-1\right)+(1)\left(\left\lceil\frac{k}{m-2\left\lfloor\frac{m+1}{3}\right\rfloor}\right\rceil\left(m-2\left\lfloor\frac{m+1}{3}\right\rfloor\right)+2\left\lfloor\frac{m+1}{3}\right\rfloor+1-k\right)$$
$$+(1)\left(\left\lceil\frac{k}{m-2\left\lfloor\frac{m+1}{3}\right\rfloor}\right\rceil\left(m-2\left\lfloor\frac{m+1}{3}\right\rfloor\right)+2\left\lfloor\frac{m+1}{3}\right\rfloor+1-k\right)+k-1-\left(m-2\left\lfloor\frac{m+1}{3}\right\rfloor\right)\left\lfloor\frac{i-1}{2}\right\rfloor$$
$$-4\left(\left\lfloor\frac{m+1}{3}\right\rfloor\right)\left(\left\lceil\frac{k}{m-2\left\lfloor\frac{m+1}{3}\right\rfloor}\right\rceil-1\right)-(1)\left(\left\lceil\frac{k}{m-2\left\lfloor\frac{m+1}{3}\right\rfloor}\right\rceil\left(m-2\left\lfloor\frac{m+1}{3}\right\rfloor\right)+2\left\lfloor\frac{m+1}{3}\right\rfloor+1-k\right)-0$$
$$=4\left(\left\lfloor\frac{m+1}{3}\right\rfloor\right)\left(\left\lceil\frac{k}{m-2\left\lfloor\frac{m+1}{3}\right\rfloor}\right\rceil-1\right)+\left\lceil\frac{k}{m-2\left\lfloor\frac{m+1}{3}\right\rfloor}\right\rceil\left(m-2\left\lfloor\frac{m+1}{3}\right\rfloor\right)+2\left\lfloor\frac{m+1}{3}\right\rfloor+1-k$$
$$+\left\lceil\frac{k}{m-2\left\lfloor\frac{m+1}{3}\right\rfloor}\right\rceil\left(m-2\left\lfloor\frac{m+1}{3}\right\rfloor\right)+2\left\lfloor\frac{m+1}{3}\right\rfloor+1-k+k-1-\left(m-2\left\lfloor\frac{m+1}{3}\right\rfloor\right)\left\lfloor\frac{i-1}{2}\right\rfloor$$
$$-4\left(\left\lfloor\frac{m+1}{3}\right\rfloor\right)\left(\left\lceil\frac{k}{m-2\left\lfloor\frac{m+1}{3}\right\rfloor}\right\rceil-1\right)-\left\lceil\frac{k}{m-2\left\lfloor\frac{m+1}{3}\right\rfloor}\right\rceil\left(m-2\left\lfloor\frac{m+1}{3}\right\rfloor\right)-2\left\lfloor\frac{m+1}{3}\right\rfloor-1+k$$
$$=2\left(\left\lfloor\frac{m+1}{3}\right\rfloor\right)\left(2\left\lceil\frac{k}{m-2\left\lfloor\frac{m+1}{3}\right\rfloor}\right\rceil-2+1+1-2\left\lceil\frac{k}{m-2\left\lfloor\frac{m+1}{3}\right\rfloor}\right\rceil+2-1\right)$$
$$+\left(m-2\left\lfloor\frac{m+1}{3}\right\rfloor\right)\left(\left\lceil\frac{k}{m-2\left\lfloor\frac{m+1}{3}\right\rfloor}\right\rceil-\left\lfloor\frac{i-1}{2}\right\rfloor\right)$$
$$=2\left(\left\lfloor\frac{m+1}{3}\right\rfloor\right)\left(2\left\lceil\frac{k}{m-2\left\lfloor\frac{m+1}{3}\right\rfloor}\right\rceil-2+1+1-2\left\lceil\frac{k}{m-2\left\lfloor\frac{m+1}{3}\right\rfloor}\right\rceil+2-1\right)$$
$$+\left(m-2\left\lfloor\frac{m+1}{3}\right\rfloor\right)\left(\left\lceil\frac{k}{m-2\left\lfloor\frac{m+1}{3}\right\rfloor}\right\rceil-\left\lfloor\frac{i-1}{2}\right\rfloor\right)$$
$$=m$$

