Total Face Irregularity Strength of type (α, β, γ) of Grid Graphs

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Abstract

We investigate new graph characteristics namely total (vertex, edge) face irregularity strength of generalized plane grid graphs G_n^m under k -labeling ϕ of type (α, β, γ) . The minimum integer k for which a vertex-edge labelled graph has distinct face weights is called the total face irregularity strength of the graph and is denoted by $tfs(G_n^m)$. In this article, the graphs $G = (V, E, F)$ under consideration are simple, finite, undirected and planar. We will estimate the exact tight lower bounds for the total face irregularity strength of some families of generalized plane grid graphs.

Keywords: Total face labeling of type (α, β, γ) , Total face irregularity strength, Cartesian product of path graphs.

1 Introduction

This complete research is purely based on simple, finite, undirected and planar graphs $G = (V, E, F)$ where $V(G)$ is the vertex set, $E(G)$ is the edge set and $F(G)$ is the face set of a particular graph G . A mapping that maps graph elements into positive integers is called *graph labeling* and these positive integers are called labels. The concept of labeling was first introduced by Chartrand et al. [9]

Suppose that k is a positive integer then a k -labeling of type (α, β, γ) , where $\alpha, \beta, \gamma \in \{0, 1\}$ is a mapping ϕ from the set of graph elements into the set of positive integers $\{1, 2, 3, \dots, k\}$. If the domain of k -labeling of type (α, β, γ) is $V(G)$, $E(G)$, $F(G)$ or $V(G) \cup E(G)$ then we say this a *vertex k -labeling of type $(1, 0, 0)$* , *edge k -labeling of type $(0, 1, 0)$* , *face k -labeling of type $(0, 0, 1)$* or *total k -labeling of type $(1, 1, 0)$* respectively.

The other possible cases are $V(G) \cup F(G)$, $E(G) \cup F(G)$ and $V(G) \cup E(G) \cup F(G)$ which are defined as *vertex-face k -labeling of type $(1, 0, 1)$* , *edge-face k -labeling of type $(0, 1, 1)$* and *entire k -labeling of type $(1, 1, 1)$* respectively. But keep in mind that the trivial case $(\alpha, \beta, \gamma) = (0, 0, 0)$ is never allowed. In this research, we have worked on total labeling in which we label both, vertices and edges, but the ultimate focus is always on *face weight* of the graph. *The weight of a vertex* is the sum of labels of that vertex itself and the labels of its adjacent edges, *the weight of a graph edge* is the sum of label of that edge itself and the labels of its adjacent vertices, *the weight of a face* is the sum of labels of that face itself and the labels of its surrounding edges and vertices. In general, the weight of any face f of a plane graph G under k -labeling ϕ of type (α, β, γ) can be defined as,

$$Wt_{\phi(\alpha, \beta, \gamma)}(f) = \alpha \sum_{v \sim f} \phi(v) + \beta \sum_{e \sim f} \phi(e) + \gamma \phi(f) \quad (1)$$

A k -labeling ϕ of type (α, β, γ) of the plane graph G is called *face irregular k -labeling of type (α, β, γ)* of the plane graph G , if for every two different faces $f \neq g$ and $f, g \in G$, we have

$$Wt_{\phi(\alpha, \beta, \gamma)}(f) \neq Wt_{\phi(\alpha, \beta, \gamma)}(g)$$

In general, for a labelled graph G , the minimum integer k for which the graph G admits a face irregular k -labeling of type (α, β, γ) is called the *face irregularity strength of type (α, β, γ)* of the plane graph G and it is denoted by $fs_{(\alpha, \beta, \gamma)}(G)$ and particularly, for a vertex-edge labelled graph, the minimum integer k for which the graph G admits a face irregular k -labeling of type (α, β, γ) is called the *total face irregularity strength of type (α, β, γ)* of the plane graph G and it is denoted by $tfs_{(\alpha, \beta, \gamma)}(G)$. A deep survey on irregularity strength of graphs can be seen in [6, 10, 12, 16, 21].

In this research, under consideration graphs are grid graphs G_n^m with n rows and m columns. There are unlimited number of grid graphs with the structure G_n^m but we have limited this research on those graphs where

$\lfloor \frac{m+1}{3} \rfloor > m - 2 \lfloor \frac{m+1}{3} \rfloor$ and we aim to prove exact tight lower bounds for the total face irregularity strength under k -labeling ϕ of type (α, β, γ) of such graphs.

Theorem 1. Let $G = (V, E, F)$ be a 2-connected plane graph with n_i i -sided faces, $i \geq 3$. Let $\alpha, \beta, \gamma \in \{0, 1\}$, $a = \min\{i : n_i \neq 0\}$ and $b = \max\{i : n_i \neq 0\}$, $n_b = 1$ and $c = \max\{i : n_i \neq 0, i \leq b\}$. Then the face irregularity strength of type α, β, γ of the plane graph G is

$$fs_{(\alpha, \beta, \gamma)}(G) \geq \left\lceil \frac{(\alpha + \beta)a + \gamma + |F(G)| - 2}{(\alpha + \beta)c + \gamma} \right\rceil$$

Proof. Proof of this theorem can be seen in [5]. □

2 Example

Suppose that the graph under consideration is G_{30}^m . So,

$$k = tfs_{(1,1,0)}(G_{30}^m) = \left\lceil \frac{30m+7}{8} \right\rceil$$

and

$$\left\lfloor \frac{m+1}{3} \right\rfloor > m - 2 \left\lfloor \frac{m+1}{3} \right\rfloor$$

In order to verify the results, we claim that the graph G_{30}^m has tight lower bound $tfs_{(1,1,0)}(G_{30}^m) \geq \left\lceil \frac{30m+8}{8} \right\rceil$. If the graph has this tight lower bound then the following conditions should be true

All graph weights should be distinct,

Horizontal differences in face weights should be 1,

Vertical differences in face weights should be m ,

So, now we prove these results.

The vertices can be labeled by the following formulas

For $i = 1, 2, \dots, 22$ and $j = 1, 2, \dots, m+1$

$$f(v_i^j) = 1 + \left\lfloor \frac{m+1}{3} \right\rfloor \left\lfloor \frac{i-1}{2} \right\rfloor$$

For $i = 22, \dots, 30$ and $j = 1, 2, \dots, m+1$

$$f(v_i^j) = k$$

Horizontal edges can be labeled by the following formulas

For $i = 1, 2, \dots, 26$ and $j = 1, 2, \dots, m+1$

$$f(v_i^j v_{i+1}^j) = 1 + (m - 2 \left\lfloor \frac{m+1}{3} \right\rfloor) \left\lfloor \frac{i-1}{2} \right\rfloor$$

For $i = 27, \dots, 30$ and $j = 1, 2, \dots, m+1$

$$f(v_i^j v_{i+1}^j) = k$$

Vertical edges can be labelled by the following formulas

For $i = 1, 2, \dots, 21$ and $j = 1, 2, \dots, m-1$

$$f(v_i^j v_{i+1}^j) = \left\lceil \frac{j}{2} \right\rceil$$

For $i = 22, 23$ and $j = 1, 2, \dots, m-1$

$$f(v_i^j v_{i+1}^j) = \left\lceil \frac{j}{2} \right\rceil + \left(\left\lfloor \frac{m+1}{3} \right\rfloor \left\lceil \frac{k}{\left\lfloor \frac{m+1}{3} \right\rfloor} \right\rceil + 1 - k \right) (i - 2 \left\lceil \frac{k}{\left\lfloor \frac{m+1}{3} \right\rfloor} \right\rceil + 1)$$

For $i = 24, 25$ and $j = 1, 2, \dots, m-1$

$$f(v_i^j v_{i+1}^j) = \left\lceil \frac{j}{2} \right\rceil + 2 \left(\left\lfloor \frac{m+1}{3} \right\rfloor \left\lceil \frac{k}{\left\lfloor \frac{m+1}{3} \right\rfloor} \right\rceil \right) + 2 - 2k + \left(\left\lfloor \frac{m+1}{3} \right\rfloor \right) (i - 2 \left\lceil \frac{k}{\left\lfloor \frac{m+1}{3} \right\rfloor} \right\rceil - 1)$$

For $i = 26, 27$ and $j = 1, 3, \dots, m$; $m \equiv 1(2)$ or $j = 1, 3, \dots, m+1$; $m \equiv 0(2)$

$$f(v_i^j v_{i+1}^j) = \left\lceil \frac{j}{2} \right\rceil + 2 \left(\left\lfloor \frac{m+1}{3} \right\rfloor \right) \left(\left\lceil \frac{k}{m-2 \left\lfloor \frac{m+1}{3} \right\rfloor} \right\rceil - 1 \right) + 2 - 2k$$

$$+ \left\lceil \frac{1}{2} \left(\left\lceil \frac{k}{m-2 \left\lfloor \frac{m+1}{3} \right\rfloor} \right\rceil (m - 2 \left\lfloor \frac{m+1}{3} \right\rfloor) + 2 \left\lfloor \frac{m+1}{3} \right\rfloor + 1 - k \right) \right\rceil$$

$$\left\lceil \frac{1}{2} \left(i - 2 \left\lceil \frac{k}{m-2 \left\lfloor \frac{m+1}{3} \right\rfloor} \right\rceil + 1 \right) \right\rceil + \left\lceil \frac{1}{2} \left(\left\lceil \frac{k}{m-2 \left\lfloor \frac{m+1}{3} \right\rfloor} \right\rceil (m - 2 \left\lfloor \frac{m+1}{3} \right\rfloor) + 2 \left\lfloor \frac{m+1}{3} \right\rfloor + 1 - k \right) \right\rceil \left\lceil \frac{1}{2} \left(i - 2 \left\lceil \frac{k}{m-2 \left\lfloor \frac{m+1}{3} \right\rfloor} \right\rceil + 1 \right) \right\rceil$$

