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Fermatean fuzzy linguistic weighted averaging/geometric operators based on modified operational laws and their application in multiple attribute group decision-making

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Abstract

Fermatean fuzzy linguistic (FFL) set theory provides an efficient tool for modeling a higher level of uncertain and imprecise information, which cannot be represented using intuitionistic fuzzy linguistic (IFL)/Pythagorean fuzzy linguistic (PFL) sets. On the other hand, the linguistic scale function is the better way to consider the semantics of the linguistic terms during the evaluation process. In the present paper, we first define some new modified operational laws for Fermatean fuzzy linguistic numbers (FFLNs) based on linguistic scale function (LSF) to overcome the shortcomings of the existing operational laws and prove some important mathematical properties of them. Based on it, the work defines several new aggregation operators (AOs), namely, the FFL-weighted averaging (FFLWA) operator, the FFL-weighted geometric (FFLWG) operator, the FFL-ordered weighted averaging (FFLOWA) operator, the FFL-ordered weighted geometric (FFLOWG) operator, the FFL-hybrid averaging (FFLHA) operator and the FFL-hybrid geometric (FFLHG) operator under FFL environment. Several properties of these AOs are investigated in detail. Further, based on these operators, a multiple attribute group decision-making (MAGDM) approach with FFL information is developed. Finally, to illustrate the effectiveness of the present approach, a real-life supplier selection problem is presented where the evaluation information of the alternatives is given in terms of FFLNs.

Keywords: Fermatean fuzzy set, Fermatean fuzzy linguistic set, Fermatean fuzzy linguistic number, MAGDM, supplier selection

1. Introduction

The intuitionistic fuzzy set (IFS) theory was introduced by Atanassov [1] in 1983 to accommodate the uncertain and vague concepts more precisely in complex real-life situations. An IFS assigns each element a degree of membership (DM) and a degree of non-membership (DNM), whose sum is always less than or equal to one. It has become an important and widely studied generalization of fuzzy sets[2]. Due to the applicability and effectiveness of the IFS theory, several researchers started work in this direction and established many significant results. For aggregating different intuitionistic fuzzy numbers (IFNs), a large number of AOs have been defined by considering various aspects of available information [3–13]. Besides, several information measures have been proposed under an IF environment, including distance measure [14–20], similarity measure [21–25], entropy measure [26–31], divergence measure [27, 28, 32, 33], and inaccuracy measure [34] and applied them in different application areas.

In 2013, Yager [35] and Yager and Abbasov [36] proposed the notion of the Pythagorean fuzzy set (PFS) as a new generalization to IFS. PFSs are more effective in modeling imperfect or vague information, which cannot be represented in terms of IFSs. For example: suppose an expert provides the DM of an alternative corresponding to a criterion as 0.8 and the DNM as 0.5. As we see, the sum of both degrees is 1.3, which does not satisfy the essential condition of IFS. Further, if we consider the sum of the squares of both the degrees, i.e., $0.8^2+0.5^2$, that gives $0.89 <1$, hence this information can be represented in the form of PFS, not in IFS. In a short span, the PFS theory has become an efficient tool to solve various real-life problems [37–46].

Let us consider the above-discussed example again by assuming the DM as 0.9 and the DNM as 0.6. It is clear that we do not express this information by using IFS and PFS. To cope with this problem, Senapati and Yager[47] proposed the concept of Fermatean fuzzy set (FFS), where the DM and DNM are both real numbers lies between 0 and 1 and satisfied the condition $0\leq(DM)^3 +(DNM)^3\leq1$. The main advantage of the FFS is that it provides a better tool over IFS and PFS for handling the higher level of uncertainties arising in many real-life decision-making problems. It is easy to verify that $0.9^3+0.6^3<1$, hence FFS is an appropriate tool to capture this

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uncertain information. Further, Senapati and Yager[48] defined some operations on FFSs and discussed their application in decision-making. To aggregate different Fermatean fuzzy numbers (FFNs), Senapati and Yager[49] developed some weighted averaging/geometric AOs and utilized them to solve decision-making problems with multiple criteria.

In many real-life situations, due to the increase of complexities and uncertainties in practical decision problems, an expert feels difficulty expressing his/her preference information by exact numerical values. Besides, many attributes and criteria can be evaluated quickly and effectively in terms of linguistic values. Firstly, Zadeh [50–52] developed the idea of the linguistic term set (LTS) in 1975. For example- suppose an expert assesses the performance of a motorbike, then he/she may be used good, excellent, etc., to express his/her evaluation information because linguistic terms (LTs) are very close to human cognition. In 2010, Wang and Li[53] developed a hybrid set theory by combining the notions of LTS and IFS in a single formulation, which is known as the intuitionistic linguistic fuzzy sets (ILFS). In the literature, several research studies have been conducted under the ILF environment. Liu[54] proposed some generalized dependent AOs with intuitionistic linguistic fuzzy numbers (ILNs) and studied their application in decision-making. Liu and Wang[55] defined some intuitionistic linguistic generalized power aggregation operators. Su et al.[56] studied ordered weighted distance averaging operators with ILF information. Yu et al.[57] presented an extended TODIM method for solving MAGDM problems with ILNs.

Recently, Liu et al.[58] generalized the notion of ILFSs and introduced the Fermatean fuzzy linguistic set (FFLS) theory by integrating the idea of LTS with FFS. Besides, a MCDM approach was formulated for solving decision problems with FFL information. Further, Liu et al.[59] defined some new distance and similarity measures between FFLSs based on linguistic scale function (LSF) and utilized them in the development of TODIM and TOPSIS methods. FFLS theory has a broader scope of applications in different practical areas. However, a limited investigation has been conducted on FFLSs and their applications. It is also worth noting that the operational laws defined by Liu et al.[59] for FFLNs are not valid in general. Therefore, it is significant to pay attention to the research studies under the FFL environment. The main objective of this work is to define the modified operational laws for Fermatean fuzzy linguistic numbers (FFLNs) and study different AOs based on them to aggregate FFL information. For doing so, firstly, the work defines some new modified operational laws for FFLNs based on LSF, which overcome the drawbacks of the existing operational laws. We also study several essential properties of the proposed modified operational laws. Then, the paper develops several new AOs such as the FFL-weighted averaging (FFLWA) operator, the FFL-weighted geometric (FFLWG) operator, the FFL-ordered weighted averaging (FFLOWA) operator, the FFL-ordered weighted geometric (FFLOWG) operator, the FFL-hybrid averaging (FFLHA) operator and the FFL-hybrid geometric (FFLHG) operator for aggregating different FFLNs. Several properties of the proposed AOs are discussed and proved. Further, a decision-making approach is formulated to solve MAGDM problems under the FFL environment.

The rest of the manuscript is organized as follows: In Section 2 we briefly review some preliminary results on linguistic variables (LVs), LSFs, FFS, FFLS and discuss some significant drawbacks of the FFL operational laws defined by Liu et al.[59]. Section 3 presents modified algebraic operational laws for FFLNs based on LSF and proves several important properties of FFLNs using proposed operation laws. Then, we define the FFLWA, FFLWG, FFLOWA, FFLOWG, FFLHA, and FFLHG AOs to aggregate different FFLNs. In Section 4 based on the developed AOs, a MAGDM approach is formulated for solving real-life decision problems with FFL information. Then, a real-life supplier selection problem is given to illustrate the decision-making steps and effectiveness of the developed approach. In Section 5 we conclude the paper and discuss some future works.

2. Preliminaries

2.1: Linguistic Variables

The linguistic variable provides a useful tool to represent qualitative information in terms of linguistic values. According to Herrera and Martínez [60], the linguistic variable can be defined as follows:

Definition 1[60]**:** Let $\hat{L} = \{\ell_d \mid d = 0,1\ldots,2t\}$ be a totally ordered discrete LTS with the odd cardinality. Any level ℓ_d denotes a possible value for a linguistic variable and t is a positive integer. The LTS should meet the following properties:

i. $\ell_i \le \ell_j \Leftrightarrow i \le j$;

ii. $neg(\ell_d) = \ell_{2t-d}$;

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iii. $\max(\ell_i,\ell_j)=\ell_i \Leftrightarrow i \geq j$; iv. $\min(\ell_i,\ell_j)=\ell_i \Leftrightarrow i \leq j$;

where *neg* denotes the negation operator.

For example, a well-known set of seven linguistic terms can be defined as:

$\hat{L}=\{\ell_0=\mathrm{N}(\text{none}),\ell_1=\mathrm{VL}(\text{very low}),\ell_2=\mathrm{L}(\text{low}),\ell_3=\mathrm{M}(\text{medium}),\ell_4=\mathrm{H}(\text{ high }),\ell_5=\mathrm{VH}(\text{very high}),\ell_6=\mathrm{P}(\text{perfect})\}$.

Further, Xu[61] defined the extended continuous LTS $\hat{L}_{[0,2t]}=\{\ell_d \mid \ell_0 \leq \ell_d \leq \ell_{2t}, d\in[0,2t]\}$, where, if $\ell_d \in \hat{L}$, then ℓ_d is called the original linguistic term (OLT), otherwise ℓ_d is called the virtual linguistic term (VLT). However, $\ell_d \in \hat{L}$ is usually used by the decision-makers to evaluate attributes/alternatives while $\ell_d \in \hat{L}_{[0,2t]}$ only appears in the calculation process.

Definition 2[61]**:** Let $\ell_\alpha,\ell_\beta \in \hat{L}_{[0,2t]}$ and $\lambda,\lambda_1,\lambda_2 \in [0,1]$, then some operational laws are given as follows

(i). $\ell_\alpha \oplus \ell_\beta = \ell_{\alpha+\beta}$;

(ii). $\ell_\alpha \otimes \ell_\beta = \ell_{\alpha\times\beta}$;

(iii). $\lambda\ell_\alpha = \ell_{\lambda\alpha}$;

(iv). $\lambda(\ell_\alpha \oplus \ell_\beta)=\lambda\ell_\alpha \oplus \lambda\ell_\beta$;

(v). $(\lambda_1+\lambda_2)\ell_\alpha = \lambda_1\ell_\alpha \oplus \lambda_2\ell_\alpha$.

2.2: Linguistic scale function

In the evaluation process, an expert uses LTs directly rather than their corresponding semantics. In general, the simplest way to deal with LTs is to use the levels of LTs directly. However, in different semantics decision-making environments, LTs have some differences in expressing evaluations. To resolve these issues, Wang et al.[62] defined the LSF to deal with linguistic information. According to the decision-making environment, experts can choose different linguistic scale functions, which express available linguistic information more flexibly and precisely in different semantic situations.

Definition 3[62]**:** Let $\hat{L}=\{\ell_d \mid d=0,1,2,\ldots,2t\}$ be a discrete LTS with the odd cardinality and $\kappa_d \in [0,1]$ be a real number, then the LSF φ can be defined as

$$\varphi:\ell_d \to \kappa_d,\ d=0,1,2,\ldots,2t., \tag{1}$$

where φ is a strictly monotonically increasing function with respect to subscript d.

In general, there are three different linguistic scaling functions, given as

***LSF*1**[63]**:** *When the semantics of linguistic terms are uniformly (balanced) distributed, i.e., the absolute semantic gap (ASG) between any adjacent LTs is always equal.*

$$\varphi_1(\ell_d)=\kappa_d=\frac{d}{2t},\ d=0,1,2,\ldots,2t. \tag{2}$$

***LSF* 2**[63]**:** *When the ASG between two semantics of the adjacent LTs increases with the extension from ℓ_t to both ends of LTS.*

$$\varphi_2(\ell_d)=\kappa_d=\begin{cases}\dfrac{\theta^t-\theta^{t-d}}{2(\theta^t-1)}, & d=0,1,2,\ldots,t,\\[2ex] \dfrac{\theta^t+\theta^{d-t}-2}{2(\theta^t-1)}, & d=t+1,t+2,\ldots,2t,\end{cases} \tag{3}$$

where θ is a threshold, which can be determined by a subjective method according to the specific problem, and it should be greater than or equal to 1. If the LTS is a set of seven terms, then $\theta\in[1.37,1.40]$.

***LSF* 3**[63]**:** *When the ASG between two semantics of the adjacent LTs decreases with the extension from ℓ_t to both ends of LTS.*

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$$\varphi_3(\ell_d)=\kappa_d=\begin{cases}\dfrac{t^{\rho}-(t-d)^{\rho}}{2t^{\rho}}, & d=0,1,2,\ldots,t,\\ \dfrac{t^{\tau}+(d-t)^{\tau}}{2t^{\tau}}, & d=t+1,t+2,\ldots,2t,\end{cases} \tag{4}$$

where $\rho,\tau\in[0,1]$ are determined according to the specific problem. If the LTS is a set of seven terms, then $\rho=\tau=0.8$.

Example 1: Let $\hat{L}=\left\{\begin{array}{l}\ell_0=\mathrm{N}(\text{none}),\ell_1=\mathrm{VL}(\text{very low}),\ell_2=\mathrm{L}(\text{low}),\ell_3=\mathrm{M}(\text{medium}),\\ \ell_4=\mathrm{H}(\text{high}),\ell_5=\mathrm{VH}(\text{very high}),\ell_6=\mathrm{P}(\text{perfect})\end{array}\right\}$ be a LTS with seven terms. Figs. 1, 2, and 3 show the balanced distribution of $\hat{L}$, unbalanced distribution of $\hat{L}$ in an increasing trend and the unbalanced distribution of $\hat{L}$ in a decreasing trend, respectively. Besides, Fig. 4 represents the relationships between LTs of $\hat{L}$ and their corresponding semantics under different situations.