For $i=2\left\lceil\frac{k}{m-2\left\lfloor\frac{m+1}{3}\right\rfloor}\right\rceil+1$ and $j=1,2,\ldots,m$

$Wt_{\phi(1,1,0)}\left(f_{i+1}^{j}\right)-Wt_{\phi(1,1,0)}\left(f_{i}^{j}\right)=$

$$= \phi\left(v_{i+2}^{j}\right) + \phi\left(v_{i+2}^{j+1}\right) + \phi\left(v_{i+1}^{j}v_{i+2}^{j}\right) + \phi\left(v_{i+2}^{j}v_{i+2}^{j+1}\right) + \phi\left(v_{i+1}^{j+1}v_{i+2}^{j+1}\right) - \phi\left(v_{i}^{j}\right) - \phi\left(v_{i}^{j+1}\right) - \phi\left(v_{i}^{j}v_{i}^{j+1}\right)$$
$$- \phi\left(v_{i}^{j}v_{i+1}^{j}\right) - \phi\left(v_{i}^{j+1}v_{i+1}^{j+1}\right)$$
$$= \left\lceil\frac{j}{2}\right\rceil + \left\lceil\frac{j+1}{2}\right\rceil + 6 - 6k + 2m\left\lceil\frac{k}{m-2\left\lfloor\frac{m+1}{3}\right\rfloor}\right\rceil + \left\lceil\frac{1}{2}\left(i+1-2\left\lceil\frac{k}{m-2\left\lfloor\frac{m+1}{3}\right\rfloor}\right\rceil - 1\right)\right\rceil\left(\left\lfloor\frac{m}{2}\right\rfloor + \left\lceil\frac{m}{2}\right\rceil\right)$$
$$+ \left\lfloor\frac{1}{2}\left(i+1-2\left\lceil\frac{k}{m-2\left\lfloor\frac{m+1}{3}\right\rfloor}\right\rceil - 1\right)\right\rfloor\left(\left\lceil\frac{m}{2}\right\rceil + \left\lfloor\frac{m}{2}\right\rfloor\right) + k - k - \left\lceil\frac{j}{2}\right\rceil - \left\lceil\frac{j+1}{2}\right\rceil$$
$$- 4\left(\left\lfloor\frac{m+1}{3}\right\rfloor\right)\left(\left\lceil\frac{k}{m-2\left\lfloor\frac{m+1}{3}\right\rfloor}\right\rceil - 1\right) - 4 + 4k - \left\lceil\frac{1}{2}\left(i-2\left\lceil\frac{k}{m-2\left\lfloor\frac{m+1}{3}\right\rfloor}\right\rceil + 1\right)\right\rceil$$
$$\left(\left\lfloor\frac{\left(\left\lceil\frac{k}{m-2\left\lfloor\frac{m+1}{3}\right\rfloor}\right\rceil\left(m-2\left\lfloor\frac{m+1}{3}\right\rfloor\right) + 2\left\lfloor\frac{m+1}{3}\right\rfloor + 1 - k\right)}{2}\right\rfloor + \left\lceil\frac{\left(\left\lceil\frac{k}{m-2\left\lfloor\frac{m+1}{3}\right\rfloor}\right\rceil\left(m-2\left\lfloor\frac{m+1}{3}\right\rfloor\right) + 2\left\lfloor\frac{m+1}{3}\right\rfloor + 1 - k\right)}{2}\right\rceil\right)$$
$$- \left\lfloor\frac{1}{2}\left(i-2\left\lceil\frac{k}{m-2\left\lfloor\frac{m+1}{3}\right\rfloor}\right\rceil + 1\right)\right\rfloor$$