For $i = 26, 27$ and $j = 2, 4, \dots, m+1$; $m \equiv 1(2)$ or $j = 2, 4, \dots, m$; $m \equiv 0(2)$

$$f(v_i^j v_{i+1}^j) = \left\lceil \frac{j}{2} \right\rceil + 2 \left(\left\lfloor \frac{m+1}{3} \right\rfloor \right) \left(\left\lceil \frac{k}{m-2 \left\lfloor \frac{m+1}{3} \right\rfloor} \right\rceil - 1 \right) + 2 - 2k$$

$$+ \left\lceil \frac{1}{2} \left(\left\lceil \frac{k}{m-2 \lfloor \frac{m+1}{3} \rfloor} \right\rceil (m-2 \lfloor \frac{m+1}{3} \rfloor) + 2 \lfloor \frac{m+1}{3} \rfloor + 1 - k \right) \right\rceil$$

$$\left\lceil \frac{1}{2} (i-2 \left\lceil \frac{k}{m-2 \lfloor \frac{m+1}{3} \rfloor} \right\rceil + 1) \right\rceil + \left\lceil \frac{1}{2} \left(\left\lceil \frac{k}{m-2 \lfloor \frac{m+1}{3} \rfloor} \right\rceil (m-2 \lfloor \frac{m+1}{3} \rfloor) + 2 \lfloor \frac{m+1}{3} \rfloor + 1 - k \right) \right\rceil \left\lceil \frac{1}{2} (i-2 \left\lceil \frac{k}{m-2 \lfloor \frac{m+1}{3} \rfloor} \right\rceil + 1) \right\rceil$$

For $i = 28, 29$ and $j = 1, 3, \dots, m$; $m \equiv 1(2)$ or $j = 1, 3, \dots, m+1$; $m \equiv 0(2)$

$$f(v_i^j v_{i+1}^j) = \left\lceil \frac{j}{2} \right\rceil + 3 - 3k + m \left\lceil \frac{k}{m-2 \lfloor \frac{m+1}{3} \rfloor} \right\rceil + \left\lfloor \frac{m}{2} \right\rfloor \left\lceil \frac{1}{2} (i-2 \left\lceil \frac{k}{m-2 \lfloor \frac{m+1}{3} \rfloor} \right\rceil - 1) \right\rceil + \left\lceil \frac{m}{2} \right\rceil \left\lceil \frac{1}{2} (i-2 \left\lceil \frac{k}{m-2 \lfloor \frac{m+1}{3} \rfloor} \right\rceil - 1) \right\rceil$$

For $i = 28, 29$ and $j = 2, 4, \dots, m+1$; $m \equiv 1(2)$ or $j = 2, 4, \dots, m$; $m \equiv 0(2)$

$$f(v_i^j v_{i+1}^j) = \left\lceil \frac{j}{2} \right\rceil + 3 - 3k + m \left\lceil \frac{k}{m-2 \lfloor \frac{m+1}{3} \rfloor} \right\rceil + \left\lceil \frac{m}{2} \right\rceil \left\lceil \frac{1}{2} (i-2 \left\lceil \frac{k}{m-2 \lfloor \frac{m+1}{3} \rfloor} \right\rceil - 1) \right\rceil + \left\lfloor \frac{m}{2} \right\rfloor \left\lceil \frac{1}{2} (i-2 \left\lceil \frac{k}{m-2 \lfloor \frac{m+1}{3} \rfloor} \right\rceil - 1) \right\rceil$$

By calculation, one can easily check that all the differences of horizontal faces are equal to one, that is $Wt_{\phi(1,1,0)}(f_i^{j+1}) - Wt_{\phi(1,1,0)}(f_i^j) = 1$ and all the differences of vertical faces are equal to m , that is $Wt_{\phi(1,1,0)}(f_{i+1}^j) - Wt_{\phi(1,1,0)}(f_i^j) = m$. This shows that our grid graph with 30 rows and m columns, having total face irregularity strength equal to $\lceil \frac{30m+7}{8} \rceil$, holding condition $\lfloor \frac{m+1}{3} \rfloor > m-2 \lfloor \frac{m+1}{3} \rfloor$, is showing true results because all weights are distinct and the horizontal differences of weights are equal to one while vertical differences of weights are equal to m .

Now we extend this work for graphs G_n^m with n rows and m columns. We will find the exact tight lower bound and will prove the results by showing that all the weights are distinct, the differences of horizontal faces are equal to one, and the differences of vertical faces are equal to m .

3 Total face irregularity strength of grid graphs

Theorem 2. Let $G_n^m = P_{n+1} \square P_{m+1}$ is a generalized grid graph with $n \geq 2$ rows and $m \geq 2$ columns, then

$$tfs_{(1,1,0)}(G_n^m) = \left\lceil \frac{mn+7}{8} \right\rceil$$

Proof. SAMAJH NAI A RAHI K KESE PRESENT KRU □

The weights of the face f under k -labeling ϕ of type $(1, 1, 0)$ can be defined as

$$Wt_{\phi(1,1,0)}(f_i^j) = \phi(v_i^j) + \phi(v_i^{j+1}) + \phi(v_{i+1}^j) + \phi(v_{i+1}^{j+1}) + \phi(v_i^j v_{i+1}^{j+1}) + \phi(v_i^{j+1} v_{i+1}^j) + \phi(v_{i+1}^j v_i^{j+1}) + \phi(v_{i+1}^{j+1} v_i^j)$$

The horizontal difference in weights can be defined as

$$\begin{aligned} Wt_{\phi(1,1,0)}(f_i^{j+1}) - Wt_{\phi(1,1,0)}(f_i^j) &= \phi(v_i^{j+1}) + \phi(v_i^{j+2}) + \phi(v_{i+1}^{j+1}) + \phi(v_{i+1}^{j+2}) + \phi(v_i^{j+1} v_{i+1}^{j+2}) + \phi(v_i^{j+1} v_{i+1}^{j+1}) \\ &\quad + \phi(v_{i+1}^{j+1} v_i^{j+2}) + \phi(v_{i+1}^{j+2} v_i^{j+1}) - \phi(v_i^j) - \phi(v_i^{j+1}) - \phi(v_{i+1}^j) - \phi(v_{i+1}^{j+1}) \\ &\quad - \phi(v_i^j v_{i+1}^{j+1}) - \phi(v_i^{j+1} v_{i+1}^j) - \phi(v_{i+1}^j v_i^{j+1}) - \phi(v_{i+1}^{j+1} v_i^j) \\ &= \phi(v_i^{j+2}) + \phi(v_{i+1}^{j+2}) + \phi(v_i^{j+1} v_{i+1}^{j+2}) + \phi(v_{i+1}^{j+1} v_i^{j+2}) + \phi(v_{i+1}^{j+2} v_i^{j+1}) \\ &\quad - \phi(v_i^j) - \phi(v_{i+1}^j) - \phi(v_i^j v_{i+1}^{j+1}) - \phi(v_{i+1}^j v_i^{j+1}) - \phi(v_{i+1}^{j+1} v_i^j) \end{aligned}$$

Theorem 3. If G_n^m is a grid graph which is generalized for n rows and m columns with $\lfloor \frac{m+1}{3} \rfloor > m-2 \lfloor \frac{m+1}{3} \rfloor$ under k -labeling ϕ of type $(1, 1, 0)$ then the differences of weights between any two horizontal adjacent faces will always be one.

Proof. To prove this theorem, it will be sufficient to show that among all the intervals of labeling, we have $Wt_{\phi(1,1,0)}(f_i^{j+1}) - Wt_{\phi(1,1,0)}(f_i^j) = 1$. In order to do this, we will proceed as follows:

$$\text{For } i = 1, 2, 3, \dots, 2 \left\lceil \frac{k}{\lfloor \frac{m+1}{3} \rfloor} \right\rceil - 1 \text{ and } j = 1, 2, 3, \dots, m-1$$

$$\begin{aligned} Wt_{\phi(1,1,0)}(f_i^{j+1}) - Wt_{\phi(1,1,0)}(f_i^j) &= \phi(v_i^{j+2}) + \phi(v_{i+1}^{j+2}) + \phi(v_i^{j+1} v_{i+1}^{j+2}) + \phi(v_{i+1}^{j+1} v_i^{j+2}) + \phi(v_i^{j+2} v_{i+1}^{j+2}) \\ &\quad - \phi(v_i^j) - \phi(v_{i+1}^j) - \phi(v_i^j v_{i+1}^{j+1}) - \phi(v_{i+1}^j v_i^{j+1}) - \phi(v_{i+1}^{j+1} v_i^j) \\ &= \phi(v_i^{j+2} v_{i+1}^{j+2}) - \phi(v_i^j v_{i+1}^j) \\ &= \left\lceil \frac{j+2}{2} \right\rceil - \left\lceil \frac{j}{2} \right\rceil \end{aligned}$$