Fig 1: The uniformly distributed linguistic terms set

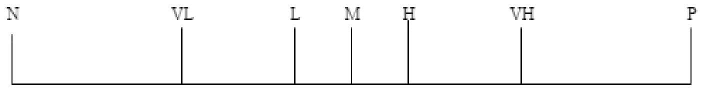

Fig 2: The semantics of the unbalanced distributed LTS in ASG increasing trend

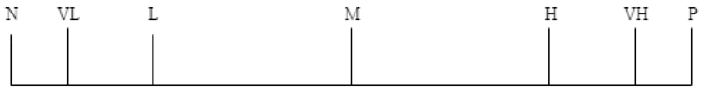

Fig 3: The unbalanced distributed LTS in ASG decreasing trend

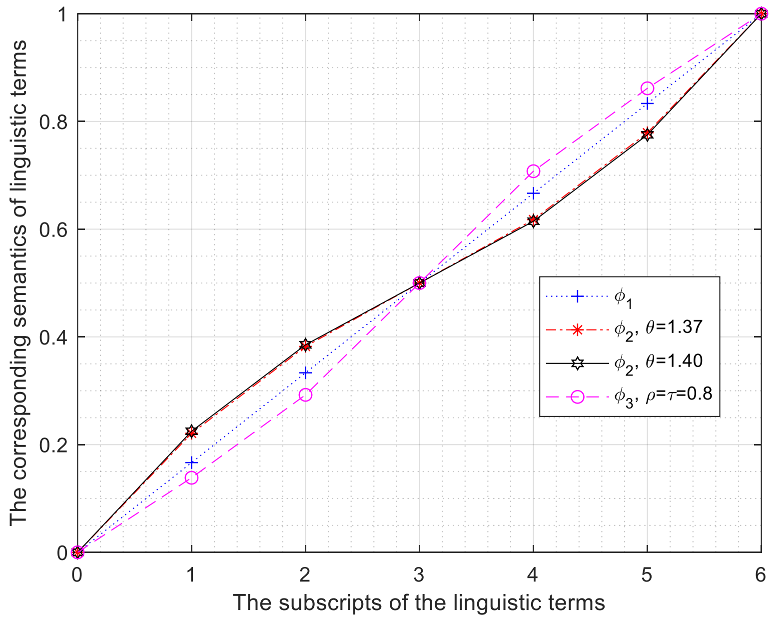

Fig 4: The relationships between LTs and their corresponding semantics under different situations

Meanwhile, to avoid an information loss and to facilitate the calculation process, the LSF φ can be further generalized to an extended continuous LTS as follows:

Definition 4[62]**:** Let $\hat{L}_{[0,2t]}=\{\ell_d \mid \ell_0 \le \ell_d \le \ell_{2t}, d\in[0,2t]\}$ be an extended continuous LTS and $\kappa_d \in[0,1]$ be a real number, then the linguistic scale function (LSF) φ^* is defined as

$$\varphi^*:\hat{L}_{[0,2t]}\to\kappa_d, \tag{5}$$

where φ^* is also a strictly monotonically increasing function, and its inverse is expressed as φ^{*-1}.

Example 2: Let $\hat{L}_{[0,6]}=\{\ell_d \mid d\in[0,6]\}$ be a continuous LTS, then the inverse corresponding to the LSFs φ_1^*, φ_2^* and φ_3^* can be obtained as follows:

(1) If $\varphi_1^*(\ell_d)=\kappa_d=\frac{d}{6}$ $(d=[0,6])$, then $\varphi_1^{*-1}(\kappa_d)=\ell_{6\times\kappa_d}$ $(\kappa_d\in[0,1])$.

(2) If $\varphi_2^*(\ell_d)=\kappa_d=\begin{cases}\dfrac{\theta^3-\theta^{3-d}}{2(\theta^3-1)}, & 0\le d\le 3\\ \dfrac{\theta^3+\theta^{d-3}-2}{2(\theta^3-1)}, & 3<d\le 6\end{cases}$, then $\varphi_2^{*-1}(\kappa_d)=\begin{cases}\ell_{3-\log_\theta\left[\theta^3-(2\theta^3-2)\kappa_d\right]}, & \kappa_d\in[0,0.5],\\ \ell_{3+\log_\theta\left[(2\theta^3-2)\kappa_d-\theta^3+2\right]}, & \kappa_d\in(0.5,1].\end{cases}$

(3) If $\varphi_3^*(\ell_d)=\kappa_d=\begin{cases}\dfrac{3^\rho-(3-d)^\rho}{2\times3^\rho}, & 0\le d\le 3\\ \dfrac{3^\tau+(d-3)^\tau}{2\times3^\tau}, & 3<d\le 6\end{cases}$, then $\varphi_3^{*-1}(\kappa_d)=\begin{cases}\ell_{3-\left[3^\rho-2\times3^\rho\times\kappa_d\right]^{1/\rho}}, & \kappa_d\in[0,0.5],\\ \ell_{3+\left[2\times3^\tau\times\kappa_d-3^\tau\right]^{1/\tau}}, & \kappa_d\in(0.5,1].\end{cases}$

2.3: Fermatean fuzzy linguistic set

Definition 5[47]**:** A FFS F in a fixed set $X=\{x_1,x_2,\ldots,x_n\}$ is given by

$$F=\left\{\left\langle x_j,\xi_F(x_j),\psi_F(x_j)\right\rangle \mid x_j\in X\right\}, \tag{6}$$

where $\xi_F(x_j)$ and $\psi_F(x_j)$ denote, respectively, the DM and DNM of $x_j\in X$ to the set F, with the conditions

$$\xi_F: X\to[0,1],\ \psi_F: X\to[0,1] \text{ and } 0\le(\xi_F(x_j))^3+(\psi_F(x_j))^3\le1\ \forall\, x\in X.$$

For all $x_j\in X$, the corresponding degree of hesitancy (DH) is defined as $\zeta_F(x_j)=\sqrt[3]{1-(\xi_F(x_j))^3-(\psi_F(x_j))^3}$. In the interest of simplicity, Senapati and Yager [47] called the pair $\left\langle\xi_F(x_j),\psi_F(x_j)\right\rangle$ a FFLN and denoted by $\alpha=\langle\xi_\alpha,\psi_\alpha\rangle$, which satisfies the conditions $\xi_\alpha\in[0,1],\psi_\alpha\in[0,1]$ and $0\le(\xi_\alpha)^3+(\psi_\alpha)^3\le1$.

Definition 6[58]**:** Let $X=\{x_1,x_2,\ldots,x_n\}$ be a fixed set and $\hat{L}_{[0,2t]}=\{\ell_d \mid \ell_0\le\ell_d\le\ell_{2t}, d\in[0,2t]\}$ be an extended continuous LTS, then a FFLS can be defined as

$$F=\left\{\left\langle x_j,\ell_{\sigma_F(x_j)},\xi_F(x_j),\psi_F(x_j)\right\rangle \mid x_j\in X\right\}, \tag{7}$$

where $\ell_{\sigma_F(x_j)}\in\hat{L}_{[0,2t]}$, $\xi_F: X\to[0,1]$ and $\psi_F: X\to[0,1]$, satisfying $0\le(\xi_F(x_j))^3+(\psi_F(x_j))^3\le1\ \forall\, x_j\in X$. The numbers $\xi_F(x_j)$ and $\psi_F(x_j)$ represent, respectively, the DM and DNM of $x_j\in X$ to the linguistic term $\ell_{\sigma_F(x_j)}$. For all $x_j\in X$, if $\zeta_F(x_j)=\sqrt[3]{1-(\xi_F(x_j))^3-(\psi_F(x_j))^3}$, then $\zeta_F(x_j)$ is called the DH of $x_j\in X$ to $\ell_{\sigma_F(x_j)}$.

Note that when $\xi_F(x_j)=1$ and $\psi_F(x_j)=0\ \forall\, x_j\in X$, the FFLS reduces to the LTS. In particular, when X has only one element, the FFLS is reduced into $\left\langle\ell_{\sigma_F(x)},\xi_F(x),\psi_F(x)\right\rangle$. For convenience, the triplet $\left\langle\ell_{\sigma_F(x)},\xi_F(x),\psi_F(x)\right\rangle$ is

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called a FFLN and simply denoted by $\wp = \left\langle \ell_{\sigma(\wp)}, \xi_\wp, \psi_\wp \right\rangle$, which meets the conditions $\xi_\wp \in [0,1], \psi_\wp \in [0,1]$ and $0 \le (\xi_\wp)^3 + (\psi_\wp)^3 \le 1$. We indicate the collection of all FFLNs by Ω.

Definition 7[59]**:** Let $\hat{L}_{[0,2t]}$ be an extended continuous LTS, $\wp = \left\langle \ell_{\sigma(\wp)}, \xi_\wp, \psi_\wp \right\rangle$, $\wp_1 = \left\langle \ell_{\sigma(\wp_1)}, \xi_{\wp_1}, \psi_{\wp_1} \right\rangle$ and $\wp_2 = \left\langle \ell_{\sigma(\wp_2)}, \xi_{\wp_2}, \psi_{\wp_2} \right\rangle$ be any three FFLNs, where $\ell_{\sigma(\wp)}, \ell_{\sigma(\wp_1)}, \ell_{\sigma(\wp_2)} \in \hat{L}_{[0,2t]}$. Further, consider that φ^* and φ^{*-1} denote a linguistic scale function and its inverse function, respectively. Then, by using the LSF, some algebraic operational laws on FFLNs were defined by Liu et al. [59] as follows:

(i). $\wp_1 \oplus \wp_2 = \left\langle \varphi^{*-1}\left(\varphi^*\left(\ell_{\sigma(\wp_1)}\right) + \varphi^*\left(\ell_{\sigma(\wp_2)}\right)\right), \sqrt[3]{\xi_{\wp_1}^3 + \xi_{\wp_2}^3 - \xi_{\wp_1}^3 \xi_{\wp_2}^3}, \psi_{\wp_1}\psi_{\wp_2} \right\rangle$;

(ii). $\wp_1 \otimes \wp_2 = \left\langle \varphi^{*-1}\left(\varphi^*\left(\ell_{\sigma(\wp_1)}\right)\varphi^*\left(\ell_{\sigma(\wp_2)}\right)\right), \xi_{\wp_1}\xi_{\wp_2}, \sqrt[3]{\psi_{\wp_1}^3 + \psi_{\wp_2}^3 - \psi_{\wp_1}^3 \psi_{\wp_2}^3} \right\rangle$;

(iii). $\lambda\wp = \left\langle \varphi^{*-1}\left(\lambda\varphi^*\left(\ell_{\sigma(\wp)}\right)\right), \sqrt[3]{1-\left(1-\xi_\wp^3\right)^\lambda}, \left(\psi_\wp\right)^\lambda \right\rangle$, $\lambda \ge 0$;

(iv). $\wp^\lambda = \left\langle \varphi^{*-1}\left(\left(\varphi^*\left(\ell_{\sigma(\wp)}\right)\right)^\lambda\right), \left(\xi_\wp\right)^\lambda, \sqrt[3]{1-\left(1-\psi_\wp^3\right)^\lambda} \right\rangle$, $\lambda \ge 0$;

(v). $neg(\wp) = \left\langle \varphi^{*-1}\left(\varphi^*\left(\ell_{2t}\right) - \varphi^*\left(\ell_{\sigma(\wp)}\right)\right), \psi_\wp, \xi_\wp \right\rangle$.

Definition 8[59]**:** Let $\wp = \left\langle \ell_{\sigma(\wp)}, \xi_\wp, \psi_\wp \right\rangle$ be a FFLN and φ^* be a LSF, the score and accuracy functions of $\wp$ are defined as

$$\mathfrak{S}(\wp) = \varphi^*\left(\ell_{\sigma(\wp)}\right) \times \left(\frac{\xi_\wp^3 + 1 - \psi_\wp^3}{2}\right) \quad \text{and} \quad \mathfrak{A}(\wp) = \varphi^*\left(\ell_{\sigma(\wp)}\right) \times \left(\xi_\wp^3 + \psi_\wp^3\right). \tag{8}$$

For any two FFLNs $\wp_1 = \left\langle \ell_{\sigma(\wp_1)}, \xi_{\wp_1}, \psi_{\wp_1} \right\rangle$ and $\wp_2 = \left\langle \ell_{\sigma(\wp_2)}, \xi_{\wp_2}, \psi_{\wp_2} \right\rangle$, the comparison rules between $\wp_1$ and $\wp_2$ are given as

(i). If $\mathfrak{S}(\wp_1) > \mathfrak{S}(\wp_2)$, then $\wp_1 \succ \wp_2$;

(ii). If $\mathfrak{S}(\wp_1) = \mathfrak{S}(\wp_2)$, then: (a) $\mathfrak{A}(\wp_1) > \mathfrak{A}(\wp_2)$, then $\wp_1 \succ \wp_2$; (b) $\mathfrak{A}(\wp_1) = \mathfrak{A}(\wp_2)$, then $\wp_1 = \wp_2$.

<u>Some shortcomings of the operational laws given in Definition 7:</u>

Here, we consider a numerical example in order to show the shortcomings of the operations on FFLNs defined by Liu et al.[59].

Example 3: Let $\hat{L}_{[0,6]} = \left\{\ell_d \mid d \in [0,6]\right\}$ be an extended continuous LTS, $\wp_1 = \langle \ell_3, 0.3, 0.6\rangle$, $\wp_2 = \langle \ell_5, 0.5, 0.7\rangle$ $\wp_3 = \langle \ell_1, 0, 0.5\rangle$, $\wp_4 = \langle \ell_3, 0, 0.7\rangle$, $\wp_5 = \langle \ell_4, 0.8, 0\rangle$ and $\wp_6 = \langle \ell_6, 0.6, 0\rangle$ be six FFLNs. If $\varphi^*\left(\ell_{\sigma(a)}\right) = \varphi_2^*\left(\ell_{\sigma(a)}\right)(\theta = 1.4)$ and $\lambda = 4$, then according to the operational laws given in Definition 7, we have

(i). $\wp_1 \oplus \wp_2 = \left\langle \varphi_2^{*-1}\left(\varphi_2^*(\ell_3) + \varphi_2^*(\ell_5)\right), \sqrt[3]{0.3^3 + 0.5^3 - 0.3^3 0.5^3}, 0.6 \times 0.7 \right\rangle$

$$= \left\langle \varphi_2^{*-1}(0.5000 + 0.7752), 0.5279, 0.4200 \right\rangle = \left\langle \varphi_2^{*-1}(1.2752), 0.5279, 0.4200 \right\rangle.$$

Here, we see that $\varphi_2^*(\ell_3) + \varphi_2^*(\ell_5) = 1.2752 > 1$, therefore, $\varphi_2^{*-1}\left(\varphi_2^*(\ell_3) + \varphi_2^*(\ell_5)\right) = \varphi_2^{*-1}(1.2752)$ is undefined.