$$\left(\left\lceil\frac{\left(\left\lceil\frac{k}{m-2\left\lfloor\frac{m+1}{3}\right\rfloor}\right\rceil\left(m-2\left\lfloor\frac{m+1}{3}\right\rfloor\right) + 2\left\lfloor\frac{m+1}{3}\right\rfloor + 1 - k\right)}{2}\right\rceil + \left\lfloor\frac{\left(\left\lceil\frac{k}{m-2\left\lfloor\frac{m+1}{3}\right\rfloor}\right\rceil\left(m-2\left\lfloor\frac{m+1}{3}\right\rfloor\right) + 2\left\lfloor\frac{m+1}{3}\right\rfloor + 1 - k\right)}{2}\right\rfloor\right)$$
$$= 6 - 6k + 2m\left\lceil\frac{k}{m-2\left\lfloor\frac{m+1}{3}\right\rfloor}\right\rceil + (1)(m) + 0 - 4\left(\left\lfloor\frac{m+1}{3}\right\rfloor\right)\left(\left\lceil\frac{k}{m-2\left\lfloor\frac{m+1}{3}\right\rfloor}\right\rceil - 1\right) - 4 + 4k$$
$$- (1)\left(\left\lceil\frac{k}{m-2\left\lfloor\frac{m+1}{3}\right\rfloor}\right\rceil\left(m-2\left\lfloor\frac{m+1}{3}\right\rfloor\right) + 2\left\lfloor\frac{m+1}{3}\right\rfloor + 1 - k\right)$$
$$- (1)\left(\left\lceil\frac{k}{m-2\left\lfloor\frac{m+1}{3}\right\rfloor}\right\rceil\left(m-2\left\lfloor\frac{m+1}{3}\right\rfloor\right) + 2\left\lfloor\frac{m+1}{3}\right\rfloor + 1 - k\right)$$
$$= 6 - 6k + 2m\left\lceil\frac{k}{m-2\left\lfloor\frac{m+1}{3}\right\rfloor}\right\rceil + m - 4\left(\left\lfloor\frac{m+1}{3}\right\rfloor\right)\left(\left\lceil\frac{k}{m-2\left\lfloor\frac{m+1}{3}\right\rfloor}\right\rceil - 1\right) - 4 + 4k$$
$$- \left\lceil\frac{k}{m-2\left\lfloor\frac{m+1}{3}\right\rfloor}\right\rceil\left(m-2\left\lfloor\frac{m+1}{3}\right\rfloor\right) - 2\left\lfloor\frac{m+1}{3}\right\rfloor - 1 + k$$
$$- \left\lceil\frac{k}{m-2\left\lfloor\frac{m+1}{3}\right\rfloor}\right\rceil\left(m-2\left\lfloor\frac{m+1}{3}\right\rfloor\right) - 2\left\lfloor\frac{m+1}{3}\right\rfloor - 1 + k$$
$$= 2 - 2k + 2m\left\lceil\frac{k}{m-2\left\lfloor\frac{m+1}{3}\right\rfloor}\right\rceil + m - 4\left\lfloor\frac{m+1}{3}\right\rfloor\left\lceil\frac{k}{m-2\left\lfloor\frac{m+1}{3}\right\rfloor}\right\rceil + 4\left\lfloor\frac{m+1}{3}\right\rfloor$$
$$- 2\left\lceil\frac{k}{m-2\left\lfloor\frac{m+1}{3}\right\rfloor}\right\rceil\left(m-2\left\lfloor\frac{m+1}{3}\right\rfloor\right) - 4\left\lfloor\frac{m+1}{3}\right\rfloor - 2 + 2k$$
$$= m + \left\lceil\frac{k}{m-2\left\lfloor\frac{m+1}{3}\right\rfloor}\right\rceil\left(2m - 4\left\lfloor\frac{m+1}{3}\right\rfloor - 2m + 4\left\lfloor\frac{m+1}{3}\right\rfloor\right) + 4\left\lfloor\frac{m+1}{3}\right\rfloor - 4\left\lfloor\frac{m+1}{3}\right\rfloor$$
$$= m$$