$$= 1$$

$$\text{For } i = 2 \left\lfloor \frac{k}{\left\lfloor \frac{m+1}{3} \right\rfloor} \right\rfloor, 2 \left\lfloor \frac{k}{\left\lfloor \frac{m+1}{3} \right\rfloor} \right\rfloor + 1 \text{ and } j = 1, 2, 3, \dots, m-1$$

$$\begin{aligned} & Wt_{\phi(1,1,0)}(f_i^{j+1}) - Wt_{\phi(1,1,0)}(f_i^j) \\ &= \phi(v_i^{j+2}) + \phi(v_{i+1}^{j+2}) + \phi(v_i^{j+1}v_{i+1}^{j+2}) + \phi(v_{i+1}^{j+1}v_{i+1}^{j+2}) + \phi(v_i^{j+2}v_{i+1}^{j+2}) \\ &\quad - \phi(v_i^j) - \phi(v_{i+1}^j) - \phi(v_i^jv_{i+1}^{j+1}) - \phi(v_{i+1}^jv_{i+1}^{j+1}) - \phi(v_i^jv_{i+1}^{j+1}) \\ &= \left\lfloor \frac{j+2}{2} \right\rfloor + \left(\left\lfloor \frac{m+1}{3} \right\rfloor \left\lfloor \frac{k}{\left\lfloor \frac{m+1}{3} \right\rfloor} \right\rfloor + 1 - k \right) (i - 2 \left\lfloor \frac{k}{\left\lfloor \frac{m+1}{3} \right\rfloor} \right\rfloor + 1) \\ &\quad - \left\lfloor \frac{j}{2} \right\rfloor - \left(\left\lfloor \frac{m+1}{3} \right\rfloor \left\lfloor \frac{k}{\left\lfloor \frac{m+1}{3} \right\rfloor} \right\rfloor + 1 - k \right) (i - 2 \left\lfloor \frac{k}{\left\lfloor \frac{m+1}{3} \right\rfloor} \right\rfloor + 1) \\ &= \left\lfloor \frac{j+2}{2} \right\rfloor - \left\lfloor \frac{j}{2} \right\rfloor \\ &= 1 \end{aligned}$$

$$\text{For } i = 2 \left\lfloor \frac{k}{\left\lfloor \frac{m+1}{3} \right\rfloor} \right\rfloor + 2, \dots, 2 \left\lfloor \frac{k}{\left\lfloor \frac{m+1}{3} \right\rfloor} \right\rfloor - 1 \text{ and } j = 1, 2, 3, \dots, m-1$$

$$\begin{aligned} & Wt_{\phi(1,1,0)}(f_i^{j+1}) - Wt_{\phi(1,1,0)}(f_i^j) \\ &= \phi(v_i^{j+2}) + \phi(v_{i+1}^{j+2}) + \phi(v_i^{j+1}v_{i+1}^{j+2}) + \phi(v_{i+1}^{j+1}v_{i+1}^{j+2}) + \phi(v_i^{j+2}v_{i+1}^{j+2}) \\ &\quad - \phi(v_i^j) - \phi(v_{i+1}^j) - \phi(v_i^jv_{i+1}^{j+1}) - \phi(v_{i+1}^jv_{i+1}^{j+1}) - \phi(v_i^jv_{i+1}^{j+1}) \\ &= \left\lfloor \frac{j+2}{2} \right\rfloor + 2 \left(\left\lfloor \frac{m+1}{3} \right\rfloor \left\lfloor \frac{k}{\left\lfloor \frac{m+1}{3} \right\rfloor} \right\rfloor \right) + 2 - 2k + \left(\left\lfloor \frac{m+1}{3} \right\rfloor \right) (i - 2 \left\lfloor \frac{k}{\left\lfloor \frac{m+1}{3} \right\rfloor} \right\rfloor - 1) \\ &\quad - \left\lfloor \frac{j}{2} \right\rfloor - 2 \left(\left\lfloor \frac{m+1}{3} \right\rfloor \left\lfloor \frac{k}{\left\lfloor \frac{m+1}{3} \right\rfloor} \right\rfloor \right) - 2 + 2k + \left(\left\lfloor \frac{m+1}{3} \right\rfloor \right) (i - 2 \left\lfloor \frac{k}{\left\lfloor \frac{m+1}{3} \right\rfloor} \right\rfloor - 1) \\ &= \left\lfloor \frac{j+2}{2} \right\rfloor - \left\lfloor \frac{j}{2} \right\rfloor \\ &= 1 \end{aligned}$$

$$\begin{aligned} \text{For } \quad i &= 2 \left\lfloor \frac{k}{m-2 \left\lfloor \frac{m+1}{3} \right\rfloor} \right\rfloor, 2 \left\lfloor \frac{k}{m-2 \left\lfloor \frac{m+1}{3} \right\rfloor} \right\rfloor + 1 \text{ and} \\ j &= 1, 3, \dots, m; m \equiv 1(2) \text{ or } j = 1, 3, \dots, m+1; m \equiv 0(2) \end{aligned}$$

$$\begin{aligned} & Wt_{\phi(1,1,0)}(f_i^{j+1}) - Wt_{\phi(1,1,0)}(f_i^j) \\ &= \phi(v_i^{j+2}) + \phi(v_{i+1}^{j+2}) + \phi(v_i^{j+1}v_{i+1}^{j+2}) + \phi(v_{i+1}^{j+1}v_{i+1}^{j+2}) + \phi(v_i^{j+2}v_{i+1}^{j+2}) \\ &\quad - \phi(v_i^j) - \phi(v_{i+1}^j) - \phi(v_i^jv_{i+1}^{j+1}) - \phi(v_{i+1}^jv_{i+1}^{j+1}) - \phi(v_i^jv_{i+1}^{j+1}) \\ &= \left\lfloor \frac{j+2}{2} \right\rfloor + 2 \left\lfloor \frac{m+1}{3} \right\rfloor \left(\left\lfloor \frac{k}{m-2 \left\lfloor \frac{m+1}{3} \right\rfloor} \right\rfloor - 1 \right) + 2 - 2k \\ &\quad + \left[\frac{1}{2} \left(\left\lfloor \frac{k}{m-2 \left\lfloor \frac{m+1}{3} \right\rfloor} \right\rfloor (m-2 \left\lfloor \frac{m+1}{3} \right\rfloor) + 2 \left\lfloor \frac{m+1}{3} \right\rfloor + 1 - k \right) \left[\frac{1}{2} (i - 2 \left\lfloor \frac{k}{m-2 \left\lfloor \frac{m+1}{3} \right\rfloor} \right\rfloor + 1) \right] \right. \\ &\quad \left. + \left[\frac{1}{2} \left(\left\lfloor \frac{k}{m-2 \left\lfloor \frac{m+1}{3} \right\rfloor} \right\rfloor (m-2 \left\lfloor \frac{m+1}{3} \right\rfloor) + 2 \left\lfloor \frac{m+1}{3} \right\rfloor + 1 - k \right) \left[\frac{1}{2} (i - 2 \left\lfloor \frac{k}{m-2 \left\lfloor \frac{m+1}{3} \right\rfloor} \right\rfloor + 1) \right] \right. \right. \\ &\quad \left. \left. - \left\lfloor \frac{j}{2} \right\rfloor - 2 \left(\left\lfloor \frac{m+1}{3} \right\rfloor \right) \left(\left\lfloor \frac{k}{m-2 \left\lfloor \frac{m+1}{3} \right\rfloor} \right\rfloor - 1 \right) - 2 + 2k \right] \right] \end{aligned}$$

$$\begin{aligned}
& - \left\lfloor \frac{1}{2} \left(\left\lfloor \frac{k}{m-2 \lfloor \frac{m+1}{3} \rfloor} \right\rfloor (m-2 \lfloor \frac{m+1}{3} \rfloor) + 2 \left\lfloor \frac{m+1}{3} \right\rfloor + 1 - k \right) \right\rfloor \left\lfloor \frac{1}{2} (i-2 \lfloor \frac{k}{m-2 \lfloor \frac{m+1}{3} \rfloor} \rfloor + 1) \right\rfloor \\
& - \left\lfloor \frac{1}{2} \left(\left\lfloor \frac{k}{m-2 \lfloor \frac{m+1}{3} \rfloor} \right\rfloor (m-2 \lfloor \frac{m+1}{3} \rfloor) + 2 \left\lfloor \frac{m+1}{3} \right\rfloor + 1 - k \right) \right\rfloor \left\lfloor \frac{1}{2} (i-2 \lfloor \frac{k}{m-2 \lfloor \frac{m+1}{3} \rfloor} \rfloor + 1) \right\rfloor \\
& = \left\lfloor \frac{j+2}{2} \right\rfloor - \left\lfloor \frac{j}{2} \right\rfloor \\
& = 1
\end{aligned}$$

$$\begin{aligned}
\text{For } i &= 2 \left\lfloor \frac{k}{m-2 \lfloor \frac{m+1}{3} \rfloor} \right\rfloor, 2 \left\lfloor \frac{k}{m-2 \lfloor \frac{m+1}{3} \rfloor} \right\rfloor + 1 \text{ and} \\
j &= 2, 4, \dots, m+1; m \equiv 1(2) \text{ or } j = 2, 4, \dots, m; m \equiv 0(2)
\end{aligned}$$