(ii). $\wp_1 \otimes \wp_3 = \left\langle \varphi_2^{*-1}\left(\varphi_2^*(\ell_3)\varphi_2^*(\ell_1)\right), 0.3 \times 0, \sqrt[3]{0.6^3 + 0.5^3 - 0.6^3 \times 0.5^3} \right\rangle$

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$$=\left\langle \varphi_2^{*-1}\left(0.5000\times 0.2248\right),0.0000,0.6797\right\rangle=\left\langle \ell_{0.4619},0.0000,0.6797\right\rangle, \tag{9}$$

and

$$\wp_1\otimes\wp_4=\left\langle \varphi_2^{*-1}\left(\varphi_2^{*}\left(\ell_3\right)\varphi_2^{*}\left(\ell_3\right)\right),0.3\times 0,\sqrt[3]{0.6^3+0.7^3-0.6^3\times 0.7^3}\right\rangle$$

$$=\left\langle \varphi_2^{*-1}\left(0.5000+0.5000\right),0.0000,0.7856\right\rangle=\left\langle \ell_6,0.0000,0.7856\right\rangle. \tag{10}$$

From Eq. (9) and (10), it is clear that there is no effect of nonmembership values onto the membership values of $\wp_1\otimes\wp_3$ and $\wp_1\otimes\wp_4$. This outcome does not match our intuition.

(iii). $$\wp_2\oplus\wp_5=\left\langle \varphi_2^{*-1}\left(\varphi_2^{*}\left(\ell_5\right)+\varphi_2^{*}\left(\ell_4\right)\right),\sqrt[3]{0.5^3+0.8^3-0.5^3\times 0.8^3},0.7\times 0,\right\rangle$$

$$=\left\langle \varphi_2^{*-1}\left(0.7752+0.6147\right),0.8306,0.0000\right\rangle=\left\langle \varphi_2^{*-1}\left(1.3899\right),0.8306,0.0000\right\rangle, \tag{11}$$

and

$$\wp_2\oplus\wp_6=\left\langle \varphi_2^{*-1}\left(\varphi_2^{*}\left(\ell_5\right)+\varphi_2^{*}\left(\ell_6\right)\right),\sqrt[3]{0.5^3+0.6^3-0.5^3\times 0.6^3},0.7\times 0\right\rangle$$

$$=\left\langle \varphi_2^{*-1}\left(0.7752+1.0000\right),0.6797,0.0000\right\rangle=\left\langle \varphi_2^{*-1}\left(1.7752\right),0.6797,0.0000\right\rangle. \tag{12}$$

The obtained resulting values in Eq. (11) and Eq. (12) indicate that there is no effect of the membership values on the nonmembership values of $\wp_1\otimes\wp_3$ and $\wp_1\otimes\wp_4$. Also, $\varphi_2^{*-1}(1.3899)$ and $\varphi_2^{*-1}(1.7752)$ are undefined.

(iv) $$4\wp_1=\left\langle \varphi_2^{*-1}\left(4\times\varphi_2^{*}\left(\ell_3\right)\right),\sqrt[3]{1-\left(1-0.3^3\right)^4},\left(0.6\right)^4\right\rangle=\left\langle \varphi_2^{*-1}\left(2.0000\right),0.4698,0.1296\right\rangle, \tag{13}$$

$$4\wp_2=\left\langle \varphi_2^{*-1}\left(4\times\varphi_2^{*}\left(\ell_5\right)\right),\sqrt[3]{1-\left(1-0.5^3\right)^4},\left(0.7\right)^4\right\rangle=\left\langle \varphi_2^{*-1}\left(3.1008\right),0.7452,0.2401\right\rangle, \tag{14}$$

From Eq. (13) and Eq. (14), we can see that $\varphi_2^{*-1}(2.0000)$ and $\varphi_2^{*-1}(3.1008)$ are undefined because here $\kappa_d>1$. Hence $4\wp_1$ and $4\wp_2$ are not FFLNs.

Based on the above analysis, we conclude that the operational laws defined in Definition 7 are not suitable for FFLNs. Therefore, in order to nullify the above shortcomings, it is necessary to modify these operational laws. In the next section, we first define some new modified operational laws for FFLNs based on LSF and discuss their properties in detail. Then, we introduce some aggregation operators for aggregating different FFLNs.

3. Fermatean fuzzy aggregation operators

3.1: *Improved operational laws for FFLNs based on LSF*

Here, we define some improved operational laws for FFLNs, which overcome the shortcomings of the existing operations.

Definition 9: Let $\hat{L}_{[0,2t]}$ be an extended continuous LTS, $\wp=\left\langle \ell_{\sigma(\wp)},\xi_{\wp},\psi_{\wp}\right\rangle$, $\wp_1=\left\langle \ell_{\sigma(\wp_1)},\xi_{\wp_1},\psi_{\wp_1}\right\rangle$ and $\wp_2=\left\langle s_{\sigma(\wp_2)},\xi_{\wp_2},\psi_{\wp_2}\right\rangle$ be three FFLNs, where $\ell_{\sigma(\wp)},\ell_{\sigma(\wp_1)},\ell_{\sigma(\wp_2)}\in\hat{L}_{[0,2t]}$. Further, consider that φ^{*} and φ^{*-1} denote a linguistic scale function and its inverse function, respectively. The improved operational laws between them based on LSFs are defined as

(i). $$\wp_1\oplus\wp_2=\left\langle \varphi^{*-1}\left(\varphi^{*}\left(\ell_{\sigma(\wp_1)}\right)+\varphi^{*}\left(\ell_{\sigma(\wp_2)}\right)-\varphi^{*}\left(\ell_{\sigma(\wp_1)}\right)\varphi^{*}\left(\ell_{\sigma(\wp_2)}\right)\right),\sqrt[3]{\xi_{\wp_1}^3+\xi_{\wp_2}^3-\xi_{\wp_1}^3\xi_{\wp_2}^3},\ \sqrt[3]{\psi_{\wp_1}^3+\psi_{\wp_2}^3-\psi_{\wp_1}^3\psi_{\wp_2}^3-\psi_{\wp_1}^3\xi_{\wp_2}^3-\xi_{\wp_1}^3\psi_{\wp_2}^3}\right\rangle$$

$$=\left\langle \varphi^{*-1}\left(1-\left(1-\varphi^{*}\left(\ell_{\sigma(\wp_1)}\right)\right)\left(1-\varphi^{*}\left(\ell_{\sigma(\wp_2)}\right)\right)\right),\sqrt[3]{1-\left(1-\xi_{\wp_1}^3\right)\left(1-\xi_{\wp_2}^3\right)},\ \sqrt[3]{\left(1-\xi_{\wp_1}^3\right)\left(1-\xi_{\wp_2}^3\right)-\left(1-\left(\xi_{\wp_1}^3+\psi_{\wp_1}^3\right)\right)\left(1-\left(\xi_{\wp_2}^3+\psi_{\wp_2}^3\right)\right)}\right\rangle;$$

(ii). $\wp_1 \otimes \wp_2 = \left\langle \varphi^{*-1}\left(\varphi^*\left(\ell_{\sigma(\wp_1)}\right)\varphi^*\left(\ell_{\sigma(\wp_2)}\right)\right), \sqrt[3]{\xi_{\wp_1}^3+\xi_{\wp_2}^3-\xi_{\wp_1}^3\xi_{\wp_2}^3-\xi_{\wp_1}^3\psi_{\wp_2}^3-\psi_{\wp_1}^3\xi_{\wp_2}^3}, \sqrt[3]{\psi_{\wp_1}^3+\psi_{\wp_2}^3-\psi_{\wp_1}^3\psi_{\wp_2}^3}\right\rangle$

$$= \left\langle \varphi^{*-1}\left(\varphi^*\left(\ell_{\sigma(\wp_1)}\right)\varphi^*\left(\ell_{\sigma(\wp_2)}\right)\right), \sqrt[3]{\left(1-\psi_{\wp_1}^3\right)\left(1-\psi_{\wp_2}^3\right)-\left(1-\left(\xi_{\wp_1}^3+\psi_{\wp_1}^3\right)\right)\left(1-\left(\xi_{\wp_2}^3+\psi_{\wp_2}^3\right)\right)}, \sqrt[3]{1-\left(1-\psi_{\wp_1}^3\right)\left(1-\psi_{\wp_2}^3\right)}\right\rangle;$$

(iii). $\lambda\tilde{*}\wp = \left\langle \varphi^{*-1}\left(1-\left(1-\varphi^*\left(\ell_{\sigma(\wp)}\right)\right)^\lambda\right), \sqrt[3]{1-\left(1-\xi_\wp^3\right)^\lambda}, \sqrt[3]{\left(1-\xi_\wp^3\right)^\lambda-\left(1-\left(\xi_\wp^3+\psi_\wp^3\right)\right)^\lambda}\right\rangle, \ \lambda>0;$

(iv). $\wp^\wedge\lambda = \left\langle \varphi^{*-1}\left(\left(\varphi^*\left(\ell_{\sigma(\wp)}\right)\right)^\lambda\right), \sqrt[3]{\left(1-\psi_\wp^3\right)^\lambda-\left(1-\left(\xi_\wp^3+\psi_\wp^3\right)\right)^\lambda}, \sqrt[3]{1-\left(1-\psi_\wp^3\right)^\lambda}\right\rangle, \ \lambda>0.$

Theorem 1: The numbers $\wp_1\oplus\wp_2$, $\wp_1\otimes\wp_2$, $\lambda\tilde{*}\wp$, and $\wp^\wedge\lambda$ are also FFLNs.

Proof: Here, we shall prove only $\wp_1\oplus\wp_2$ and $\lambda\tilde{*}\wp$ are FFLNs, while others can be shown similarly. Since $\wp_i = \left\langle \ell_{\sigma(\wp_i)}, \xi_{\wp_i}, \psi_{\wp_i}\right\rangle (i=1,2)$ are two FFLNs, where $\ell_{\sigma(\wp_i)} \in \hat{L}_{[0,2t]}$, $\xi_{\wp_i}, \psi_{\wp_i} \in [0,1]$ and $0 \le \xi_{\wp_i}^3 + \psi_{\wp_i}^3 \le 1$, $i=1,2$. For $\ell_{\sigma(\wp_1)}, \ell_{\sigma(\wp_2)} \in \hat{L}_{[0,2t]}$, based on the definition of the LSFs, we know $0 \le \varphi^*\left(\ell_{\sigma(\wp_1)}\right), \varphi^*\left(\ell_{\sigma(\wp_2)}\right) \le 1$. Then $0 \le 1-\left(1-\varphi^*\left(\ell_{\sigma(\wp_1)}\right)\right)\left(1-\varphi^*\left(\ell_{\sigma(\wp_2)}\right)\right) \le 1 \Rightarrow \varphi^{*-1}\left(1-\left(1-\varphi^*\left(\ell_{\sigma(\wp_1)}\right)\right)\left(1-\varphi^*\left(\ell_{\sigma(\wp_2)}\right)\right)\right) \in \hat{L}_{[0,2t]}$. Now $0 \le \xi_{\wp_1}, \xi_{\wp_2} \le 1$, which implies $0 \le \left(1-\xi_{\wp_1}^3\right)\left(1-\xi_{\wp_2}^3\right) \le 1 \Leftrightarrow 0 \le \sqrt[3]{1-\left(1-\xi_{\wp_1}^3\right)\left(1-\xi_{\wp_2}^3\right)} \le 1$. Moreover, because $1-\xi_{\wp_1}^3 \ge 1-\xi_{\wp_1}^3-\psi_{\wp_1}^3$ and $1-\xi_{\wp_2}^3 \ge 1-\xi_{\wp_2}^3-\psi_{\wp_2}^3$, then $0 \le \sqrt[3]{\left(1-\xi_{\wp_1}^3\right)\left(1-\xi_{\wp_2}^3\right)-\left(1-\left(\xi_{\wp_1}^3+\psi_{\wp_1}^3\right)\right)\left(1-\left(\xi_{\wp_2}^3+\psi_{\wp_2}^3\right)\right)} \le 1$.

Further $\left(\sqrt[3]{1-\begin{pmatrix}\left(1-\xi_{\wp_1}^3\right)\\ \left(1-\xi_{\wp_2}^3\right)\end{pmatrix}}\right)^3 + \left(\sqrt[3]{\begin{pmatrix}\left(1-\xi_{\wp_1}^3\right)\\ \left(1-\xi_{\wp_2}^3\right)\end{pmatrix}-\begin{pmatrix}\left(1-\left(\xi_{\wp_1}^3+\psi_{\wp_1}^3\right)\right)\\ \left(1-\left(\xi_{\wp_2}^3+\psi_{\wp_2}^3\right)\right)\end{pmatrix}}\right)^3 = 1-\left(1-\left(\xi_{\wp_1}^3+\psi_{\wp_1}^3\right)\right)\left(1-\left(\xi_{\wp_2}^3+\psi_{\wp_2}^3\right)\right) \le 1$. Thus $a_1\oplus a_2$ is a FFLN.

For any $\lambda>0$, $0 \le 1-\left(1-\varphi^*\left(\ell_{\sigma(\wp)}\right)\right)^\lambda \le 1$, which gives $\varphi^{*-1}\left(1-\left(1-\varphi^*\left(\ell_{\sigma(\wp)}\right)\right)^\lambda\right) \in \hat{L}_{[0,2t]}$. Also, $0 \le \xi_{\wp_1}, \xi_{\wp_2}, \psi_{\wp_1}, \psi_{\wp_2} \le 1$, which implies $0 \le \sqrt[3]{1-\left(1-\xi_\wp^3\right)^\lambda} \le 1$ and $0 \le \sqrt[3]{\left(1-\xi_\wp^3\right)^\lambda-\left(1-\left(\xi_\wp^3+\psi_\wp^3\right)\right)^\lambda} \le 1$. Further $\left(\sqrt[3]{1-\left(1-\xi_\wp^3\right)^\lambda}\right)^3 + \left(\sqrt[3]{\left(1-\xi_\wp^3\right)^\lambda-\left(1-\left(\xi_\wp^3+\psi_\wp^3\right)\right)^\lambda}\right)^3 = 1-\left(1-\left(\xi_\wp^3+\psi_\wp^3\right)\right)^\lambda \le 1$. Hence $\lambda\tilde{*}\wp$ is a FFLN.