For $i = 2\left\lceil\frac{k}{m-2\left\lfloor\frac{m+1}{3}\right\rfloor}\right\rceil + 2, \cdots, n-1$ and $j = 1, 2, \ldots, m$

$$Wt_{\phi(1,1,0)}\left(f_{i+1}^{j}\right) - Wt_{\phi(1,1,0)}\left(f_{i}^{j}\right) =$$
$$= \phi\left(v_{i+2}^{j}\right) + \phi\left(v_{i+2}^{j+1}\right) + \phi\left(v_{i+1}^{j}v_{i+2}^{j}\right) + \phi\left(v_{i+2}^{j}v_{i+2}^{j+1}\right) + \phi\left(v_{i+1}^{j+1}v_{i+2}^{j+1}\right) - \phi\left(v_{i}^{j}\right) - \phi\left(v_{i}^{j+1}\right) - \phi\left(v_{i}^{j}v_{i}^{j+1}\right)$$
$$- \phi\left(v_{i}^{j}v_{i+1}^{j}\right) - \phi\left(v_{i}^{j+1}v_{i+1}^{j+1}\right)$$

$$
\begin{aligned}
&= \left\lceil \frac{j}{2} \right\rceil + \left\lceil \frac{j+1}{2} \right\rceil + 6 - 6k + 2m \left\lceil \frac{k}{m-2\left\lfloor \frac{m+1}{3} \right\rfloor} \right\rceil + \left\lceil \frac{1}{2}\left(i+1-2\left\lceil \frac{k}{m-2\left\lfloor \frac{m+1}{3} \right\rfloor} \right\rceil - 1 \right) \right\rceil \left(\left\lfloor \frac{m}{2} \right\rfloor + \left\lceil \frac{m}{2} \right\rceil \right) \\
&+ \left\lfloor \frac{1}{2}\left(i+1-2\left\lceil \frac{k}{m-2\left\lfloor \frac{m+1}{3} \right\rfloor} \right\rceil - 1 \right) \right\rfloor \left(\left\lceil \frac{m}{2} \right\rceil + \left\lfloor \frac{m}{2} \right\rfloor \right) + k - k - \left\lceil \frac{j}{2} \right\rceil - \left\lceil \frac{j+1}{2} \right\rceil - 6 + 6k \\
&- 2m \left\lceil \frac{k}{m-2\left\lfloor \frac{m+1}{3} \right\rfloor} \right\rceil - \left\lceil \frac{1}{2}\left(i-2\left\lceil \frac{k}{m-2\left\lfloor \frac{m+1}{3} \right\rfloor} \right\rceil - 1 \right) \right\rceil \left(\left\lfloor \frac{m}{2} \right\rfloor + \left\lceil \frac{m}{2} \right\rceil \right) \\
&- \left\lfloor \frac{1}{2}\left(i-2\left\lceil \frac{k}{m-2\left\lfloor \frac{m+1}{3} \right\rfloor} \right\rceil - 1 \right) \right\rfloor \left(\left\lceil \frac{m}{2} \right\rceil + \left\lfloor \frac{m}{2} \right\rfloor \right) \\
&= m \left\lceil \frac{1}{2}\left(i+1-2\left\lceil \frac{k}{m-2\left\lfloor \frac{m+1}{3} \right\rfloor} \right\rceil - 1 \right) \right\rceil + m \left\lfloor \frac{1}{2}\left(i+1-2\left\lceil \frac{k}{m-2\left\lfloor \frac{m+1}{3} \right\rfloor} \right\rceil - 1 \right) \right\rfloor \\
&- m \left\lceil \frac{1}{2}\left(i-2\left\lceil \frac{k}{m-2\left\lfloor \frac{m+1}{3} \right\rfloor} \right\rceil - 1 \right) \right\rceil - m \left\lfloor \frac{1}{2}\left(i-2\left\lceil \frac{k}{m-2\left\lfloor \frac{m+1}{3} \right\rfloor} \right\rceil - 1 \right) \right\rfloor \\
&= m \left\lceil \frac{1}{2}\left(i-2\left\lceil \frac{k}{m-2\left\lfloor \frac{m+1}{3} \right\rfloor} \right\rceil \right) \right\rceil + m \left\lfloor \frac{1}{2}\left(i-2\left\lceil \frac{k}{m-2\left\lfloor \frac{m+1}{3} \right\rfloor} \right\rceil \right) \right\rfloor \\
&- m \left\lceil \frac{1}{2}\left(i-2\left\lceil \frac{k}{m-2\left\lfloor \frac{m+1}{3} \right\rfloor} \right\rceil - 1 \right) \right\rceil - m \left\lfloor \frac{1}{2}\left(i-2\left\lceil \frac{k}{m-2\left\lfloor \frac{m+1}{3} \right\rfloor} \right\rceil - 1 \right) \right\rfloor \\
&= m \left\{ \left\lceil \frac{\left(i-2\left\lceil \frac{k}{m-2\left\lfloor \frac{m+1}{3} \right\rfloor} \right\rceil \right)}{2} \right\rceil + \left\lfloor \frac{\left(i-2\left\lceil \frac{k}{m-2\left\lfloor \frac{m+1}{3} \right\rfloor} \right\rceil \right)}{2} \right\rfloor \right\} \\
&- m \left\{ \left\lceil \frac{\left(i-2\left\lceil \frac{k}{m-2\left\lfloor \frac{m+1}{3} \right\rfloor} \right\rceil - 1 \right)}{2} \right\rceil + \left\lfloor \frac{\left(i-2\left\lceil \frac{k}{m-2\left\lfloor \frac{m+1}{3} \right\rfloor} \right\rceil - 1 \right)}{2} \right\rfloor \right\} \\
&= m \left(i-2\left\lceil \frac{k}{m-2\left\lfloor \frac{m+1}{3} \right\rfloor} \right\rceil \right) - m \left(i-2\left\lceil \frac{k}{m-2\left\lfloor \frac{m+1}{3} \right\rfloor} \right\rceil - 1 \right) \\
&= m \left(i-2\left\lceil \frac{k}{m-2\left\lfloor \frac{m+1}{3} \right\rfloor} \right\rceil - i + 2\left\lceil \frac{k}{m-2\left\lfloor \frac{m+1}{3} \right\rfloor} \right\rceil + 1 \right) \\
&= m
\end{aligned}
$$