$$\begin{aligned}
& Wt_{\phi(1,1,0)}(f_i^{j+1}) - Wt_{\phi(1,1,0)}(f_i^j) \\
& = \phi(v_i^{j+2}) + \phi(v_{i+1}^{j+2}) + \phi(v_i^{j+1}v_i^{j+2}) + \phi(v_{i+1}^{j+1}v_{i+1}^{j+2}) + \phi(v_i^{j+2}v_{i+1}^{j+2}) \\
& - \phi(v_i^j) - \phi(v_{i+1}^j) - \phi(v_i^jv_i^{j+1}) - \phi(v_i^jv_{i+1}^j) - \phi(v_{i+1}^jv_{i+1}^{j+1}) \\
& = \left\lfloor \frac{j+2}{2} \right\rfloor + 2 \left(\left\lfloor \frac{m+1}{3} \right\rfloor \right) \left(\left\lfloor \frac{k}{m-2 \lfloor \frac{m+1}{3} \rfloor} \right\rfloor - 1 \right) + 2 - 2k \\
& + \left\lfloor \frac{1}{2} \left(\left\lfloor \frac{k}{m-2 \lfloor \frac{m+1}{3} \rfloor} \right\rfloor (m-2 \lfloor \frac{m+1}{3} \rfloor) + 2 \left\lfloor \frac{m+1}{3} \right\rfloor + 1 - k \right) \right\rfloor \left\lfloor \frac{1}{2} (i-2 \lfloor \frac{k}{m-2 \lfloor \frac{m+1}{3} \rfloor} \rfloor + 1) \right\rfloor \\
& + \left\lfloor \frac{1}{2} \left(\left\lfloor \frac{k}{m-2 \lfloor \frac{m+1}{3} \rfloor} \right\rfloor (m-2 \lfloor \frac{m+1}{3} \rfloor) + 2 \left\lfloor \frac{m+1}{3} \right\rfloor + 1 - k \right) \right\rfloor \left\lfloor \frac{1}{2} (i-2 \lfloor \frac{k}{m-2 \lfloor \frac{m+1}{3} \rfloor} \rfloor + 1) \right\rfloor \\
& - \left\lfloor \frac{j}{2} \right\rfloor - 2 \left(\left\lfloor \frac{m+1}{3} \right\rfloor \right) \left(\left\lfloor \frac{k}{m-2 \lfloor \frac{m+1}{3} \rfloor} \right\rfloor - 1 \right) - 2 + 2k \\
& - \left\lfloor \frac{1}{2} \left(\left\lfloor \frac{k}{m-2 \lfloor \frac{m+1}{3} \rfloor} \right\rfloor (m-2 \lfloor \frac{m+1}{3} \rfloor) + 2 \left\lfloor \frac{m+1}{3} \right\rfloor + 1 - k \right) \right\rfloor \left\lfloor \frac{1}{2} (i-2 \lfloor \frac{k}{m-2 \lfloor \frac{m+1}{3} \rfloor} \rfloor + 1) \right\rfloor \\
& - \left\lfloor \frac{1}{2} \left(\left\lfloor \frac{k}{m-2 \lfloor \frac{m+1}{3} \rfloor} \right\rfloor (m-2 \lfloor \frac{m+1}{3} \rfloor) + 2 \left\lfloor \frac{m+1}{3} \right\rfloor + 1 - k \right) \right\rfloor \left\lfloor \frac{1}{2} (i-2 \lfloor \frac{k}{m-2 \lfloor \frac{m+1}{3} \rfloor} \rfloor + 1) \right\rfloor \\
& = \left\lfloor \frac{j+2}{2} \right\rfloor - \left\lfloor \frac{j}{2} \right\rfloor \\
& = 1
\end{aligned}$$

$$\begin{aligned}
\text{For } i &= 2 \left\lfloor \frac{k}{m-2 \lfloor \frac{m+1}{3} \rfloor} \right\rfloor + 2, \dots, n-1 \text{ and} \\
j &= 1, 3, \dots, m; m \equiv 1(2) \text{ or } j = 1, 3, \dots, m+1; m \equiv 0(2)
\end{aligned}$$

$$\begin{aligned}
& Wt_{\phi(1,1,0)}(f_i^{j+1}) - Wt_{\phi(1,1,0)}(f_i^j) \\
& = \phi(v_i^{j+2}) + \phi(v_{i+1}^{j+2}) + \phi(v_i^{j+1}v_i^{j+2}) + \phi(v_{i+1}^{j+1}v_{i+1}^{j+2}) + \phi(v_i^{j+2}v_{i+1}^{j+2}) \\
& - \phi(v_i^j) - \phi(v_{i+1}^j) - \phi(v_i^jv_i^{j+1}) - \phi(v_i^jv_{i+1}^j) - \phi(v_{i+1}^jv_{i+1}^{j+1}) \\
& = \left\lfloor \frac{j+2}{2} \right\rfloor + 3 - 3k + m \left\lfloor \frac{k}{m-2 \lfloor \frac{m+1}{3} \rfloor} \right\rfloor + \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{1}{2} (i-2 \lfloor \frac{k}{m-2 \lfloor \frac{m+1}{3} \rfloor} \rfloor - 1) \right\rfloor
\end{aligned}$$

$$\begin{aligned}
& + \left\lceil \frac{m}{2} \right\rceil \left[\frac{1}{2} \left(i - 2 \left\lfloor \frac{k}{m-2 \lfloor \frac{m+1}{3} \rfloor} \right\rfloor - 1 \right) \right] - \left\lfloor \frac{j}{2} \right\rfloor - 3 + 3k - m \left\lfloor \frac{k}{m-2 \lfloor \frac{m+1}{3} \rfloor} \right\rfloor \\
& - \left\lfloor \frac{m}{2} \right\rfloor \left[\frac{1}{2} \left(i - 2 \left\lfloor \frac{k}{m-2 \lfloor \frac{m+1}{3} \rfloor} \right\rfloor - 1 \right) \right] - \left\lceil \frac{m}{2} \right\rceil \left[\frac{1}{2} \left(i - 2 \left\lfloor \frac{k}{m-2 \lfloor \frac{m+1}{3} \rfloor} \right\rfloor - 1 \right) \right] \\
& = \left\lfloor \frac{j+2}{2} \right\rfloor - \left\lfloor \frac{j}{2} \right\rfloor \\
& = 1
\end{aligned}$$

$$\begin{aligned}
\text{For } i &= 2 \left\lfloor \frac{k}{m-2 \lfloor \frac{m+1}{3} \rfloor} \right\rfloor + 2, \dots, n-1 \text{ and} \\
j &= 2, 4, \dots, m+1; m \equiv 1(2) \text{ or } j = 2, 4, \dots, m; m \equiv 0(2)
\end{aligned}$$

We prove the following weights

$$\begin{aligned}
& Wt_{\phi(1,1,0)}(f_i^{j+1}) - Wt_{\phi(1,1,0)}(f_i^j) \\
& = \phi(v_i^{j+2}) + \phi(v_{i+1}^{j+2}) + \phi(v_i^{j+1}v_i^{j+2}) + \phi(v_{i+1}^{j+1}v_{i+1}^{j+2}) + \phi(v_i^{j+2}v_{i+1}^{j+2}) \\
& - \phi(v_i^j) - \phi(v_{i+1}^j) - \phi(v_i^jv_i^{j+1}) - \phi(v_i^jv_{i+1}^j) - \phi(v_{i+1}^jv_{i+1}^{j+1}) \\
& = \left\lfloor \frac{j+2}{2} \right\rfloor + 3 - 3k + m \left\lfloor \frac{k}{m-2 \lfloor \frac{m+1}{3} \rfloor} \right\rfloor + \left\lceil \frac{m}{2} \right\rceil \left[\frac{1}{2} \left(i - 2 \left\lfloor \frac{k}{m-2 \lfloor \frac{m+1}{3} \rfloor} \right\rfloor - 1 \right) \right] \\
& + \left\lfloor \frac{m}{2} \right\rfloor \left[\frac{1}{2} \left(i - 2 \left\lfloor \frac{k}{m-2 \lfloor \frac{m+1}{3} \rfloor} \right\rfloor - 1 \right) \right] - \left\lfloor \frac{j}{2} \right\rfloor - 3 + 3k - m \left\lfloor \frac{k}{m-2 \lfloor \frac{m+1}{3} \rfloor} \right\rfloor \\
& - \left\lceil \frac{m}{2} \right\rceil \left[\frac{1}{2} \left(i - 2 \left\lfloor \frac{k}{m-2 \lfloor \frac{m+1}{3} \rfloor} \right\rfloor - 1 \right) \right] - \left\lfloor \frac{m}{2} \right\rfloor \left[\frac{1}{2} \left(i - 2 \left\lfloor \frac{k}{m-2 \lfloor \frac{m+1}{3} \rfloor} \right\rfloor - 1 \right) \right] \\
& = \left\lfloor \frac{j+2}{2} \right\rfloor - \left\lfloor \frac{j}{2} \right\rfloor \\
& = 1
\end{aligned}$$