This completes the proof. ■

Example 4: Let $\hat{L}_{[0,6]} = \left\{\ell_d \mid d \in [0,6]\right\}$ be an extended continuous LTS, $\wp = \langle \ell_2, 0.4, 0.5\rangle$, $\wp_1 = \langle \ell_3, 0.3, 0.6\rangle$, $\wp_2 = \langle \ell_5, 0.5, 0.7\rangle$ be three FFLNs and $\lambda = 5$. Then according to the modified operation laws, we obtained the following results as shown in Table 1:

Table 1: Values of different operations

Operation	$\varphi^*\left(\ell_{\sigma(\wp)}\right)=\varphi_1^*\left(\ell_{\sigma(\wp)}\right)$	$\varphi^*\left(\ell_{\sigma(\wp)}\right)=\varphi_2^*\left(\ell_{\sigma(\wp)}\right)$ and $\theta=1.4$	$\varphi^*\left(\ell_{\sigma(\wp)}\right)=\varphi_3^*\left(\ell_{\sigma(\wp)}\right)$ and $\rho=\tau=0.8$
$\wp_1\oplus\wp_2$	$\langle \ell_{5.4996}, 0.5297, 0.7655\rangle$	$\langle \ell_{5.5418}, 0.5297, 0.7655\rangle$	$\langle \ell_{5.4896}, 0.5297, 0.7655\rangle$
$\wp_1\otimes\wp_2$	$\langle \ell_{2.5002}, 0.4826, 0.7856\rangle$	$\langle \ell_{5.5418}, 0.4826, 0.7856\rangle$	$\langle \ell_{5.4896}, 0.4826, 0.7856\rangle$
$\lambda\tilde{*}\wp$	$\langle \ell_{5.2098}, 0.6149, 0.6945\rangle$	$\langle \ell_{5.6440}, 0.6149, 0.6945\rangle$	$\langle \ell_{4.7348}, 0.6149, 0.6945\rangle$
$\wp^\wedge\lambda$	$\langle \ell_{0.0246}, 0.5355, 0.7452\rangle$	$\langle \ell_{0.0323}, 0.5355, 0.7452\rangle$	$\langle \ell_{0.0157}, 0.5355, 0.7452\rangle$

Further, if we consider Example 3 again and utilize the improved operational laws summarized in Definition 9, Table 2 presents the obtained results.

Table 2: Calculation results of different operations

Operation	$\varphi^*\left(\ell_{\sigma(\wp)}\right)=\varphi_2^*\left(\ell_{\sigma(\wp)}\right)$ and $\theta=1.4$
$\wp_1\oplus\wp_2$	$\langle\ell_{5.5418},0.5279,0.7655\rangle$
$\wp_1\otimes\wp_3$	$\langle\ell_{5.5418},0.2869,0.6797\rangle$
$\wp_1\otimes\wp_4$	$\langle\ell_{6.0000},0.2608,0.7856\rangle$
$\wp_2\oplus\wp_5$	$\langle\ell_{5.6534},0.8306,0.5541\rangle$
$\wp_2\oplus\wp_6$	$\langle\ell_{6.0000},0.6797,0.5541\rangle$
$4\tilde{*}\wp_1$	$\langle\ell_{5.7540},0.4698,0.8281\rangle$
$4\tilde{*}\wp_2$	$\langle\ell_{5.9902},0.7452,0.7969\rangle$

The obtained calculation results verify that the improved operational laws are more reasonable and realistic as per our intuition.

Theorem 2: Let $\hat{L}_{[0,2t]}$ be an extended continuous LTS, and $\wp_1=\left\langle\ell_{\sigma(\wp_1)},\xi_{\wp_1},\psi_{\wp_1}\right\rangle$, $\wp_2=\left\langle\ell_{\sigma(\wp_2)},\xi_{\wp_2},\psi_{\wp_2}\right\rangle$ and $\wp_3=\left\langle\ell_{\sigma(\wp_3)},\xi_{\wp_3},\psi_{\wp_3}\right\rangle$ be three FFLNs, where $\ell_{\sigma(\wp_1)},\ell_{\sigma(\wp_2)},\ell_{\sigma(\wp_3)}\in\hat{L}_{[0,2t]}$. The following results hold:

(i). $\wp_1\oplus\wp_2=\wp_2\oplus\wp_1$;

(ii). $\wp_1\otimes\wp_2=\wp_2\otimes\wp_1$;

(iii). $\left(\wp_1\oplus\wp_2\right)\oplus\wp_3=\wp_1\oplus\left(\wp_2\oplus\wp_3\right)$;

(iv). $\left(\wp_1\otimes\wp_2\right)\otimes\wp_3=\wp_1\otimes\left(\wp_2\otimes\wp_3\right)$.

Proof: Results follow directly from Definition 9, so we omit the proofs of them.

Theorem 3: Let $\hat{L}_{[0,2t]}$ be an extended continuous LTS, $\wp=\left\langle\ell_{\sigma(\wp)},\xi_{\wp},\psi_{\wp}\right\rangle$ $\wp_1=\left\langle\ell_{\sigma(\wp_1)},\xi_{\wp_1},\psi_{\wp_1}\right\rangle$, and $\wp_2=\left\langle\ell_{\sigma(\wp_2)},\xi_{\wp_2},\psi_{\wp_2}\right\rangle$ be three FFLNs and $\lambda,\lambda_1,\lambda_2>0$, where $\ell_{\sigma(\wp)},\ell_{\sigma(\wp_2)},\ell_{\sigma(\wp_2)}\in\hat{L}_{[0,2t]}$, then

(i). $\left(\lambda\tilde{*}\wp_1\right)\oplus\left(\lambda\tilde{*}\wp_2\right)=\lambda\tilde{*}\left(\wp_1\oplus\wp_2\right)$;

(ii). $\left(\lambda_1\tilde{*}\wp\right)\oplus\left(\lambda_2\,\tilde{*}\,\wp\right)=\left(\lambda_1+\lambda_2\right)\tilde{*}\wp$

(iii). $\left(\wp_1\wedge\lambda\right)\otimes\left(\wp_2\wedge\lambda\right)=\left(\wp_1\otimes\wp_2\right)\wedge\lambda$;

(iv). $\left(\wp\wedge\lambda_1\right)\otimes\left(\wp\wedge\lambda_2\right)=\wp\wedge\left(\lambda_1+\lambda_2\right)$;

(v). $\lambda_1\tilde{*}\left(\lambda_2\tilde{*}\wp\right)=\left(\lambda_1\lambda_2\right)\tilde{*}\wp$;

(vi). $\left(\wp\wedge\lambda_1\right)\wedge\lambda_2=\wp\wedge\left(\lambda_1\lambda_2\right)$.

(vii). $neg\left(\wp_1\oplus\wp_2\right)=neg\left(\wp_1\right)\otimes neg\left(\wp_2\right)$

(viii). $neg\left(\wp_1\otimes\wp_2\right)=neg\left(\wp_1\right)\oplus neg\left(\wp_2\right)$;

(ix). $\left(neg\left(\wp\right)\right)\wedge\lambda=neg\left(\lambda\tilde{*}\wp\right)$;

(x). $\lambda\tilde{*}\left(neg\left(\wp\right)\right)=neg\left(\wp\wedge\lambda\right)$.

Proof: Here, we only prove the parts (i), (iii), (v), (vii), and (ix); the others can be proved similarly.

(i) From Definition 9, we have

$$\lambda\tilde{*}\wp_1=\left\langle\varphi^{*-1}\left(1-\left(1-\varphi^*\left(\ell_{\sigma(\wp_1)}\right)\right)^{\lambda}\right),\sqrt[3]{1-\left(1-\xi_{\wp_1}^3\right)^{\lambda}},\sqrt[3]{\left(1-\xi_{\wp_1}^3\right)^{\lambda}-\left(1-\left(\xi_{\wp_1}^3+\psi_{\wp_1}^3\right)\right)^{\lambda}}\right\rangle, \tag{15}$$

and

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$$\lambda \tilde{*} \wp_2 = \left\langle \varphi^{*-1}\left(1-\left(1-\varphi^{*}\left(\ell_{\sigma(\wp_2)}\right)\right)^{\lambda}\right), \sqrt[3]{1-\left(1-\xi_{\wp_2}^{3}\right)^{\lambda}}, \sqrt[3]{\left(1-\xi_{\wp_2}^{3}\right)^{\lambda}-\left(1-\left(\xi_{\wp_2}^{3}+\psi_{\wp_2}^{3}\right)\right)^{\lambda}} \right\rangle, \tag{16}$$

Using Eq. (15) and Eq. (16), we get

$$\left(\lambda \tilde{*} \wp_1\right) \oplus \left(\lambda \tilde{*} \wp_2\right)$$

$$= \left\langle \begin{array}{l} \varphi^{*-1}\left(1-\left(1-\varphi^{*}\left(\varphi^{*-1}\left(1-\left(1-\varphi^{*}\left(\ell_{\sigma(\wp_1)}\right)\right)^{\lambda}\right)\right)\right)\left(1-\varphi^{*}\left(\varphi^{*-1}\left(1-\left(1-\varphi^{*}\left(\ell_{\sigma(\wp_2)}\right)\right)^{\lambda}\right)\right)\right)\right), \sqrt[3]{1-\left(1-\left(\sqrt[3]{1-\left(1-\xi_{\wp_1}^{3}\right)^{\lambda}}\right)^{3}\right)\left(1-\left(\sqrt[3]{1-\left(1-\xi_{\wp_1}^{3}\right)^{\lambda}}\right)^{3}\right)}, \\ \sqrt[3]{\begin{array}{l} \left(1-\left(\sqrt[3]{1-\left(1-\xi_{\wp_1}^{3}\right)^{\lambda}}\right)^{3}\right)\left(1-\left(\sqrt[3]{1-\left(1-\xi_{\wp_2}^{3}\right)^{\lambda}}\right)^{3}\right) \\ -\left(1-\left(\left(\sqrt[3]{1-\left(1-\xi_{\wp_1}^{3}\right)^{\lambda}}\right)^{3}+\left(\sqrt[3]{\left(1-\xi_{\wp_1}^{3}\right)^{\lambda}-\left(1-\left(\xi_{\wp_1}^{3}+\psi_{\wp_1}^{3}\right)\right)^{\lambda}}\right)^{3}\right)\right)\left(1-\left(\left(\sqrt[3]{1-\left(1-\xi_{\wp_2}^{3}\right)^{\lambda}}\right)^{3}+\left(\sqrt[3]{\left(1-\xi_{\wp_2}^{3}\right)^{\lambda}-\left(1-\left(\xi_{\wp_2}^{3}+\psi_{\wp_2}^{3}\right)\right)^{\lambda}}\right)^{3}\right)\right) \end{array}} \end{array} \right\rangle$$

$$= \left\langle \varphi^{*-1}\left(1-\left(\left(1-\varphi^{*}\left(\ell_{\sigma(\wp_1)}\right)\right)\left(1-\varphi^{*}\left(\ell_{\sigma(\wp_2)}\right)\right)\right)^{\lambda}\right), \sqrt[3]{1-\left(\left(1-\xi_{\wp_1}^{3}\right)\left(1-\xi_{\wp_2}^{3}\right)\right)^{\lambda}}, \sqrt[3]{\left(\left(1-\xi_{\wp_1}^{3}\right)\left(1-\xi_{\wp_2}^{3}\right)\right)^{\lambda}-\left(\left(1-\left(\xi_{\wp_1}^{3}+\psi_{\wp_1}^{3}\right)\right)\left(1-\left(\xi_{\wp_2}^{3}+\psi_{\wp_2}^{3}\right)\right)\right)^{\lambda}} \right\rangle$$

$$= \lambda \tilde{*} \left(\wp_1 \oplus \wp_2\right).$$

(iii) According to Definition 9, we have

$$\wp_1 {}^\wedge \lambda = \left\langle \varphi^{*-1}\left(\left(\varphi^{*}\left(\ell_{\sigma(\wp_1)}\right)\right)^{\lambda}\right), \sqrt[3]{\left(1-\psi_{\wp_1}^{3}\right)^{\lambda}-\left(1-\left(\xi_{\wp_1}^{3}+\psi_{\wp_1}^{3}\right)\right)^{\lambda}}, \sqrt[3]{1-\left(1-\psi_{\wp_1}^{3}\right)^{\lambda}} \right\rangle, \tag{17}$$

and

$$\wp_2 {}^\wedge \lambda = \left\langle \varphi^{*-1}\left(\left(\varphi^{*}\left(\ell_{\sigma(\wp_2)}\right)\right)^{\lambda}\right), \sqrt[3]{\left(1-\psi_{\wp_2}^{3}\right)^{\lambda}-\left(1-\left(\xi_{\wp_2}^{3}+\psi_{\wp_2}^{3}\right)\right)^{\lambda}}, \sqrt[3]{1-\left(1-\psi_{\wp_2}^{3}\right)^{\lambda}} \right\rangle. \tag{18}$$