□

Theorem 4. *Let $G_n^m = P_{n+1} \square P_{m+1}$ is a generalized grid graph for positive integers n, m with $n \geq 2$ rows, $m \geq 2$ columns, and $\left\lfloor \frac{m+1}{3} \right\rfloor > m - 2\left\lfloor \frac{m+1}{3} \right\rfloor$ then*

$$
tfs_{(1,1,0)}\left(G_n^m\right) = \left\lceil \frac{mn+7}{8} \right\rceil
$$

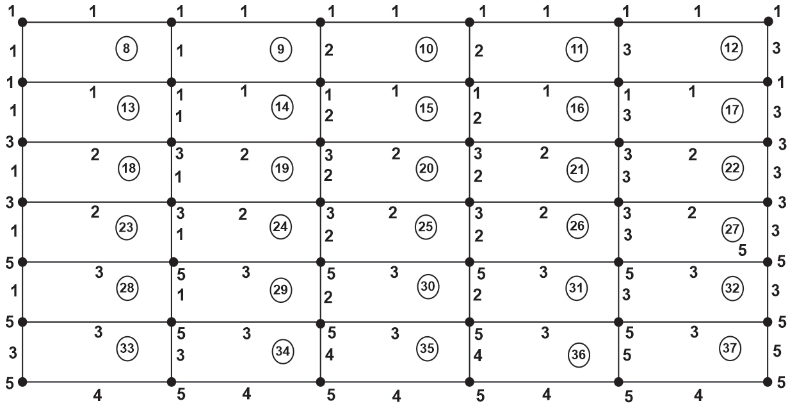
Proof. Total face irregularity strength of grid graph G_n^m under labeling ϕ of type $(1,1,0)$ will be equal to $\left\lceil \frac{mn+7}{8} \right\rceil$ if

- all the differences in weights of the horizontally adjacent faces is 1,
- all the differences in weights of the vertically adjacent faces is m

From theorems 2 and 3, we see that above two conditions are satisfied.
This leads proof of theorem 4. □

Example 2.1. *Show that the total face irregularity strength of grid graph G_6^5, under a $k-$labeling of type $(1,1,0)$ is 5.*

Proof. The graph under consideration is $G_6^5 = P_7 \square P_6$. A $k-$labeling of type $(1,1,0)$ for the grid graph G_6^5 can be illustrated as



Total face irregular 5-labeling of type (1,1,0) of grid graph G_6^5

Here $k = \lceil \frac{30+7}{8} \rceil = 5$, $\lfloor \frac{m+1}{3} \rfloor = 2$, $\left\lceil \frac{k}{\lfloor \frac{m+1}{3} \rfloor} \right\rceil = 3$, $m - 2\lfloor \frac{m+1}{3} \rfloor = 1$, $\left\lceil \frac{k}{m-2\lfloor \frac{m+1}{3} \rfloor} \right\rceil = 5$

We can see that $\lfloor \frac{m+1}{3} \rfloor > m - 2\lfloor \frac{m+1}{3} \rfloor$. In order to show that $tfs_{(1,1,0)}(G_6^5) = 5$, it is sufficient to prove that all the horizontal differences in face weights are 1 and all the vertical differences in face weights are m. Now we prove these results.