□

The weights of the face f under k -labeling ϕ of type $(1, 1, 0)$ can be defined as

$$Wt_{\phi(1,1,0)}(f_i^j) = \phi(v_i^j) + \phi(v_{i+1}^{j+1}) + \phi(v_{i+1}^j) + \phi(v_{i+1}^{j+1}) + \phi(v_i^jv_i^{j+1}) + \phi(v_i^jv_{i+1}^j) + \phi(v_{i+1}^{j+1}v_{i+1}^{j+1}) + \phi(v_i^{j+1}v_{i+1}^{j+1})$$

The vertical difference in weights can be defined as

$$\begin{aligned}
Wt_{\phi(1,1,0)}(f_{i+1}^j) - Wt_{\phi(1,1,0)}(f_i^j) &= \phi(v_{i+1}^j) + \phi(v_{i+1}^{j+1}) + \phi(v_{i+2}^j) + \phi(v_{i+2}^{j+1}) + \phi(v_{i+1}^jv_{i+1}^{j+1}) + \phi(v_{i+1}^jv_{i+2}^j) \\
& + \phi(v_{i+2}^jv_{i+2}^{j+1}) + \phi(v_{i+1}^{j+1}v_{i+2}^{j+1}) - \phi(v_i^j) - \phi(v_i^{j+1}) - \phi(v_{i+1}^j) - \phi(v_{i+1}^{j+1}) \\
& - \phi(v_i^jv_i^{j+1}) - \phi(v_i^jv_{i+1}^j) - \phi(v_{i+1}^jv_{i+1}^{j+1}) - \phi(v_i^{j+1}v_{i+1}^{j+1}) \\
& = \phi(v_{i+2}^j) + \phi(v_{i+2}^{j+1}) + \phi(v_{i+1}^jv_{i+2}^j) + \phi(v_{i+2}^jv_{i+2}^{j+1}) + \phi(v_{i+1}^{j+1}v_{i+2}^{j+1}) \\
& - \phi(v_i^j) - \phi(v_i^{j+1}) - \phi(v_i^jv_i^{j+1}) - \phi(v_i^jv_{i+1}^j) - \phi(v_{i+1}^{j+1}v_{i+1}^{j+1})
\end{aligned}$$

Theorem 4. If G_n^m is a grid graph which is generalized for n rows and m columns with $\lfloor \frac{m+1}{3} \rfloor > m - 2 \lfloor \frac{m+1}{3} \rfloor$ under k -labeling ϕ of type $(1, 1, 0)$ then the differences of weights between any two vertically adjacent faces will always be m (will always be equal to number of columns).

Proof. To prove this theorem, it will be sufficient to show that among all the intervals of labeling, we have $Wt_{\phi(1,1,0)}(f_{i+1}^j) - Wt_{\phi(1,1,0)}(f_i^j) = m$. In order to do this, we will proceed as follows:

$$\text{For } i = 1, 2, \dots, 2 \left\lfloor \frac{k}{\lfloor \frac{m+1}{3} \rfloor} \right\rfloor - 2; j = 1, 2, \dots, m$$

The difference in weights will be calculated as follows

$$\begin{aligned}
& Wt_{\phi(1,1,0)}(f_{i+1}^j) - Wt_{\phi(1,1,0)}(f_i^j) = \\
& = \phi(v_{i+2}^j) + \phi(v_{i+2}^{j+1}) + \phi(v_{i+1}^j v_{i+2}^j) + \phi(v_{i+2}^j v_{i+2}^{j+1}) + \phi(v_{i+1}^{j+1} v_{i+2}^{j+1}) - \phi(v_i^j) - \phi(v_i^{j+1}) - \phi(v_i^j v_i^{j+1}) \\
& - \phi(v_i^j v_{i+1}^j) - \phi(v_i^{j+1} v_{i+1}^{j+1}) \\
& = 1 + \left\lfloor \frac{m+1}{3} \right\rfloor \left\lfloor \frac{i+1}{2} \right\rfloor + 1 + \left\lfloor \frac{m+1}{3} \right\rfloor \left\lfloor \frac{i+1}{2} \right\rfloor + \left\lfloor \frac{j}{2} \right\rfloor + 1 + (m-2 \left\lfloor \frac{m+1}{3} \right\rfloor) \left\lfloor \frac{i+1}{2} \right\rfloor + \left\lfloor \frac{j+1}{2} \right\rfloor \\
& - 1 - \left\lfloor \frac{m+1}{3} \right\rfloor \left\lfloor \frac{i-1}{2} \right\rfloor - 1 - \left\lfloor \frac{m+1}{3} \right\rfloor \left\lfloor \frac{i-1}{2} \right\rfloor - 1 - (m-2 \left\lfloor \frac{m+1}{3} \right\rfloor) \left\lfloor \frac{i-1}{2} \right\rfloor - \left\lfloor \frac{j}{2} \right\rfloor - \left\lfloor \frac{j+1}{2} \right\rfloor \\
& = 2 \left\lfloor \frac{m+1}{3} \right\rfloor \left(\left\lfloor \frac{i+1}{2} \right\rfloor - \left\lfloor \frac{i-1}{2} \right\rfloor \right) + (m-2 \left\lfloor \frac{m+1}{3} \right\rfloor) \left(\left\lfloor \frac{i+1}{2} \right\rfloor - \left\lfloor \frac{i-1}{2} \right\rfloor \right) \\
& = 2 \left\lfloor \frac{m+1}{3} \right\rfloor + m - 2 \left\lfloor \frac{m+1}{3} \right\rfloor \\
& = m
\end{aligned}$$

For $i = 2 \left\lfloor \frac{k}{\left\lfloor \frac{m+1}{3} \right\rfloor} \right\rfloor - 1$; $j = 1, 2, \dots, m$

The vertical weight differences are as follows:

$$\begin{aligned}
& Wt_{\phi(1,1,0)}(f_{i+1}^j) - Wt_{\phi(1,1,0)}(f_i^j) = \\
& = \phi(v_{i+2}^j) + \phi(v_{i+2}^{j+1}) + \phi(v_{i+1}^j v_{i+2}^j) + \phi(v_{i+2}^j v_{i+2}^{j+1}) + \phi(v_{i+1}^{j+1} v_{i+2}^{j+1}) - \phi(v_i^j) - \phi(v_i^{j+1}) - \phi(v_i^j v_i^{j+1}) \\
& - \phi(v_i^j v_{i+1}^j) - \phi(v_i^{j+1} v_{i+1}^{j+1}) \\
& = k + k + \left\lfloor \frac{j}{2} \right\rfloor + \left(\left\lfloor \frac{m+1}{3} \right\rfloor \left\lfloor \frac{k}{\left\lfloor \frac{m+1}{3} \right\rfloor} \right\rfloor + 1 - k \right) (i - 2 \left\lfloor \frac{k}{\left\lfloor \frac{m+1}{3} \right\rfloor} \right\rfloor + 2) + 1 + (m - 2 \left\lfloor \frac{m+1}{3} \right\rfloor) \left\lfloor \frac{i+1}{2} \right\rfloor \\
& + \left\lfloor \frac{j+1}{2} \right\rfloor + \left(\left\lfloor \frac{m+1}{3} \right\rfloor \left\lfloor \frac{k}{\left\lfloor \frac{m+1}{3} \right\rfloor} \right\rfloor + 1 - k \right) (i - 2 \left\lfloor \frac{k}{\left\lfloor \frac{m+1}{3} \right\rfloor} \right\rfloor + 2) - 1 - \left\lfloor \frac{m+1}{3} \right\rfloor \left\lfloor \frac{i-1}{2} \right\rfloor - 1 \\
& - \left\lfloor \frac{m+1}{3} \right\rfloor \left\lfloor \frac{i-1}{2} \right\rfloor - 1 - (m - 2 \left\lfloor \frac{m+1}{3} \right\rfloor) \left\lfloor \frac{i-1}{2} \right\rfloor - \left\lfloor \frac{j}{2} \right\rfloor - \left\lfloor \frac{j+1}{2} \right\rfloor \\
& = 2k + 2 \left(\left\lfloor \frac{m+1}{3} \right\rfloor \left\lfloor \frac{k}{\left\lfloor \frac{m+1}{3} \right\rfloor} \right\rfloor + 1 - k \right) \left((2 \left\lfloor \frac{k}{\left\lfloor \frac{m+1}{3} \right\rfloor} \right\rfloor - 1) - 2 \left\lfloor \frac{k}{\left\lfloor \frac{m+1}{3} \right\rfloor} \right\rfloor + 2 \right) \\
& + (m - 2 \left\lfloor \frac{m+1}{3} \right\rfloor) \left(\left\lfloor \frac{i+1}{2} \right\rfloor + \left\lfloor \frac{i-1}{2} \right\rfloor \right) - 2 \left\lfloor \frac{m+1}{3} \right\rfloor \left\lfloor \frac{i-1}{2} \right\rfloor - 2 \\
& = 2k + 2 \left\lfloor \frac{m+1}{3} \right\rfloor \left\lfloor \frac{k}{\left\lfloor \frac{m+1}{3} \right\rfloor} \right\rfloor + 2 - 2k + m - 2 \left\lfloor \frac{m+1}{3} \right\rfloor - 2 \left\lfloor \frac{m+1}{3} \right\rfloor \left\lfloor \frac{i-1}{2} \right\rfloor - 2 \\
& = m + 2 \left\lfloor \frac{m+1}{3} \right\rfloor \left(\left\lfloor \frac{k}{\left\lfloor \frac{m+1}{3} \right\rfloor} \right\rfloor - 1 - \left\lfloor \frac{k}{\left\lfloor \frac{m+1}{3} \right\rfloor} \right\rfloor - 1 \right) \\
& = m
\end{aligned}$$