By Eq. (17) and Eq. (18), we get

$$\left(\wp_1 {}^\wedge \lambda\right) \otimes \left(\wp_2 {}^\wedge \lambda\right) = \left\langle \begin{array}{c} \varphi^{*-1}\left(\varphi^{*}\left(\varphi^{*-1}\left(\left(\varphi^{*}\left(\ell_{\sigma(\wp_1)}\right)\right)^{\lambda}\right)\right)\varphi^{*}\left(\varphi^{*-1}\left(\left(\varphi^{*}\left(\ell_{\sigma(\wp_2)}\right)\right)^{\lambda}\right)\right)\right), \\ \sqrt[3]{\left(\begin{array}{l} \left(1-\left(\sqrt[3]{1-\left(1-\psi_{\wp_1}^{3}\right)^{\lambda}}\right)^{3}\right) \\ \left(1-\left(\sqrt[3]{1-\left(1-\psi_{\wp_2}^{3}\right)^{\lambda}}\right)^{3}\right) \end{array}\right) - \left(\begin{array}{l} \left(1-\left(\begin{array}{r} \left(\sqrt[3]{\left(1-\psi_{\wp_1}^{3}\right)^{\lambda}-\left(1-\left(\xi_{\wp_1}^{3}+\psi_{\wp_1}^{3}\right)\right)^{\lambda}}\right)^{3} \\ +\left(\sqrt[3]{1-\left(1-\psi_{\wp_1}^{3}\right)^{\lambda}}\right)^{3} \end{array}\right)\right) \\ \left(1-\left(\begin{array}{r} \left(\sqrt[3]{\left(1-\psi_{\wp_2}^{3}\right)^{\lambda}-\left(1-\left(\xi_{\wp_2}^{3}+\psi_{\wp_2}^{3}\right)\right)^{\lambda}}\right)^{3} \\ +\left(\sqrt[3]{1-\left(1-\psi_{\wp_2}^{3}\right)^{\lambda}}\right)^{3} \end{array}\right)\right) \end{array}\right)}, \\ \sqrt[3]{1-\left(1-\left(\sqrt[3]{1-\left(1-\psi_{\wp_1}^{3}\right)^{\lambda}}\right)^{3}\right)\left(1-\left(\sqrt[3]{1-\left(1-\psi_{\wp_2}^{3}\right)^{\lambda}}\right)^{3}\right)} \end{array} \right\rangle$$

$$= \left\langle \begin{array}{r} \varphi^{*-1}\left(\left(\varphi^{*}\left(\ell_{\sigma(\wp_1)}\right)\varphi^{*}\left(\ell_{\sigma(\wp_2)}\right)\right)^{\lambda}\right), \sqrt[3]{\left(\left(1-\psi_{\wp_1}^{3}\right)\left(1-\psi_{\wp_2}^{3}\right)\right)^{\lambda}-\left(\left(1-\left(\xi_{\wp_1}^{3}+\psi_{\wp_1}^{3}\right)\right)\left(1-\left(\xi_{\wp_2}^{3}+\psi_{\wp_2}^{3}\right)\right)\right)^{\lambda}}, \\ \sqrt[3]{1-\left(\left(1-\psi_{\wp_1}^{3}\right)\left(1-\psi_{\wp_2}^{3}\right)\right)^{\lambda}} \end{array} \right\rangle;$$

$$= \left(\wp_1 \otimes \wp_2\right) {}^\wedge \lambda.$$

(v) For two positive real numbers λ_1 and λ_2, we have

$$\lambda_1 \tilde{*} \left(\lambda_2 \tilde{*} \wp\right) = \lambda_1 \left(\left\langle \varphi^{*-1}\left(1-\left(1-\varphi^{*}\left(\ell_{\sigma(\wp)}\right)\right)^{\lambda_2}\right), \sqrt[3]{1-\left(1-\xi_{\wp}^{3}\right)^{\lambda_2}}, \sqrt[3]{\left(1-\xi_{\wp}^{3}\right)^{\lambda_2}-\left(1-\left(\xi_{\wp}^{3}+\psi_{\wp}^{3}\right)\right)^{\lambda_2}} \right\rangle \right),$$

$$=\left\langle \begin{array}{l} \varphi^{*-1}\left(1-\left(1-\varphi^{*}\left(\varphi^{*-1}\left(1-\left(1-\varphi^{*}\left(\ell_{\sigma(\wp)}\right)\right)^{\lambda_2}\right)\right)\right)^{\lambda_1}\right), \sqrt[3]{1-\left(1-\left(\sqrt[3]{1-\left(1-\xi_{\wp}^{3}\right)^{\lambda_2}}\right)^{3}\right)^{\lambda_1}}, \\ \sqrt[3]{\left(1-\left(\sqrt[3]{1-\left(1-\xi_{\wp}^{3}\right)^{\lambda_2}}\right)^{3}\right)^{\lambda_1}-\left(1-\left(\left(\sqrt[3]{1-\left(1-\xi_{\wp}^{3}\right)^{\lambda_2}}\right)^{3}+\left(\sqrt[3]{\left(1-\xi_{\wp}^{3}\right)^{\lambda_2}-\left(1-\left(\xi_{\wp}^{3}+\psi_{\wp}^{3}\right)\right)^{\lambda_2}}\right)^{3}\right)\right)^{\lambda_1}} \end{array} \right\rangle$$

$$=\left\langle \varphi^{*-1}\left(1-\left(1-\varphi^{*}\left(\ell_{\sigma(\wp)}\right)\right)^{\lambda_1\lambda_2}\right), \sqrt[3]{1-\left(1-\xi_{\wp}^{3}\right)^{\lambda_1\lambda_2}}, \sqrt[3]{\left(1-\xi_{\wp}^{3}\right)^{\lambda_1\lambda_2}-\left(1-\left(\xi_{\wp}^{3}+\psi_{\wp}^{3}\right)\right)^{\lambda_1\lambda_2}} \right\rangle$$

$$=\left(\lambda_1\lambda_2\right)\tilde{*}\wp .$$

(vii) From Definitions 7 and 9, we have

$$neg\left(\wp_1 \oplus \wp_2\right)=\left\langle \begin{array}{l} \varphi^{*-1}\left(\varphi^{*}\left(\ell_{2t}\right)-\varphi^{*}\left(\varphi^{*-1}\left(1-\left(1-\varphi^{*}\left(\ell_{\sigma(\wp_1)}\right)\right)\left(1-\varphi^{*}\left(\ell_{\sigma(\wp_2)}\right)\right)\right)\right)\right), \\ \sqrt[3]{\left(1-\xi_{\wp_1}^{3}\right)\left(1-\xi_{\wp_2}^{3}\right)-\left(1-\left(\xi_{\wp_1}^{3}+\psi_{\wp_1}^{3}\right)\right)\left(1-\left(\xi_{\wp_2}^{3}+\psi_{\wp_2}^{3}\right)\right)}, \sqrt[3]{1-\left(1-\xi_{\wp_1}^{3}\right)\left(1-\xi_{\wp_2}^{3}\right)} \end{array} \right\rangle$$

$$=\left\langle \begin{array}{l} \varphi^{*-1}\left(\varphi^{*}\left(\ell_{2t}\right)-\left(1-\left(1-\varphi^{*}\left(\ell_{\sigma(\wp_1)}\right)\right)\left(1-\varphi^{*}\left(\ell_{\sigma(\wp_2)}\right)\right)\right)\right), \\ \sqrt[3]{\left(1-\xi_{\wp_1}^{3}\right)\left(1-\xi_{\wp_2}^{3}\right)-\left(1-\left(\xi_{\wp_1}^{3}+\psi_{\wp_1}^{3}\right)\right)\left(1-\left(\xi_{\wp_2}^{3}+\psi_{\wp_2}^{3}\right)\right)}, \sqrt[3]{1-\left(1-\xi_{\wp_1}^{3}\right)\left(1-\xi_{\wp_2}^{3}\right)} \end{array} \right\rangle$$

$$=\left\langle \begin{array}{l} \varphi^{*-1}\left(\left(\varphi^{*}\left(\ell_{2t}\right)-\varphi^{*}\left(\ell_{\sigma(\wp_1)}\right)\right)\left(\varphi^{*}\left(\ell_{2t}\right)-\varphi^{*}\left(\ell_{\sigma(\wp_2)}\right)\right)\right), \\ \sqrt[3]{\left(1-\xi_{\wp_1}^{3}\right)\left(1-\xi_{\wp_2}^{3}\right)-\left(1-\left(\psi_{\wp_1}^{3}+\xi_{\wp_1}^{3}\right)\right)\left(1-\left(\psi_{\wp_2}^{3}+\xi_{\wp_2}^{3}\right)\right)}, \sqrt[3]{1-\left(1-\xi_{\wp_1}^{3}\right)\left(1-\xi_{\wp_2}^{3}\right)} \end{array} \right\rangle;$$

$$=neg\left(\wp_1\right)\otimes neg\left(\wp_2\right).$$

(ix) $\left(neg\left(\wp\right)\right)\wedge\lambda=\left(\left\langle \varphi^{*-1}\left(\varphi^{*}\left(\ell_{2t}\right)-\varphi^{*}\left(\ell_{\sigma(\wp)}\right)\right),\psi_{\wp},\xi_{\wp}\right\rangle\right)^{\lambda}$

$$=\left\langle \varphi^{*-1}\left(\left(\varphi^{*}\left(\ell_{2t}\right)-\varphi^{*}\left(\ell_{\sigma(\wp)}\right)\right)^{\lambda}\right), \sqrt[3]{\left(1-\xi_{\wp}^{3}\right)^{\lambda}-\left(1-\left(\xi_{\wp}^{3}+\psi_{\wp}^{3}\right)\right)^{\lambda}}, \sqrt[3]{1-\left(1-\xi_{\wp}^{3}\right)^{\lambda}} \right\rangle$$

$$=\left\langle \varphi^{*-1}\left(\varphi^{*}\left(\ell_{2t}\right)-\varphi^{*}\left(\varphi^{*-1}\left(1-\left(1-\varphi^{*}\left(\ell_{\sigma(\wp)}\right)\right)^{\lambda}\right)\right)\right), \sqrt[3]{\left(1-\xi_{\wp}^{3}\right)^{\lambda}-\left(1-\left(\xi_{\wp}^{3}+\psi_{\wp}^{3}\right)\right)^{\lambda}}, \sqrt[3]{1-\left(1-\xi_{\wp}^{3}\right)^{\lambda}} \right\rangle$$

$$=neg\left(\left\langle \varphi^{*-1}\left(1-\left(1-\varphi^{*}\left(\ell_{\sigma(\wp)}\right)\right)^{\lambda}\right), \sqrt[3]{1-\left(1-\xi_{\wp}^{3}\right)^{\lambda}}, \sqrt[3]{\left(1-\xi_{\wp}^{3}\right)^{\lambda}-\left(1-\left(\xi_{\wp}^{3}+\psi_{\wp}^{3}\right)\right)^{\lambda}} \right\rangle\right)$$

$$=neg\left(\lambda\tilde{*}\wp\right).$$

Hence, the theorem proved. ■

Next, by utilizing proposed improved operational laws on FFLNs, we propose some arithmetic and geometric aggregation operators for fusing a collection of FFLNs $\wp_i=\left\langle \ell_{\sigma(\wp_i)},\xi_{\wp_i},\psi_{\wp_i}\right\rangle (i=1,2,\ldots,n)$.

3.2: *FFL-weighted average (FFLWA) operator*

The weighted average (WA) is the most commonly used mean operator in a wide range of application areas. Here, we extend the idea of WA to the FFL information environment and propose the following formal definition.

Definition 10: Let $\wp_i=\left\langle \ell_{\sigma(\wp_i)},\xi_{\wp_i},\psi_{\wp_i}\right\rangle (i=1,2,\ldots,n)$ be a collection of FFLNs. The FFL-weighted average (FFLWA) operator is a mapping $\text{FFLWA}:\Omega^n\to\Omega$, such that

$$\text{FFLWA}(\wp_1,\wp_2,\ldots,\wp_n)=\mathop{\oplus}_{i=1}^{n}\left(w_i\tilde{*}\wp_i\right), \tag{19}$$

where $w=(w_1,w_2,\ldots,w_n)^T$ is the weight vector of $\wp_i$ with $w_i\in[0,1]$, $\sum_{i=1}^{n}w_i=1$. Especially when $w=\left(\frac{1}{n},\frac{1}{n},\ldots,\frac{1}{n}\right)^T$, the FFLWA operator reduces to FFL-average (FFLA) operator, which is defined as

$$\text{FFLWA}(\wp_1,\wp_2,\ldots,\wp_n)=\frac{1}{n}\tilde{*}\left(\mathop{\oplus}_{i=1}^{n}\wp_i\right). \tag{20}$$

Theorem 4: *Let* $\wp_i=\left\langle \ell_{\sigma(\wp_i)},\xi_{\wp_i},\psi_{\wp_i}\right\rangle (i=1,2,\ldots,n)$ *be a collection of* n *FFLNs and* $w=(w_1,w_2,\ldots,w_n)^T$ *be the weight vector of* $\wp_i$ *with* $w_i\in[0,1]$, $\sum_{i=1}^{n}w_i=1$, *then the aggregated value by using the FFLWA operator is also a FFLN and*

$$\text{FFLWA}(\wp_1,\wp_2,\ldots,\wp_n)=\left\langle \begin{array}{c}\varphi^{*-1}\left(1-\prod_{i=1}^{n}\left(1-\varphi^{*}\left(\ell_{\sigma(\wp_i)}\right)\right)^{w_i}\right),\sqrt[3]{1-\prod_{i=1}^{n}\left(1-\xi_{\wp_i}^3\right)^{w_i}},\\ \sqrt[3]{\prod_{i=1}^{n}\left(1-\xi_{\wp_i}^3\right)^{w_i}-\prod_{i=1}^{n}\left(1-\left(\xi_{\wp_i}^3+\psi_{\wp_i}^3\right)\right)^{w_i}}\end{array}\right\rangle. \tag{21}$$

Proof: The first result directly holds from Theorem 1. Using the principle of mathematical induction, we shall prove the result stated in Eq. (21). Firstly, for $n=2$, by Definition 9, we get

$$\left.\begin{array}{l} w_1\tilde{*}\wp_1=\left\langle \varphi^{*-1}\left(1-\left(1-\varphi^{*}\left(\ell_{\sigma(\wp_1)}\right)\right)^{w_1}\right),\sqrt[3]{1-\left(1-\xi_{\wp_1}^3\right)^{w_1}},\sqrt[3]{\left(1-\xi_{\wp_1}^3\right)^{w_1}-\left(1-\left(\xi_{\wp_1}^3+\psi_{\wp_1}^3\right)\right)^{w_1}}\right\rangle,\\ w_2\tilde{*}\wp_2=\left\langle \varphi^{*-1}\left(1-\left(1-\varphi^{*}\left(\ell_{\sigma(\wp_2)}\right)\right)^{w_2}\right),\sqrt[3]{1-\left(1-\xi_{\wp_2}^3\right)^{w_2}},\sqrt[3]{\left(1-\xi_{\wp_2}^3\right)^{w_2}-\left(1-\left(\xi_{\wp_2}^3+\psi_{\wp_2}^3\right)\right)^{w_2}}\right\rangle\end{array}\right\}. \tag{22}$$

Hence

$$\text{FFLWA}(\wp_1,\wp_2)=\left\langle \begin{array}{c}\varphi^{*-1}\left(1-\prod_{i=1}^{2}\left(1-\varphi^{*}\left(\ell_{\sigma(\wp_i)}\right)\right)^{w_i}\right),\sqrt[3]{1-\prod_{i=1}^{2}\left(1-\xi_{\wp_i}^3\right)^{w_i}},\\ \sqrt[3]{\prod_{i=1}^{2}\left(1-\xi_{\wp_i}^3\right)^{w_i}-\prod_{i=1}^{n}\left(1-\left(\xi_{\wp_i}^3+\psi_{\wp_i}^3\right)\right)^{w_i}}\end{array}\right\rangle. \tag{23}$$

Hence, the result is valid for $n=2$.