Horizontal differences in face weights can be calculated as follows:

$$\begin{aligned}
&\text{For } \; i = 1,2,3,4,5 \text{ and } j = 1,2,3,4\\
&Wt_{\phi(1,1,0)}\left(f_i^{j+1}\right) - Wt_{\phi(1,1,0)}\left(f_i^{j}\right) =\\
&= \phi\left(v_i^{j+2}\right) + \phi\left(v_{i+1}^{j+2}\right) + \phi\left(v_i^{j+1}v_i^{j+2}\right) + \phi\left(v_{i+1}^{j+1}v_{i+1}^{j+2}\right) + \phi\left(v_i^{j+2}v_{i+1}^{j+2}\right)\\
&- \phi\left(v_i^{j}\right) - \phi\left(v_{i+1}^{j}\right) - \phi\left(v_i^{j}v_i^{j+1}\right) - \phi\left(v_i^{j}v_{i+1}^{j}\right) - \phi\left(v_{i+1}^{j}v_{i+1}^{j+1}\right)\\
&= 1 \;\; \textit{for every} \;\; i = 1,2,3,4,5 \text{ and } j = 1,2,3,4
\end{aligned}$$

$$\begin{aligned}
&\text{For } \; i = 6,7 \text{ and } j = 1,2,3,4\\
&Wt_{\phi(1,1,0)}\left(f_i^{j+1}\right) - Wt_{\phi(1,1,0)}\left(f_i^{j}\right) =\\
&= \phi\left(v_i^{j+2}\right) + \phi\left(v_{i+1}^{j+2}\right) + \phi\left(v_i^{j+1}v_i^{j+2}\right) + \phi\left(v_{i+1}^{j+1}v_{i+1}^{j+2}\right) + \phi\left(v_i^{j+2}v_{i+1}^{j+2}\right)\\
&- \phi\left(v_i^{j}\right) - \phi\left(v_{i+1}^{j}\right) - \phi\left(v_i^{j}v_i^{j+1}\right) - \phi\left(v_i^{j}v_{i+1}^{j}\right) - \phi\left(v_{i+1}^{j}v_{i+1}^{j+1}\right)\\
&= 1 \;\; \textit{for every} \;\; i = 6,7 \text{ and } j = 1,2,3,4
\end{aligned}$$

Vertical differences in face weights can be calculated as follows:

$$\begin{aligned}
&\text{For } \; i = 1,2,3,4 \text{ and } j = 1,2,3,4,5\\
&Wt_{\phi(1,1,0)}\left(f_{i+1}^{j}\right) - Wt_{\phi(1,1,0)}\left(f_i^{j}\right) =\\
&= \phi\left(v_{i+2}^{j}\right) + \phi\left(v_{i+2}^{j+1}\right) + \phi\left(v_{i+1}^{j}v_{i+2}^{j}\right) + \phi\left(v_{i+2}^{j}v_{i+2}^{j+1}\right) + \phi\left(v_{i+1}^{j+1}v_{i+2}^{j+1}\right) - \phi\left(v_i^{j}\right) - \phi\left(v_i^{j+1}\right) - \phi\left(v_i^{j}v_i^{j+1}\right)\\
&- \phi\left(v_i^{j}v_{i+1}^{j}\right) - \phi\left(v_i^{j+1}v_{i+1}^{j+1}\right)\\
&= 5 \;\; \textit{for every} \;\; i = 1,2,3,4 \text{ and } j = 1,2,3,4,5
\end{aligned}$$