For $i = 2 \left\lfloor \frac{k}{\left\lfloor \frac{m+1}{3} \right\rfloor} \right\rfloor$; $j = 1, 2, \dots, m$

$$\begin{aligned}
& Wt_{\phi(1,1,0)}(f_{i+1}^j) - Wt_{\phi(1,1,0)}(f_i^j) = \\
& = \phi(v_{i+2}^j) + \phi(v_{i+2}^{j+1}) + \phi(v_{i+1}^j v_{i+2}^j) + \phi(v_{i+2}^j v_{i+2}^{j+1}) + \phi(v_{i+1}^{j+1} v_{i+2}^{j+1}) - \phi(v_i^j) - \phi(v_i^{j+1}) - \phi(v_i^j v_i^{j+1}) \\
& - \phi(v_i^j v_{i+1}^j) - \phi(v_i^{j+1} v_{i+1}^{j+1}) \\
& = k + k + \left\lfloor \frac{j}{2} \right\rfloor + \left(\left\lfloor \frac{m+1}{3} \right\rfloor \left\lfloor \frac{k}{\left\lfloor \frac{m+1}{3} \right\rfloor} \right\rfloor + 1 - k \right) (i - 2 \left\lfloor \frac{k}{\left\lfloor \frac{m+1}{3} \right\rfloor} \right\rfloor + 2) + 1 + (m - 2 \left\lfloor \frac{m+1}{3} \right\rfloor) \left\lfloor \frac{i+1}{2} \right\rfloor \\
& + \left\lfloor \frac{j+1}{2} \right\rfloor + \left(\left\lfloor \frac{m+1}{3} \right\rfloor \left\lfloor \frac{k}{\left\lfloor \frac{m+1}{3} \right\rfloor} \right\rfloor + 1 - k \right) (i - 2 \left\lfloor \frac{k}{\left\lfloor \frac{m+1}{3} \right\rfloor} \right\rfloor + 2) - 1 - \left\lfloor \frac{m+1}{3} \right\rfloor \left\lfloor \frac{i-1}{2} \right\rfloor - 1 \\
& - \left\lfloor \frac{m+1}{3} \right\rfloor \left\lfloor \frac{i-1}{2} \right\rfloor - 1 - (m - 2 \left\lfloor \frac{m+1}{3} \right\rfloor) \left\lfloor \frac{i-1}{2} \right\rfloor - \left\lfloor \frac{j}{2} \right\rfloor \\
& = m
\end{aligned}$$

For $i = 2 \left\lceil \frac{k}{\left\lfloor \frac{m+1}{3} \right\rfloor} \right\rceil + 1$; $j = 1, 2, \dots, m$

$$\text{For } i = 2 \left\lceil \frac{k}{\frac{m+1}{3}} \right\rceil + 2, \dots, 2 \left\lceil \frac{k}{m-2} \frac{m+1}{3} \right\rceil - 2; j = 1, 2, \dots, m$$

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$$\begin{aligned}
& -\phi(v_i^j v_{i+1}^j) - \phi(v_i^{j+1} v_{i+1}^{j+1}) \\
& = \left\lfloor \frac{j}{2} \right\rfloor + \left\lfloor \frac{j+1}{2} \right\rfloor + 4\left(\left\lfloor \frac{m+1}{3} \right\rfloor \left\lfloor \frac{k}{\left\lfloor \frac{m+1}{3} \right\rfloor} \right\rfloor\right) + 4 - 4k + 2\left(\left\lfloor \frac{m+1}{3} \right\rfloor\right)(i - 2 \left\lfloor \frac{k}{\left\lfloor \frac{m+1}{3} \right\rfloor} \right\rfloor) + 1 \\
& + (m - 2 \left\lfloor \frac{m+1}{3} \right\rfloor) \left\lfloor \frac{i+1}{2} \right\rfloor - 1 - (m - 2 \left\lfloor \frac{m+1}{3} \right\rfloor) \left\lfloor \frac{i-1}{2} \right\rfloor - \left\lfloor \frac{j}{2} \right\rfloor - \left\lfloor \frac{j+1}{2} \right\rfloor \\
& - 4\left(\left\lfloor \frac{m+1}{3} \right\rfloor \left\lfloor \frac{k}{\left\lfloor \frac{m+1}{3} \right\rfloor} \right\rfloor\right) - 4 + 4k - 2\left(\left\lfloor \frac{m+1}{3} \right\rfloor\right)(i - 2 \left\lfloor \frac{k}{\left\lfloor \frac{m+1}{3} \right\rfloor} \right\rfloor - 1) \\
& = 2\left(\left\lfloor \frac{m+1}{3} \right\rfloor\right)(i - 2 \left\lfloor \frac{k}{\left\lfloor \frac{m+1}{3} \right\rfloor} \right\rfloor - i + 2 \left\lfloor \frac{k}{\left\lfloor \frac{m+1}{3} \right\rfloor} \right\rfloor + 1) + (m - 2 \left\lfloor \frac{m+1}{3} \right\rfloor) \left(\left\lfloor \frac{i+1}{2} \right\rfloor - \left\lfloor \frac{i-1}{2} \right\rfloor\right) \\
& = 2 \left\lfloor \frac{m+1}{3} \right\rfloor + m - 2 \left\lfloor \frac{m+1}{3} \right\rfloor \\
& = m
\end{aligned}$$

$$\text{For } i = 2 \left\lfloor \frac{k}{m-2 \left\lfloor \frac{m+1}{3} \right\rfloor} \right\rfloor - 1; j = 1, 2, \dots, m$$

$$\begin{aligned}
& Wt_{\phi(1,1,0)}(f_{i+1}^j) - Wt_{\phi(1,1,0)}(f_i^j) = \\
& = \phi(v_{i+2}^j) + \phi(v_{i+2}^{j+1}) + \phi(v_{i+1}^j v_{i+2}^j) + \phi(v_{i+2}^j v_{i+2}^{j+1}) + \phi(v_{i+1}^{j+1} v_{i+2}^{j+1}) - \phi(v_i^j) - \phi(v_i^{j+1}) - \phi(v_i^j v_{i+1}^{j+1}) \\
& - \phi(v_i^j v_{i+1}^j) - \phi(v_i^{j+1} v_{i+1}^{j+1}) \\
& = \left\lfloor \frac{j}{2} \right\rfloor + \left\lfloor \frac{j+1}{2} \right\rfloor + 4\left(\left\lfloor \frac{m+1}{3} \right\rfloor\right) \left(\left\lfloor \frac{k}{m-2 \left\lfloor \frac{m+1}{3} \right\rfloor} \right\rfloor - 1\right) + 4 - 4k + \left\lfloor \frac{1}{2}(i+1 - 2 \left\lfloor \frac{k}{m-2 \left\lfloor \frac{m+1}{3} \right\rfloor} \right\rfloor + 1) \right\rfloor \\
& \left(\left\lfloor \frac{\left\lfloor \frac{k}{m-2 \left\lfloor \frac{m+1}{3} \right\rfloor} \right\rfloor (m-2 \left\lfloor \frac{m+1}{3} \right\rfloor) + 2 \left\lfloor \frac{m+1}{3} \right\rfloor + 1 - k}{2} \right\rfloor + \left\lfloor \frac{\left\lfloor \frac{k}{m-2 \left\lfloor \frac{m+1}{3} \right\rfloor} \right\rfloor (m-2 \left\lfloor \frac{m+1}{3} \right\rfloor) + 2 \left\lfloor \frac{m+1}{3} \right\rfloor + 1 - k}{2} \right\rfloor \right) \\
& + \left\lfloor \frac{1}{2}(i+1 - 2 \left\lfloor \frac{k}{m-2 \left\lfloor \frac{m+1}{3} \right\rfloor} \right\rfloor + 1) \right\rfloor \\
& \left(\left\lfloor \frac{\left\lfloor \frac{k}{m-2 \left\lfloor \frac{m+1}{3} \right\rfloor} \right\rfloor (m-2 \left\lfloor \frac{m+1}{3} \right\rfloor) + 2 \left\lfloor \frac{m+1}{3} \right\rfloor + 1 - k}{2} \right\rfloor + \left\lfloor \frac{\left\lfloor \frac{k}{m-2 \left\lfloor \frac{m+1}{3} \right\rfloor} \right\rfloor (m-2 \left\lfloor \frac{m+1}{3} \right\rfloor) + 2 \left\lfloor \frac{m+1}{3} \right\rfloor + 1 - k}{2} \right\rfloor \right) \\
& + k - 1 - (m - 2 \left\lfloor \frac{m+1}{3} \right\rfloor) \left\lfloor \frac{i-1}{2} \right\rfloor - \left\lfloor \frac{j}{2} \right\rfloor - \left\lfloor \frac{j+1}{2} \right\rfloor - 4\left(\left\lfloor \frac{m+1}{3} \right\rfloor \left\lfloor \frac{k}{\left\lfloor \frac{m+1}{3} \right\rfloor} \right\rfloor\right) - 4 + 4k \\
& - 2\left(\left\lfloor \frac{m+1}{3} \right\rfloor\right)(i - 2 \left\lfloor \frac{k}{\left\lfloor \frac{m+1}{3} \right\rfloor} \right\rfloor - 1) \\
& = 4\left(\left\lfloor \frac{m+1}{3} \right\rfloor\right) \left(\left\lfloor \frac{k}{m-2 \left\lfloor \frac{m+1}{3} \right\rfloor} \right\rfloor - 1\right) + (1) \left(\left\lfloor \frac{k}{m-2 \left\lfloor \frac{m+1}{3} \right\rfloor} \right\rfloor (m-2 \left\lfloor \frac{m+1}{3} \right\rfloor) + 2 \left\lfloor \frac{m+1}{3} \right\rfloor + 1 - k\right) + 0 \\
& + k - 1 - (m - 2 \left\lfloor \frac{m+1}{3} \right\rfloor) \left\lfloor \frac{i-1}{2} \right\rfloor - 4\left(\left\lfloor \frac{m+1}{3} \right\rfloor \left\lfloor \frac{k}{\left\lfloor \frac{m+1}{3} \right\rfloor} \right\rfloor\right) - 2\left(\left\lfloor \frac{m+1}{3} \right\rfloor\right)(i - 2 \left\lfloor \frac{k}{\left\lfloor \frac{m+1}{3} \right\rfloor} \right\rfloor - 1) \\
& = 4\left(\left\lfloor \frac{m+1}{3} \right\rfloor\right) \left(\left\lfloor \frac{k}{m-2 \left\lfloor \frac{m+1}{3} \right\rfloor} \right\rfloor - 1\right) + \left\lfloor \frac{k}{m-2 \left\lfloor \frac{m+1}{3} \right\rfloor} \right\rfloor (m-2 \left\lfloor \frac{m+1}{3} \right\rfloor) + 2 \left\lfloor \frac{m+1}{3} \right\rfloor + 1 - k + k \\
& - 1 - (m - 2 \left\lfloor \frac{m+1}{3} \right\rfloor) \left\lfloor \frac{i-1}{2} \right\rfloor - 4\left(\left\lfloor \frac{m+1}{3} \right\rfloor \left\lfloor \frac{k}{\left\lfloor \frac{m+1}{3} \right\rfloor} \right\rfloor\right) - 2\left(\left\lfloor \frac{m+1}{3} \right\rfloor\right)(i - 2 \left\lfloor \frac{k}{\left\lfloor \frac{m+1}{3} \right\rfloor} \right\rfloor - 1) \\
& = 2\left(\left\lfloor \frac{m+1}{3} \right\rfloor\right) \left(2 \left\lfloor \frac{k}{m-2 \left\lfloor \frac{m+1}{3} \right\rfloor} \right\rfloor - 2 + 1 - 2 \left\lfloor \frac{k}{\left\lfloor \frac{m+1}{3} \right\rfloor} \right\rfloor - i + 2 \left\lfloor \frac{k}{\left\lfloor \frac{m+1}{3} \right\rfloor} \right\rfloor + 1\right) \\
& + (m - 2 \left\lfloor \frac{m+1}{3} \right\rfloor) \left(\left\lfloor \frac{k}{m-2 \left\lfloor \frac{m+1}{3} \right\rfloor} \right\rfloor - \left\lfloor \frac{i-1}{2} \right\rfloor\right)
\end{aligned}$$