Next, assume that Eq. (21) is true for $n=k$, i.e.,

$$\text{FFLWA}(\wp_1,\wp_2,\ldots,\wp_k)=\left\langle \begin{array}{c}\varphi^{*-1}\left(1-\prod_{i=1}^{k}\left(1-\varphi^{*}\left(\ell_{\sigma(\wp_i)}\right)\right)^{w_i}\right),\sqrt[3]{1-\prod_{i=1}^{k}\left(1-\xi_{\wp_i}^3\right)^{w_i}},\\ \sqrt[3]{\prod_{i=1}^{k}\left(1-\xi_{\wp_i}^3\right)^{w_i}-\prod_{i=1}^{k}\left(1-\left(\xi_{\wp_i}^3+\psi_{\wp_i}^3\right)\right)^{w_i}}\end{array}\right\rangle. \tag{24}$$

Then, for $n=k+1$, by Definition 10, we have

$$\text{FFLWA}(\wp_1,\wp_2,\ldots,\wp_k,\wp_{k+1})=\text{FFLWA}(\wp_1,\wp_2,\ldots,\wp_k)\oplus\left(w_{k+1}\tilde{*}\wp_{k+1}\right)$$

$$=\left\langle \begin{array}{c}\varphi^{*-1}\left(1-\prod_{i=1}^{k}\left(1-\varphi^{*}\left(\ell_{\sigma(\wp_i)}\right)\right)^{w_i}\right),\sqrt[3]{1-\prod_{i=1}^{k}\left(1-\xi_{\wp_i}^3\right)^{w_i}},\\ \sqrt[3]{\prod_{i=1}^{k}\left(1-\xi_{\wp_i}^3\right)^{w_i}-\prod_{i=1}^{k}\left(1-\left(\xi_{\wp_i}^3+\psi_{\wp_i}^3\right)\right)^{w_i}}\end{array}\right\rangle\oplus\left\langle \ell_{\sigma(\wp_{k+1})},\xi_{\wp_{k+1}},\psi_{\wp_{k+1}}\right\rangle$$

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$$=\left\langle \varphi^{*-1}\left(1-\prod_{i=1}^{k}\left(1-\varphi^{*}\left(\ell_{\sigma(\wp_i)}\right)\right)^{w_i}\left(1-\varphi^{*}\left(\ell_{\sigma(\wp_{k+i})}\right)\right)^{w_{k+i}}\right), \sqrt[3]{1-\prod_{i=1}^{k}\left(1-\xi_{\wp_i}^3\right)^{w_i}\left(1-\xi_{\wp_{k+1}}^3\right)^{w_{k+1}}}, \sqrt[3]{\prod_{i=1}^{k}\left(1-\xi_{\wp_i}^3\right)^{w_i}\left(1-\xi_{\wp_{k+1}}^3\right)^{w_{k+1}}-\prod_{i=1}^{k}\left(1-\left(\xi_{\wp_i}^3+\psi_{\wp_i}^3\right)\right)^{w_i}\left(1-\left(\xi_{\wp_{k+1}}^3+\psi_{\wp_{k+1}}^3\right)\right)^{w_{k+1}}}\right\rangle$$

$$=\left\langle \varphi^{*-1}\left(1-\prod_{i=1}^{k+1}\left(1-\varphi^{*}\left(\ell_{\sigma(a_i)}\right)\right)^{w_i}\right), \sqrt[3]{1-\prod_{i=1}^{k+1}\left(1-\xi_{\wp_i}^3\right)^{w_i}}, \sqrt[3]{\prod_{i=1}^{k+1}\left(1-\xi_{\wp_i}^3\right)^{w_i}-\prod_{i=1}^{k+1}\left(1-\left(\xi_{\wp_i}^3+\psi_{\wp_i}^3\right)\right)^{w_i}}\right\rangle \quad (25)$$

i.e., Eq. (21) holds for $n=k+1$. Hence, the theorem. ■

Theorem 4: The FFLWA operator, defined in Eq. (21), holds the following properties:

($\wp$1)(Idempotency): If $\wp_i=\wp=\left\langle \ell_{\sigma(\wp)},\xi_{\wp},\psi_{\wp}\right\rangle$ $\forall i$, then

$$\text{FFLWA}\left(\wp_1,\wp_2,\ldots,\wp_n\right)=\wp. \quad (26)$$

($\wp$2) (Monotonicity): Let $\wp_i=\left\langle \ell_{\sigma(\wp_i)},\xi_{\wp_i},\psi_{\wp_i}\right\rangle$ and $\aleph_i=\left\langle \ell_{\sigma(\aleph_i)},\xi_{\aleph_i},\psi_{\aleph_i}\right\rangle (i=1,2,\ldots,n)$ be two collections of FFLNs such that $\ell_{\sigma(\wp_i)}\le\ell_{\sigma(\aleph_i)},\xi_{\wp_i}\le\xi_{\aleph_i},\psi_{\wp_i}\ge\psi_{\aleph_i}\ \forall\ i$, then

$$\text{FFLWA}\left(\wp_1,\wp_2,\ldots,\wp_n\right)\le\text{FFLWA}\left(\aleph_1,\aleph_2,\ldots,\aleph_n\right). \quad (27)$$

($\wp$3) (Boundedness): Let $\wp^-=\left\langle \min\left(\ell_{\sigma(\wp_1)},\ell_{\sigma(\wp_2)},\ldots,\ell_{\sigma(\wp_n)}\right),\min\left(\xi_{\wp_1},\xi_{\wp_2},\ldots,\xi_{\wp_n}\right),\max\left(\psi_{\wp_1},\psi_{\wp_2},\ldots,\psi_{\wp_n}\right)\right\rangle$ and $\wp^+=\left\langle \max\left(\ell_{\sigma(\wp_1)},\ell_{\sigma(\wp_2)},\ldots,\ell_{\sigma(\wp_n)}\right),\max\left(\xi_{\wp_1},\xi_{\wp_2},\ldots,\xi_{\wp_n}\right),\min\left(\psi_{\wp_1},\psi_{\wp_2},\ldots,\psi_{\wp_n}\right)\right\rangle$, then

$$\wp^-\le\text{FFLWA}\left(\wp_1,\wp_2,\ldots,\wp_n\right)\le\wp^+.$$

($\wp$4): If $\aleph=\left\langle s_{\sigma(\aleph)},\xi_{\aleph},\psi_{\aleph}\right\rangle$ is another FFLN, then

$$\text{FFLWA}\left(\wp_1\oplus\aleph,\wp_2\oplus\aleph,\ldots,\wp_n\oplus\aleph\right)=\text{FFLWA}\left(\wp_1,\wp_2,\ldots,\wp_n\right)\oplus\aleph. \quad (28)$$

($\wp$5): Let $\vartheta>0$ be a real number, then

$$\text{FFLWA}\left(\vartheta\tilde{*}\wp_1,\vartheta\tilde{*}\wp_2,\ldots,\vartheta\tilde{*}\wp_n\right)=\vartheta\tilde{*}\left(\text{FFLWA}\left(\wp_1,\wp_2,\ldots,\wp_n\right)\right). \quad (29)$$

($\wp$6): Let $\aleph=\left\langle s_{\sigma(\aleph)},\xi_{\aleph},\psi_{\aleph}\right\rangle$ be another FFLN and $\vartheta>0$ be a real number, then

$$\text{FFLWA}\left(\left(\vartheta\tilde{*}\wp_1\right)\oplus\aleph,\left(\vartheta\tilde{*}\wp_2\right)\oplus\aleph,\ldots,\left(\vartheta\tilde{*}\wp_n\right)\oplus\aleph\right)=\left(\vartheta\tilde{*}\left(\text{FFLWA}\left(\wp_1,\wp_2,\ldots,\wp_n\right)\right)\right)\oplus b. \quad (30)$$

($\wp$7): Let $\wp_i=\left\langle \ell_{\sigma(\wp_i)},\xi_{\wp_i},\psi_{\wp_i}\right\rangle$ and $\aleph_i=\left\langle s_{\sigma(\aleph_i)},\xi_{\aleph_i},\psi_{\aleph_i}\right\rangle$, $(i=1,2,\ldots,n)$ be two collections of FFLNs, then

$$\text{FFLWA}\left(\wp_1\oplus\aleph_1,\wp_2\oplus\aleph_2,\ldots,\wp_n\oplus\aleph_n\right)=\text{FFLWA}\left(\wp_1,\wp_2,\ldots,\wp_n\right)\oplus\text{FFLWA}\left(\aleph_1,\aleph_2,\ldots,\aleph_n\right). \quad (31)$$

Proof: (**$\wp$1**) Assume that $\wp_i=\wp=\left\langle \ell_{\sigma(\wp)},\xi_{\wp},\psi_{\wp}\right\rangle$ $\forall i$, then

$$\text{FFLWA}\left(\wp_1,\wp_2,\ldots,\wp_n\right)=\text{FFLWA}\left(\wp,\wp,\ldots,\wp\right)$$

$$=\left\langle \varphi^{*-1}\left(1-\prod_{i=1}^{n}\left(1-\varphi^{*}\left(\ell_{\sigma(\wp)}\right)\right)^{w_i}\right), \sqrt[3]{1-\prod_{i=1}^{n}\left(1-\xi_{\wp}^3\right)^{w_i}}, \sqrt[3]{\prod_{i=1}^{n}\left(1-\xi_{\wp}^3\right)^{w_i}-\prod_{i=1}^{n}\left(1-\left(\xi_{\wp}^3+\psi_{\wp}^3\right)\right)^{w_i}}\right\rangle$$

$$= \left\langle \begin{array}{r} \varphi^{*-1}\left(1-\left(1-\varphi^{*}\left(\ell_{\sigma(\wp)}\right)\right)^{w_1+w_2+\cdots+w_n}\right), \sqrt[3]{1-\left(1-\xi_{\wp}^{3}\right)^{w_1+w_2+\cdots+w_n}}, \\ \sqrt[3]{\left(1-\xi_{\wp}^{3}\right)^{w_1+w_2+\cdots+w_n}-\left(1-\left(\xi_{\wp}^{3}+\psi_{\wp}^{3}\right)\right)^{w_1+w_2+\cdots+w_n}} \end{array} \right\rangle$$

$$= \left\langle \ell_{\sigma(\wp)}, \xi_{\wp}, \psi_{\wp} \right\rangle = \wp. \qquad \blacksquare$$

(**𝓅2**) Since $\ell_{\sigma(\wp_i)} \le \ell_{\sigma(\aleph_i)}, \xi_{\wp_i} \le \xi_{\aleph_i}, \psi_{\wp_i} \ge \psi_{\aleph_i} \forall\, i$ and φ^{*} is a strictly monotonically increasing function, then

$$\left. \begin{array}{r} \varphi^{*-1}\left(1-\prod_{i=1}^{n}\left(1-\varphi^{*}\left(\ell_{\sigma(\wp_i)}\right)\right)^{w_i}\right) \le \varphi^{*-1}\left(1-\prod_{i=1}^{n}\left(1-\varphi^{*}\left(\ell_{\sigma(\aleph_i)}\right)\right)^{w_i}\right), \\ \sqrt[3]{1-\prod_{i=1}^{n}\left(1-\xi_{\wp_i}^{3}\right)^{w_i}} \le \sqrt[3]{1-\prod_{i=1}^{n}\left(1-\xi_{\aleph_i}^{3}\right)^{w_i}}, \\ \sqrt[3]{\prod_{i=1}^{n}\left(1-\xi_{\wp_i}^{3}\right)^{w_i}-\prod_{i=1}^{n}\left(1-\left(\xi_{\wp_i}^{3}+\psi_{\wp_i}^{3}\right)\right)^{w_i}} \ge \sqrt[3]{\prod_{i=1}^{n}\left(1-\xi_{\aleph_i}^{3}\right)^{w_i}-\prod_{i=1}^{n}\left(1-\left(\xi_{\aleph_i}^{3}+\psi_{\aleph_i}^{3}\right)\right)^{w_i}} \end{array} \right\}. \tag{32}$$

According to Definition 10, we have

$$\mathrm{FFLWA}\left(\wp_1,\wp_2,\ldots,\wp_n\right) = \left\langle \begin{array}{r} \varphi^{*-1}\left(1-\prod_{i=1}^{n}\left(1-\varphi^{*}\left(\ell_{\sigma(\wp_i)}\right)\right)^{w_i}\right), \sqrt[3]{1-\prod_{i=1}^{n}\left(1-\xi_{\wp_i}^{3}\right)^{w_i}}, \\ \sqrt[3]{\prod_{i=1}^{n}\left(1-\xi_{\wp_i}^{3}\right)^{w_i}-\prod_{i=1}^{n}\left(1-\left(\xi_{\wp_i}^{3}+\psi_{\wp_i}^{3}\right)\right)^{w_i}} \end{array} \right\rangle.$$

and

$$\mathrm{FFLWA}\left(\aleph_1,\aleph_2,\ldots,\aleph_n\right) = \left\langle \begin{array}{r} \varphi^{*-1}\left(1-\prod_{i=1}^{n}\left(1-\varphi^{*}\left(\ell_{\sigma(\aleph_i)}\right)\right)^{w_i}\right), \sqrt[3]{1-\prod_{i=1}^{n}\left(1-\xi_{\aleph_i}^{3}\right)^{w_i}}, \\ \sqrt[3]{\prod_{i=1}^{n}\left(1-\xi_{\aleph_i}^{3}\right)^{w_i}-\prod_{i=1}^{n}\left(1-\left(\xi_{\aleph_i}^{3}+\psi_{\aleph_i}^{3}\right)\right)^{w_i}} \end{array} \right\rangle.$$

Therefore

$$\mathrm{FFLWA}\left(\wp_1,\wp_2,\ldots,\wp_n\right) \le \mathrm{FFLWA}\left(\aleph_1,\aleph_2,\ldots,\aleph_n\right). \qquad \blacksquare$$