$$\begin{aligned}
&\text{For } \; i = 5 \text{ and } j = 1,2,3,4,5\\
&Wt_{\phi(1,1,0)}\left(f_{i+1}^{j}\right) - Wt_{\phi(1,1,0)}\left(f_i^{j}\right) =\\
&= \phi\left(v_{i+2}^{j}\right) + \phi\left(v_{i+2}^{j+1}\right) + \phi\left(v_{i+1}^{j}v_{i+2}^{j}\right) + \phi\left(v_{i+2}^{j}v_{i+2}^{j+1}\right) + \phi\left(v_{i+1}^{j+1}v_{i+2}^{j+1}\right) - \phi\left(v_i^{j}\right) - \phi\left(v_i^{j+1}\right) - \phi\left(v_i^{j}v_i^{j+1}\right)
\end{aligned}$$

$-\phi\left(v_i^j v_{i+1}^j\right)-\phi\left(v_i^{j+1} v_{i+1}^{j+1}\right)$
$=5$ *for every* $i=5$ and $j=1,2,3,4,5$

For $i=6$ and $j=1,2,3,4,5$
$Wt_{\phi(1,1,0)}\left(f_{i+1}^j\right)-Wt_{\phi(1,1,0)}\left(f_i^j\right)=$
$=\phi\left(v_{i+2}^j\right)+\phi\left(v_{i+2}^{j+1}\right)+\phi\left(v_{i+1}^j v_{i+2}^j\right)+\phi\left(v_{i+2}^j v_{i+2}^{j+1}\right)+\phi\left(v_{i+1}^{j+1} v_{i+2}^{j+1}\right)-\phi\left(v_i^j\right)-\phi\left(v_i^{j+1}\right)-\phi\left(v_i^j v_i^{j+1}\right)$
$-\phi\left(v_i^j v_{i+1}^j\right)-\phi\left(v_i^{j+1} v_{i+1}^{j+1}\right)$
$=5$ *for every* $i=6$ and $j=1,2,3,4,5$

For $i=7$; $j=1,2,3,4,5$
$Wt_{\phi(1,1,0)}\left(f_{i+1}^j\right)-Wt_{\phi(1,1,0)}\left(f_i^j\right)=$
$=\phi\left(v_{i+2}^j\right)+\phi\left(v_{i+2}^{j+1}\right)+\phi\left(v_{i+1}^j v_{i+2}^j\right)+\phi\left(v_{i+2}^j v_{i+2}^{j+1}\right)+\phi\left(v_{i+1}^{j+1} v_{i+2}^{j+1}\right)-\phi\left(v_i^j\right)-\phi\left(v_i^{j+1}\right)-\phi\left(v_i^j v_i^{j+1}\right)$
$-\phi\left(v_i^j v_{i+1}^j\right)-\phi\left(v_i^{j+1} v_{i+1}^{j+1}\right)$
$=5$ *for every* $i=7$ and $j=1,2,3,4,5$

It shows that all the differences of horizontal faces are equal to one, that is $Wt_{\phi(1,1,0)}\left(f_i^{j+1}\right)-Wt_{\phi(1,1,0)}\left(f_i^j\right)=1$ and all the differences of vertical faces are equal to m, that is $Wt_{\phi(1,1,0)}\left(f_{i+1}^j\right)-Wt_{\phi(1,1,0)}\left(f_i^j\right)=m$. Hence, total face irregularity strength of grid graph G_6^5 is 5. $\square$

3 Conclusion

In this article, we worked on total face irregularity strength of generalized plane grid graphs G_n^m under a graph labeling of type (α,β,γ) where $\alpha,\beta\in\{0,1\}$. Finding irregularity strength of grid graphs is a complicated task because we need to verify our results on all smaller and larger graphs. The problem which is specified in this research is based on those graphs G_n^m where $\left\lfloor\frac{m+1}{3}\right\rfloor>m-2\left\lfloor\frac{m+1}{3}\right\rfloor$ and these graphs are generalized under a graph labeling of type $(1,1,0)$. In this kind of labeling, we label graph vertices and graph edges but we calculate the weight of every under consideration graph face. At the end an example is also proved to verify our results. In future, work can be done on graphs G_n^m where $\left\lfloor\frac{m+1}{3}\right\rfloor<m-2\left\lfloor\frac{m+1}{3}\right\rfloor$ and relevant examples can be proved.

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