$$\text{For } i = 2 \left\lceil \frac{k}{m-2 \left\lfloor \frac{m+1}{3} \right\rfloor} \right\rceil ; j = 1, 2, \dots, m$$

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$$\begin{aligned}
& + (m-2 \left\lfloor \frac{m+1}{3} \right\rfloor) \left(\left\lfloor \frac{k}{m-2 \left\lfloor \frac{m+1}{3} \right\rfloor} \right\rfloor - \left\lfloor \frac{i-1}{2} \right\rfloor \right) \\
& = 2 \left(\left\lfloor \frac{m+1}{3} \right\rfloor \right) \left(2 \left\lfloor \frac{k}{m-2 \left\lfloor \frac{m+1}{3} \right\rfloor} \right\rfloor - 2 + 1 + 1 - 2 \left\lfloor \frac{k}{m-2 \left\lfloor \frac{m+1}{3} \right\rfloor} \right\rfloor + 2 - 1 \right) \\
& + (m-2 \left\lfloor \frac{m+1}{3} \right\rfloor) \left(\left\lfloor \frac{k}{m-2 \left\lfloor \frac{m+1}{3} \right\rfloor} \right\rfloor - \left\lfloor \frac{i-1}{2} \right\rfloor \right) \\
& = m
\end{aligned}$$

$$\text{For } i = 2 \left\lfloor \frac{k}{m-2 \left\lfloor \frac{m+1}{3} \right\rfloor} \right\rfloor + 1; j = 1, 2, \dots, m$$

$$\begin{aligned}
& Wt_{\phi(1,1,0)}(f_{i+1}^j) - Wt_{\phi(1,1,0)}(f_i^j) = \\
& = \phi(v_{i+2}^j) + \phi(v_{i+2}^{j+1}) + \phi(v_{i+1}^j v_{i+2}^j) + \phi(v_{i+2}^j v_{i+2}^{j+1}) + \phi(v_{i+1}^{j+1} v_{i+2}^{j+1}) - \phi(v_i^j) - \phi(v_i^{j+1}) - \phi(v_i^j v_i^{j+1}) \\
& - \phi(v_i^j v_{i+1}^j) - \phi(v_i^{j+1} v_{i+1}^{j+1}) \\
& = \left\lfloor \frac{j}{2} \right\rfloor + \left\lfloor \frac{j+1}{2} \right\rfloor + 6 - 6k + 2m \left\lfloor \frac{k}{m-2 \left\lfloor \frac{m+1}{3} \right\rfloor} \right\rfloor + \left\lfloor \frac{1}{2} (i+1 - 2 \left\lfloor \frac{k}{m-2 \left\lfloor \frac{m+1}{3} \right\rfloor} \right\rfloor - 1) \right\rfloor \left(\left\lfloor \frac{m}{2} \right\rfloor + \left\lfloor \frac{m}{2} \right\rfloor \right) \\
& + \left\lfloor \frac{1}{2} (i+1 - 2 \left\lfloor \frac{k}{m-2 \left\lfloor \frac{m+1}{3} \right\rfloor} \right\rfloor - 1) \right\rfloor \left(\left\lfloor \frac{m}{2} \right\rfloor + \left\lfloor \frac{m}{2} \right\rfloor \right) + k - k - \left\lfloor \frac{j}{2} \right\rfloor - \left\lfloor \frac{j+1}{2} \right\rfloor \\
& - 4 \left(\left\lfloor \frac{m+1}{3} \right\rfloor \right) \left(\left\lfloor \frac{k}{m-2 \left\lfloor \frac{m+1}{3} \right\rfloor} \right\rfloor - 1 \right) - 4 + 4k - \left\lfloor \frac{1}{2} (i-2 \left\lfloor \frac{k}{m-2 \left\lfloor \frac{m+1}{3} \right\rfloor} \right\rfloor + 1) \right\rfloor \\
& \left(\frac{\left(\left\lfloor \frac{k}{m-2 \left\lfloor \frac{m+1}{3} \right\rfloor} \right\rfloor (m-2 \left\lfloor \frac{m+1}{3} \right\rfloor) + 2 \left\lfloor \frac{m+1}{3} \right\rfloor + 1 - k \right)}{2} \right) + \left(\frac{\left(\left\lfloor \frac{k}{m-2 \left\lfloor \frac{m+1}{3} \right\rfloor} \right\rfloor (m-2 \left\lfloor \frac{m+1}{3} \right\rfloor) + 2 \left\lfloor \frac{m+1}{3} \right\rfloor + 1 - k \right)}{2} \right) \\
& - \left\lfloor \frac{1}{2} (i-2 \left\lfloor \frac{k}{m-2 \left\lfloor \frac{m+1}{3} \right\rfloor} \right\rfloor + 1) \right\rfloor \\
& \left(\frac{\left(\left\lfloor \frac{k}{m-2 \left\lfloor \frac{m+1}{3} \right\rfloor} \right\rfloor (m-2 \left\lfloor \frac{m+1}{3} \right\rfloor) + 2 \left\lfloor \frac{m+1}{3} \right\rfloor + 1 - k \right)}{2} \right) + \left(\frac{\left(\left\lfloor \frac{k}{m-2 \left\lfloor \frac{m+1}{3} \right\rfloor} \right\rfloor (m-2 \left\lfloor \frac{m+1}{3} \right\rfloor) + 2 \left\lfloor \frac{m+1}{3} \right\rfloor + 1 - k \right)}{2} \right) \\
& = 6 - 6k + 2m \left\lfloor \frac{k}{m-2 \left\lfloor \frac{m+1}{3} \right\rfloor} \right\rfloor + (1)(m) + 0 - 4 \left(\left\lfloor \frac{m+1}{3} \right\rfloor \right) \left(\left\lfloor \frac{k}{m-2 \left\lfloor \frac{m+1}{3} \right\rfloor} \right\rfloor - 1 \right) - 4 + 4k \\
& - (1) \left(\left\lfloor \frac{k}{m-2 \left\lfloor \frac{m+1}{3} \right\rfloor} \right\rfloor (m-2 \left\lfloor \frac{m+1}{3} \right\rfloor) + 2 \left\lfloor \frac{m+1}{3} \right\rfloor + 1 - k \right) \\
& - (1) \left(\left\lfloor \frac{k}{m-2 \left\lfloor \frac{m+1}{3} \right\rfloor} \right\rfloor (m-2 \left\lfloor \frac{m+1}{3} \right\rfloor) + 2 \left\lfloor \frac{m+1}{3} \right\rfloor + 1 - k \right) \\
& = 6 - 6k + 2m \left\lfloor \frac{k}{m-2 \left\lfloor \frac{m+1}{3} \right\rfloor} \right\rfloor + m - 4 \left(\left\lfloor \frac{m+1}{3} \right\rfloor \right) \left(\left\lfloor \frac{k}{m-2 \left\lfloor \frac{m+1}{3} \right\rfloor} \right\rfloor - 1 \right) - 4 + 4k \\
& - \left\lfloor \frac{k}{m-2 \left\lfloor \frac{m+1}{3} \right\rfloor} \right\rfloor (m-2 \left\lfloor \frac{m+1}{3} \right\rfloor) - 2 \left\lfloor \frac{m+1}{3} \right\rfloor - 1 + k \\
& - \left\lfloor \frac{k}{m-2 \left\lfloor \frac{m+1}{3} \right\rfloor} \right\rfloor (m-2 \left\lfloor \frac{m+1}{3} \right\rfloor) - 2 \left\lfloor \frac{m+1}{3} \right\rfloor - 1 + k \\
& = 2 - 2k + 2m \left\lfloor \frac{k}{m-2 \left\lfloor \frac{m+1}{3} \right\rfloor} \right\rfloor + m - 4 \left\lfloor \frac{m+1}{3} \right\rfloor \left(\left\lfloor \frac{k}{m-2 \left\lfloor \frac{m+1}{3} \right\rfloor} \right\rfloor + 4 \left\lfloor \frac{m+1}{3} \right\rfloor \right) \\
& - 2 \left\lfloor \frac{k}{m-2 \left\lfloor \frac{m+1}{3} \right\rfloor} \right\rfloor (m-2 \left\lfloor \frac{m+1}{3} \right\rfloor) - 4 \left\lfloor \frac{m+1}{3} \right\rfloor - 2 + 2k
\end{aligned}$$