(**𝓅3**) It directly follows from Property 2. ■

(**𝓅4**) Since, so

$$\wp_i \oplus \aleph = \left\langle \begin{array}{r} \varphi^{*-1}\left(1-\left(1-\varphi^{*}\left(\ell_{\sigma(\wp_i)}\right)\right)\left(1-\varphi^{*}\left(\ell_{\sigma(\aleph)}\right)\right)\right), \sqrt[3]{1-\left(1-\xi_{\wp_i}^{3}\right)\left(1-\xi_{\aleph}^{3}\right)}, \\ \sqrt[3]{\left(1-\xi_{\wp_i}^{3}\right)\left(1-\xi_{\aleph}^{3}\right)-\left(1-\left(\xi_{\wp_i}^{3}+\psi_{\wp_i}^{3}\right)\right)\left(1-\left(\xi_{\aleph}^{3}+\psi_{\aleph}^{3}\right)\right)} \end{array} \right\rangle. \tag{33}$$

Therefore,

$$\mathrm{FFLWA}\left(\wp_1\oplus\aleph,\wp_2\oplus\aleph,\ldots,\wp_n\oplus\aleph\right)$$

$$= \left\langle \begin{array}{r} \varphi^{*-1}\left(1-\prod_{i=1}^{n}\left(\left(1-\varphi^{*}\left(\ell_{\sigma(\wp_i)}\right)\right)\left(1-\varphi^{*}\left(\ell_{\sigma(\aleph)}\right)\right)\right)^{w_i}\right), \sqrt[3]{1-\prod_{i=1}^{n}\left(\left(1-\xi_{\wp_i}^{3}\right)\left(1-\xi_{\aleph}^{3}\right)\right)^{w_i}}, \\ \sqrt[3]{\prod_{i=1}^{n}\left(\left(1-\xi_{\wp_i}^{3}\right)\left(1-\xi_{\aleph}^{3}\right)\right)^{w_i}-\prod_{i=1}^{n}\left(\left(1-\left(\xi_{\wp_i}^{3}+\psi_{\wp_i}^{3}\right)\right)\left(1-\left(\xi_{\aleph}^{3}+\psi_{\aleph}^{3}\right)\right)\right)^{w_i}} \end{array} \right\rangle$$

$$= \left\langle \begin{array}{r} \varphi^{*-1}\left(1-\left(\prod_{i=1}^{n}\left(\left(1-\varphi^{*}\left(\ell_{\sigma(\wp_i)}\right)\right)\right)^{w_i}\right)\left(1-\varphi^{*}\left(\ell_{\sigma(\aleph)}\right)\right)\right), \sqrt[3]{1-\left(\prod_{i=1}^{n}\left(1-\xi_{\wp_i}^{3}\right)^{w_i}\right)\left(1-\xi_{\aleph}^{3}\right)}, \\ \sqrt[3]{\left(\prod_{i=1}^{n}\left(1-\xi_{\wp_i}^{3}\right)^{w_i}\right)\left(1-\xi_{\aleph}^{3}\right)-\left(\prod_{i=1}^{n}\left(1-\left(\xi_{\wp_i}^{3}+\psi_{\wp_i}^{3}\right)\right)^{w_i}\right)\left(1-\left(\xi_{\aleph}^{3}+\psi_{\aleph}^{3}\right)\right)} \end{array} \right\rangle$$

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$$=\left\langle \varphi^{*-1}\left(1-\prod_{i=1}^{n}\left(1-\varphi^{*}\left(\ell_{\sigma(\wp_i)}\right)\right)^{w_i}\right), \sqrt[3]{1-\prod_{i=1}^{n}\left(1-\xi_{\wp_i}^{3}\right)^{w_i}}, \sqrt[3]{\prod_{i=1}^{n}\left(1-\xi_{\wp_i}^{3}\right)^{w_i}-\prod_{i=1}^{n}\left(1-\left(\xi_{\wp_i}^{3}+\psi_{\wp_i}^{3}\right)\right)^{w_i}} \right\rangle \oplus \left\langle \ell_{\sigma(\aleph)}, \xi_{\aleph}, \psi_{\aleph} \right\rangle$$

$$=\mathrm{FFLWA}\left(\wp_1,\wp_2,\ldots,\wp_n\right)\oplus\aleph. \qquad \blacksquare$$

(**℘5**) For any $\vartheta>0$, we have

$$\vartheta\tilde{*}\wp_i=\left\langle \varphi^{*-1}\left(1-\left(1-\varphi^{*}\left(\ell_{\sigma(\wp_i)}\right)\right)^{\vartheta}\right), \sqrt[3]{1-\left(1-\xi_{\wp_i}^{3}\right)^{\vartheta}}, \sqrt[3]{\left(1-\xi_{\wp_i}^{3}\right)^{\vartheta}-\left(1-\left(\xi_{\wp_i}^{3}+\psi_{\wp_i}^{3}\right)\right)^{\vartheta}} \right\rangle. \tag{34}$$

Therefore,

$$\mathrm{FFLWA}\left(\vartheta\tilde{*}\wp_1,\vartheta\tilde{*}\wp_2,\ldots,\vartheta\tilde{*}\wp_n\right)=\left\langle \varphi^{*-1}\left(1-\prod_{i=1}^{n}\left(\left(1-\varphi^{*}\left(\ell_{\sigma(\wp_i)}\right)\right)^{\vartheta}\right)^{w_i}\right), \sqrt[3]{1-\prod_{i=1}^{n}\left(\left(1-\xi_{\wp_i}^{3}\right)^{\vartheta}\right)^{w_i}}, \sqrt[3]{\prod_{i=1}^{n}\left(\left(1-\xi_{\wp_i}^{3}\right)^{\vartheta}\right)^{w_i}-\prod_{i=1}^{n}\left(\left(1-\left(\xi_{\wp_i}^{3}+\psi_{\wp_i}^{3}\right)\right)^{\vartheta}\right)^{w_i}} \right\rangle$$

$$=\left\langle \varphi^{*-1}\left(1-\prod_{i=1}^{n}\left(\left(1-\varphi^{*}\left(\ell_{\sigma(\wp_i)}\right)\right)^{w_i}\right)^{\vartheta}\right), \sqrt[3]{1-\prod_{i=1}^{n}\left(\left(1-\xi_{\wp_i}^{3}\right)^{w_i}\right)^{\vartheta}}, \sqrt[3]{\prod_{i=1}^{n}\left(\left(1-\xi_{\wp_i}^{3}\right)^{w_i}\right)^{\vartheta}-\prod_{i=1}^{n}\left(\left(1-\left(\xi_{\wp_i}^{3}+\psi_{\wp_i}^{3}\right)\right)^{w_i}\right)^{\vartheta}} \right\rangle$$

$$=\vartheta\tilde{*}\left(\mathrm{FFLWA}\left(\wp_1,\wp_2,\ldots,\wp_n\right)\right). \qquad \blacksquare$$

(**℘6**) From Property 4, we know

$$\mathrm{FFLWA}\left(\wp_1\oplus\aleph,\wp_2\oplus\aleph,\ldots,\wp_n\oplus\aleph\right)=\mathrm{FFLWA}\left(\wp_1,\wp_2,\ldots,\wp_n\right)\oplus\aleph, \tag{35}$$

and according to Property 5, we have

$$\mathrm{FFLWA}\left(\vartheta\tilde{*}\wp_1,\vartheta\tilde{*}\wp_2,\ldots,\vartheta\tilde{*}\wp_n\right)=\vartheta\tilde{*}\left(\mathrm{FFLWA}\left(\wp_1,\wp_2,\ldots,\wp_n\right)\right). \tag{36}$$

From Eq. (35) and Eq. (36), we get the desired results. $\blacksquare$

(**℘7**) Since $\wp_i,\aleph_i\in\Omega$, then

$$\wp_i\oplus\aleph_i=\left\langle \varphi^{*-1}\left(1-\left(1-\varphi^{*}\left(\ell_{\sigma(\wp_i)}\right)\right)\left(1-\varphi^{*}\left(\ell_{\sigma(\aleph_i)}\right)\right)\right), \sqrt[3]{1-\left(1-\xi_{\wp_i}^{3}\right)\left(1-\xi_{\aleph_i}^{3}\right)}, \sqrt[3]{\left(1-\xi_{\wp_i}^{3}\right)\left(1-\xi_{\aleph_i}^{3}\right)-\left(1-\left(\xi_{\wp_i}^{3}+\psi_{\aleph_i}^{3}\right)\right)\left(1-\left(\xi_{\wp_i}^{3}+\psi_{\aleph_i}^{3}\right)\right)} \right\rangle. \tag{37}$$

Therefore,

$$\mathrm{FFLWA}\left(\wp_1\oplus\aleph_1,\wp_2\oplus\aleph_2,\ldots,\wp_n\oplus\aleph_n\right)$$

$$=\left\langle \varphi^{*-1}\left(1-\prod_{i=1}^{n}\left(\left(1-\varphi^{*}\left(\ell_{\sigma(\wp_i)}\right)\right)\left(1-\varphi^{*}\left(\ell_{\sigma(\aleph_i)}\right)\right)\right)^{w_i}\right), \sqrt[3]{1-\prod_{i=1}^{n}\left(\left(1-\xi_{\wp_i}^{3}\right)\left(1-\xi_{\aleph_i}^{3}\right)\right)^{w_i}}, \sqrt[3]{\prod_{i=1}^{n}\left(\left(1-\xi_{\wp_i}^{3}\right)\left(1-\xi_{\aleph_i}^{3}\right)\right)^{w_i}-\prod_{i=1}^{n}\left(\left(1-\left(\xi_{\wp_i}^{3}+\psi_{\aleph_i}^{3}\right)\right)\left(1-\left(\xi_{\wp_i}^{3}+\psi_{\aleph_i}^{3}\right)\right)\right)^{w_i}} \right\rangle$$

$$=\left\langle \varphi^{*-1}\left(1-\prod_{i=1}^{n}\left(1-\varphi^{*}\left(\ell_{\sigma(\wp_i)}\right)\right)^{w_i}\prod_{i=1}^{n}\left(1-\varphi^{*}\left(\ell_{\sigma(\aleph_i)}\right)\right)^{w_i}\right), \sqrt[3]{1-\prod_{i=1}^{n}\left(1-\xi_{\wp_i}^{3}\right)^{w_i}\prod_{i=1}^{n}\left(1-\xi_{\aleph_i}^{3}\right)^{w_i}}, \sqrt[3]{\prod_{i=1}^{n}\left(1-\xi_{\wp_i}^{3}\right)^{w_i}\prod_{i=1}^{n}\left(1-\xi_{\aleph_i}^{3}\right)^{w_i}-\prod_{i=1}^{n}\left(1-\left(\xi_{\wp_i}^{3}+\psi_{\wp_i}^{3}\right)\right)^{w_i}\prod_{i=1}^{n}\left(1-\left(\xi_{\aleph_i}^{3}+\psi_{\aleph_i}^{3}\right)\right)^{w_i}} \right\rangle$$

$$=\left\langle \varphi^{*-1}\left(1-\prod_{i=1}^{n}\left(1-\varphi^{*}\left(\ell_{\sigma(\rho_i)}\right)\right)^{w_i}\right), \sqrt[3]{1-\prod_{i=1}^{n}\left(1-\xi_{\wp_i}^{3}\right)^{w_i}}, \sqrt[3]{\prod_{i=1}^{n}\left(1-\xi_{\wp_i}^{3}\right)^{w_i}-\prod_{i=1}^{n}\left(1-\left(\xi_{\wp_i}^{3}+\psi_{\wp_i}^{3}\right)\right)^{w_i}} \right\rangle \oplus \left\langle \varphi^{*-1}\left(1-\prod_{i=1}^{n}\left(1-\varphi^{*}\left(\ell_{\sigma(\wp_i)}\right)\right)^{w_i}\right), \sqrt[3]{1-\prod_{i=1}^{n}\left(1-\xi_{\wp_i}^{3}\right)^{w_i}}, \sqrt[3]{\prod_{i=1}^{n}\left(1-\xi_{\wp_i}^{3}\right)^{w_i}-\prod_{i=1}^{n}\left(1-\left(\xi_{\wp_i}^{3}+\psi_{\wp_i}^{3}\right)\right)^{w_i}} \right\rangle$$

$$=\text{FFLWA}\left(\wp_1,\wp_2,\ldots,\wp_n\right)\oplus\text{FFLWA}\left(\aleph_1,\aleph_2,\ldots,\aleph_n\right). \qquad \blacksquare$$

3.3 *FFL- ordered weighted average (FFLOWA) operator*

The ordered weighted averaging (OWA) operator [64] is an aggregation operator that provides a parameterized family of aggregation operators between the minimum and the maximum. In this subsection, we extend the idea of the FFLWA operator into the FFLOWA operator based on the OWA operator.

Definition 11: Let $\wp_i=\left\langle \ell_{\sigma(\wp_i)},\xi_{\wp_i},\psi_{\wp_i}\right\rangle (i=1,2,\ldots,n)$ be a collection of FFLNs, the FFLOWA operator of dimension n is a mapping $\text{FFLOWA}:\Omega^n\to\Omega$, that has an associated weight vector $\omega=(\omega_1,\omega_2,\ldots,\omega_n)^T$ such that $\omega_i\in[0,1]$ and $\sum_{i=1}^{n}\omega_i=1$, then

$$\text{FFLOWA}\left(\wp_1,\wp_2,\ldots,\wp_n\right)=\overset{n}{\underset{i=1}{\oplus}}\left(\omega_i\tilde{*}\wp_{\phi(i)}\right), \tag{38}$$

where $\wp_{\phi(i)}$ is the i^{th} largest value of $\wp_i\,(i=1,2,\ldots,n)$.

Theorem 5: *Let* $\wp_i=\left\langle \ell_{\sigma(\wp_i)},\xi_{\wp_i},\psi_{\wp_i}\right\rangle (i=1,2,\ldots,n)$ *be a collection of FFLNs, then the aggregated value by using the FFLOWA operator is also a FFLN and*

$$\text{FFLOWA}\left(\wp_1,\wp_2,\ldots,\wp_n\right)=\left\langle \varphi^{*-1}\left(1-\prod_{i=1}^{n}\left(1-\varphi^{*}\left(\ell_{\sigma(\wp_{\phi(i)})}\right)\right)^{\omega_i}\right), \sqrt[3]{1-\prod_{i=1}^{n}\left(1-\xi_{\wp_{\phi(i)}}^{3}\right)^{\omega_i}}, \sqrt[3]{\prod_{i=1}^{n}\left(1-\xi_{\wp_{\phi(i)}}^{3}\right)^{\omega_i}-\prod_{i=1}^{n}\left(1-\left(\xi_{\wp_{\phi(i)}}^{3}+\psi_{\wp_{\phi(i)}}^{3}\right)\right)^{\omega_i}} \right\rangle. \tag{39}$$

Proof: The proof of this theorem is **s**imilar to Theorem 4, so it is omitted here. ■

It can be easily proved that the FFLOWA operator holds the following properties.