$$\begin{aligned}
&= m + \left\lceil \frac{k}{m-2 \lfloor \frac{m+1}{3} \rfloor} \right\rceil (2m-4 \lfloor \frac{m+1}{3} \rfloor - 2m+4 \lfloor \frac{m+1}{3} \rfloor) + 4 \lfloor \frac{m+1}{3} \rfloor - 4 \lfloor \frac{m+1}{3} \rfloor \\
&= m
\end{aligned}$$

$$\text{For } i = 2 \left\lceil \frac{k}{m-2 \lfloor \frac{m+1}{3} \rfloor} \right\rceil + 2, \dots, n-1; j = 1, 2, \dots, m$$

$$\begin{aligned}
&Wt_{\phi(1,1,0)}(f_{i+1}^j) - Wt_{\phi(1,1,0)}(f_i^j) = \\
&= \phi(v_{i+2}^j) + \phi(v_{i+2}^{j+1}) + \phi(v_{i+1}^j v_{i+2}^j) + \phi(v_{i+2}^j v_{i+2}^{j+1}) + \phi(v_{i+1}^{j+1} v_{i+2}^{j+1}) - \phi(v_i^j) - \phi(v_i^{j+1}) - \phi(v_i^j v_i^{j+1}) \\
&- \phi(v_i^j v_{i+1}^j) - \phi(v_i^{j+1} v_{i+1}^{j+1}) \\
&= \left\lfloor \frac{j}{2} \right\rfloor + \left\lceil \frac{j+1}{2} \right\rceil + 6 - 6k + 2m \left\lceil \frac{k}{m-2 \lfloor \frac{m+1}{3} \rfloor} \right\rceil + \left\lceil \frac{1}{2}(i+1-2 \left\lceil \frac{k}{m-2 \lfloor \frac{m+1}{3} \rfloor} \right\rceil - 1) \right\rceil (\lfloor \frac{m}{2} \rfloor + \lceil \frac{m}{2} \rceil) \\
&+ \left\lceil \frac{1}{2}(i+1-2 \left\lceil \frac{k}{m-2 \lfloor \frac{m+1}{3} \rfloor} \right\rceil - 1) \right\rceil (\lceil \frac{m}{2} \rceil + \lfloor \frac{m}{2} \rfloor) + k - k - \left\lfloor \frac{j}{2} \right\rfloor - \left\lceil \frac{j+1}{2} \right\rceil - 6 + 6k \\
&- 2m \left\lceil \frac{k}{m-2 \lfloor \frac{m+1}{3} \rfloor} \right\rceil - \left\lceil \frac{1}{2}(i-2 \left\lceil \frac{k}{m-2 \lfloor \frac{m+1}{3} \rfloor} \right\rceil - 1) \right\rceil (\lfloor \frac{m}{2} \rfloor + \lceil \frac{m}{2} \rceil) \\
&- \left\lceil \frac{1}{2}(i-2 \left\lceil \frac{k}{m-2 \lfloor \frac{m+1}{3} \rfloor} \right\rceil - 1) \right\rceil (\lceil \frac{m}{2} \rceil + \lfloor \frac{m}{2} \rfloor) \\
&= m \left\lceil \frac{1}{2}(i+1-2 \left\lceil \frac{k}{m-2 \lfloor \frac{m+1}{3} \rfloor} \right\rceil - 1) \right\rceil + m \left\lceil \frac{1}{2}(i+1-2 \left\lceil \frac{k}{m-2 \lfloor \frac{m+1}{3} \rfloor} \right\rceil - 1) \right\rceil \\
&- m \left\lceil \frac{1}{2}(i-2 \left\lceil \frac{k}{m-2 \lfloor \frac{m+1}{3} \rfloor} \right\rceil - 1) \right\rceil - m \left\lceil \frac{1}{2}(i-2 \left\lceil \frac{k}{m-2 \lfloor \frac{m+1}{3} \rfloor} \right\rceil - 1) \right\rceil \\
&= m \left\lceil \frac{1}{2}(i-2 \left\lceil \frac{k}{m-2 \lfloor \frac{m+1}{3} \rfloor} \right\rceil) \right\rceil + m \left\lceil \frac{1}{2}(i-2 \left\lceil \frac{k}{m-2 \lfloor \frac{m+1}{3} \rfloor} \right\rceil) \right\rceil \\
&- m \left\lceil \frac{1}{2}(i-2 \left\lceil \frac{k}{m-2 \lfloor \frac{m+1}{3} \rfloor} \right\rceil - 1) \right\rceil - m \left\lceil \frac{1}{2}(i-2 \left\lceil \frac{k}{m-2 \lfloor \frac{m+1}{3} \rfloor} \right\rceil - 1) \right\rceil \\
&= m \left\{ \left\lceil \frac{(i-2 \left\lceil \frac{k}{m-2 \lfloor \frac{m+1}{3} \rfloor} \right\rceil)}{2} \right\rceil + \left\lceil \frac{(i-2 \left\lceil \frac{k}{m-2 \lfloor \frac{m+1}{3} \rfloor} \right\rceil)}{2} \right\rceil \right\} \\
&- m \left\{ \left\lceil \frac{(i-2 \left\lceil \frac{k}{m-2 \lfloor \frac{m+1}{3} \rfloor} \right\rceil - 1)}{2} \right\rceil + \left\lceil \frac{(i-2 \left\lceil \frac{k}{m-2 \lfloor \frac{m+1}{3} \rfloor} \right\rceil - 1)}{2} \right\rceil \right\} \\
&= m(i-2 \left\lceil \frac{k}{m-2 \lfloor \frac{m+1}{3} \rfloor} \right\rceil) - m(i-2 \left\lceil \frac{k}{m-2 \lfloor \frac{m+1}{3} \rfloor} \right\rceil - 1) \\
&= m(i-2 \left\lceil \frac{k}{m-2 \lfloor \frac{m+1}{3} \rfloor} \right\rceil) - i + 2 \left\lceil \frac{k}{m-2 \lfloor \frac{m+1}{3} \rfloor} \right\rceil + 1) \\
&= m
\end{aligned}$$

□

4 Application (ABHE CHECK KERNI HE)

The very famous problem of finding shortest path in Operational Research can be solved with the help of graph labeling of type (α, β, γ) . The objects whose distance is to found will be named as vertices, their connection with each other will be named as edges and the area between objects will be considered as face. By this way, the face will also be labelled and we will be able to get more accurate results [?].

5 Conclusion (ABHE NEW LIKHNA HE)

In this paper, we have applied new graph parameters (α, β, γ) on grid graphs, where grid graph G_n^m with n rows and m columns is the graph cartesian product of path graphs $P_{n+1} \square P_{m+1}$. To determine the face irregularity strength of type (α, β, γ) of planar graphs is a very complicated task. Five problems are explained in this research which are followed by the k -labeling of grid graphs. We investigated the tight lower bound for the face irregular strength of vertex $(1, 0, 0)$, edge $(0, 1, 0)$, face $(0, 0, 1)$, vertex-face $(1, 0, 1)$ and edge-face $(0, 1, 1)$ of grid graphs G_n^m with the help of labelings of type (α, β, γ) where $\alpha, \beta \in \{0, 1\}$ and their formulas are generalized to prove the results.

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