(𝓅1) (*Idempotency*): If $\wp_i=\wp=\left\langle \ell_{\sigma(\wp)},\xi_{\wp},\psi_{\wp}\right\rangle \forall\, i$, then

$$\text{FFLOWA}\left(\wp_1,\wp_2,\ldots,\wp_n\right)=\wp. \tag{40}$$

(𝓅2) (*Monotonicity*): Let $\wp_i=\left\langle \ell_{\sigma(\wp_i)},\xi_{\wp_i},\psi_{\wp_i}\right\rangle$ and $\aleph_i=\left\langle \ell_{\sigma(\wp_i)},\xi_{\wp_i},\psi_{\wp_i}\right\rangle (i=1,2,\ldots,n)$ be two collections of FFLNs such that $\ell_{\sigma(\wp_i)}\le\ell_{\sigma(\wp_i)},\xi_{\wp_i}\le\xi_{\aleph_i},\psi_{\wp_i}\ge\psi_{\aleph_i}\forall\, i$, then

$$\text{FFLOWA}\left(\wp_1,\wp_2,\ldots,\wp_n\right)\le\text{FFLOWA}\left(\aleph_1,\aleph_2,\ldots,\aleph_n\right). \tag{41}$$

(𝓅3) (*Boundedness*): Let

$$\wp^{-}=\left\langle \min\left(\ell_{\sigma(\wp_1)},\ell_{\sigma(\wp_2)},\ldots,\ell_{\sigma(\wp_n)}\right),\min\left(\xi_{\wp_1},\xi_{\wp_2},\ldots,\xi_{\wp_n}\right),\max\left(\psi_{\wp_1},\psi_{\wp_2},\ldots,\psi_{\wp_n}\right)\right\rangle$$

and $\wp^{+}=\left\langle \max\left(\ell_{\sigma(\wp_1)},\ell_{\sigma(\wp_2)},\ldots,\ell_{\sigma(\wp_n)}\right),\max\left(\xi_{\wp_1},\xi_{\wp_2},\ldots,\xi_{\wp_n}\right),\min\left(\psi_{\wp_1},\psi_{\wp_2},\ldots,\psi_{\wp_n}\right)\right\rangle$, then

$$\wp^{-}\le\text{FFLOWA}\left(\wp_1,\wp_2,\ldots,\wp_n\right)\le\wp^{+}. \tag{42}$$

𝓅4 (*Commutativity*): Let $\left(\wp_1',\wp_2',\ldots,\wp_n'\right)$ be any permutation of $\left(\wp_1,\wp_2,\ldots,\wp_n\right)$, then

$$\text{FFLOWA}\left(\wp_1,\wp_2,\ldots,\wp_n\right)=\text{FFLOWA}\left(\wp_1',\wp_2',\ldots,\wp_n'\right). \tag{43}$$

Further, motivated by the idea of geometric mean and ordered weighted geometric operator[65], we develop the FFLWG operator and the FFLOWG operator.

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3.4 *FFL- weighted geometric (FFLWG) operator*

This subsection extends the notion of weighted geometric mean to the FFL information environment and defines the FFL weighted geometric operator as follows:

Definition 12: Let $\wp_i = \left\langle \ell_{\sigma(\wp_i)}, \xi_{\wp_i}, \psi_{\wp_i} \right\rangle$, $(i = 1,2,\ldots,n)$ be a collection of FFLNs. The FFL-weighted geometric (FFLWG) operator is a mapping $\text{FFLWG}: \Omega^n \to \Omega$, such that

$$\text{FFLWG}(\wp_1, \wp_2, \ldots, \wp_n) = \overset{n}{\underset{i=1}{\otimes}} \left(\wp_i \wedge w_i \right), \tag{44}$$

where $w = (w_1, w_2, \ldots, w_n)^T$ denotes the weight vector of $\wp_i$ with $w_i \in [0,1]$, $\sum_{i=1}^{n} w_i = 1$. Especially, in the case of $w = \left(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n} \right)^T$, the FFLWG operator is reduced into FFLG operator expressed as

$$\text{FFLG}(a_1, a_2, \ldots, a_n) = \overset{n}{\underset{i=1}{\otimes}} \left(\wp_i \wedge \frac{1}{n} \right). \tag{45}$$

Theorem 6: *Let* $\wp_i = \left\langle \ell_{\sigma(\wp_i)}, \xi_{\wp_i}, \psi_{\wp_i} \right\rangle$, $(i = 1,2,\ldots,n)$ *be a collection of FFLNs, then the aggregated value by using the FFLWG operator is also a FFLN and*

$$\text{FFLWG}(\wp_1, \wp_2, \ldots, \wp_n) = \left\langle \varphi^{*-1}\left(\prod_{i=1}^{n} \left(\varphi^*\left(\ell_{\sigma(\wp_i)} \right) \right)^{w_i} \right), \sqrt[3]{\prod_{i=1}^{n} \left(1 - \psi_{\wp_i}^3\right)^{w_i} - \prod_{i=1}^{n} \left(1 - \left(\xi_{\wp_i}^3 + \psi_{\wp_i}^3\right)\right)^{w_i}}, \sqrt[3]{1 - \prod_{i=1}^{n} \left(1 - \psi_{\wp_i}^3\right)^{w_i}} \right\rangle. \tag{46}$$

Proof: Based on improved operational laws on FFLNs mentioned in Definition 9, Theorem 6 is evident from Theorems 4.

Theorem 7: The FFLWG operator satisfies the following properties:

(𝓅1) (*Idempotency*): If $\wp_i = \wp = \left\langle \ell_{\sigma(\wp)}, \xi_{\wp}, \psi_{\wp} \right\rangle \ \forall\ i$, then

$$\text{FFLWG}(\wp_1, \wp_2, \ldots, \wp_n) = \wp. \tag{47}$$

(𝓅2) (*Monotonicity*): Let $\wp_i = \left\langle \ell_{\sigma(\wp_i)}, \xi_{\wp_i}, \psi_{\wp_i} \right\rangle$ and $\aleph_i = \left\langle \ell_{\sigma(\aleph_i)}, \xi_{\aleph_i}, \psi_{\aleph_i} \right\rangle (i = 1,2,\ldots,n)$ be two collections of FFLNs such that $\ell_{\sigma(\wp_i)} \le \ell_{\sigma(\aleph_i)}, \xi_{\wp_i} \le \xi_{\aleph_i}, \psi_{\wp_i} \ge \psi_{\aleph_i} \forall\ i$, then

$$\text{FFLWG}(\wp_1, \wp_2, \ldots, \wp_n) \le \text{FFLWG}(\aleph_1, \aleph_2, \ldots, \aleph_n). \tag{48}$$

(𝓅3) (*Boundedness*): Let $\wp^- = \left\langle \min\left(\ell_{\sigma(\wp_1)}, \ell_{\sigma(\wp_2)}, \ldots, \ell_{\sigma(\wp_n)} \right), \min\left(\xi_{\wp_1}, \xi_{\wp_2}, \ldots, \xi_{\wp_n} \right), \max\left(\psi_{\wp_1}, \psi_{\wp_2}, \ldots, \psi_{\wp_n} \right) \right\rangle$ and $\wp^+ = \left\langle \max\left(\ell_{\sigma(\wp_1)}, \ell_{\sigma(\wp_2)}, \ldots, \ell_{\sigma(\wp_n)} \right), \max\left(\xi_{\wp_1}, \xi_{\wp_2}, \ldots, \xi_{\wp_n} \right), \min\left(\psi_{\wp_1}, \psi_{\wp_2}, \ldots, \psi_{\wp_n} \right) \right\rangle$, then

$$\wp^- \le \text{FFLWG}(\wp_1, \wp_2, \ldots, \wp_n) \le \wp^+. \tag{49}$$

(𝓅4): If $\aleph = \left\langle \ell_{\sigma(\aleph)}, \xi_{\aleph}, \psi_{\aleph} \right\rangle$ is another FFLN, then

$$\text{FFLWG}\left(\wp_1 \otimes \aleph, \wp_2 \otimes \aleph, \ldots, \wp_n \otimes \aleph \right) = \text{FFLWG}(\wp_1, \wp_2, \ldots, \wp_n) \otimes \aleph. \tag{50}$$

(𝓅5): If $\vartheta > 0$ is a real number, then

$$\text{FFLWG}\left(\wp_1 \wedge \vartheta, \wp_2 \wedge \vartheta, \ldots, \wp_n \wedge \vartheta \right) = \left(\text{FFLWFG}(\wp_1, \wp_2, \ldots, \wp_n) \right) \wedge \vartheta. \tag{51}$$

(𝓅6): Let $\aleph = \left\langle \ell_{\sigma(\aleph)}, \xi_{\aleph}, \psi_{\aleph} \right\rangle$ be another FFLN and $\vartheta > 0$ be a real number, then

$$\text{FFLWG}\left(\left(\wp_1 \wedge \vartheta \right) \otimes \aleph, \left(\wp_2 \wedge \vartheta \right) \otimes \aleph, \ldots, \left(\wp_n \wedge \vartheta \right) \otimes \aleph \right) = \left(\left(\text{FFLWG}(\wp_1, \wp_2, \ldots, \wp_n) \right) \wedge \vartheta \right) \otimes \aleph. \tag{52}$$

($\wp$7): Let $\wp_i = \left\langle \ell_{\sigma(\wp_i)}, \xi_{\wp_i}, \psi_{\wp_i} \right\rangle$ and $\aleph_i = \left\langle \ell_{\sigma(\aleph_i)}, \xi_{\aleph_i}, \psi_{\aleph_i} \right\rangle$, $(i = 1, 2, \ldots, n)$ be two collections of FFLNs, then

$$\text{FFLWG}\left(\wp_1 \otimes \aleph_1, \wp_2 \otimes \aleph_2, \ldots, \wp_n \otimes \aleph_n\right) = \text{FFLWG}\left(\wp_1, \wp_2, \ldots, \wp_n\right) \otimes \text{FFLWG}\left(\aleph_1, \aleph_2, \ldots, \aleph_n\right). \tag{53}$$

Proof: Here, we prove the properties 4 and 5 only, and others can proceed likewise.

($\wp$ 4) From Definition 9, we have

$$\wp_i \otimes \aleph = \left\langle \varphi^{*-1}\left(\varphi^*\left(\ell_{\sigma(\wp_i)}\right)\varphi^*\left(\ell_{\sigma(\aleph)}\right)\right), \sqrt[3]{\left(1-\psi_{\wp_i}^3\right)\left(1-\psi_{\aleph}^3\right)-\left(1-\left(\xi_{\wp_i}^3+\psi_{\wp_i}^3\right)\right)\left(1-\left(\xi_{\aleph}^3+\psi_{\aleph}^3\right)\right)}, \sqrt[3]{1-\left(1-\psi_{\wp_i}^3\right)\left(1-\psi_{\aleph}^3\right)} \right\rangle. \tag{54}$$

Therefore

$$\begin{aligned}
&\text{FFLWG}\left(\wp_1 \otimes \aleph, \wp_2 \otimes \aleph, \ldots, \wp_n \otimes \aleph\right) \\
&= \left\langle \varphi^{*-1}\left(\prod_{i=1}^n \left(\varphi^*\left(\ell_{\sigma(\wp_i)}\right)\varphi^*\left(\ell_{\sigma(\aleph)}\right)\right)^{w_i}\right), \sqrt[3]{\prod_{i=1}^n \left(\left(1-\psi_{\wp_i}^3\right)\left(1-\psi_{\aleph}^3\right)\right)^{w_i} - \prod_{i=1}^n \left(\left(1-\left(\xi_{\wp_i}^3+\psi_{\wp_1}^3\right)\right)\left(1-\left(\xi_{\aleph}^3+\psi_{\aleph}^3\right)\right)\right)^{w_i}}, \sqrt[3]{1-\prod_{i=1}^n \left(\left(1-\psi_{\wp}^3\right)\left(1-\psi_{\aleph}^3\right)\right)^{w_i}} \right\rangle \\
&= \left\langle \varphi^{*-1}\left(\prod_{i=1}^n \left(\varphi^*\left(\ell_{\sigma(\wp_i)}\right)\right)^{w_i} \varphi^*\left(\ell_{\sigma(\aleph)}\right)\right), \sqrt[3]{\left(\prod_{i=1}^n \left(1-\psi_{\wp_i}^3\right)^{w_i}\right)\left(1-\psi_{\aleph_i}^3\right) - \left(\prod_{i=1}^n \left(1-\left(\xi_{\wp_i}^3+\psi_{\wp_1}^3\right)\right)^{w_i}\right)\left(1-\left(\xi_{\aleph_i}^3+\psi_{\aleph_i}^3\right)\right)}, \sqrt[3]{1-\left(\prod_{i=1}^n \left(1-\psi_{\wp_i}^3\right)^{w_i}\right)\left(1-\psi_{\aleph_i}^3\right)} \right\rangle \\
&= \left\langle \varphi^{*-1}\left(\prod_{i=1}^n \left(\varphi^*\left(\ell_{\sigma(\wp_i)}\right)\right)^{w_i}\right), \sqrt[3]{\prod_{i=1}^n \left(1-\psi_{\wp_i}^3\right)^{w_i} - \prod_{i=1}^n \left(1-\left(\xi_{\wp_i}^3+\psi_{\wp_i}^3\right)\right)^{w_i}}, \sqrt[3]{1-\prod_{i=1}^n \left(1-\psi_{\wp_i}^3\right)^{w_i}} \right\rangle \otimes \left\langle \ell_{\sigma(\aleph)}, \xi_{\aleph}, \psi_{\aleph} \right\rangle \\
&= \text{FFLWG}\left(\wp_1, \wp_2, \ldots, \wp_n\right) \otimes \aleph \qquad \blacksquare
\end{aligned}$$

($\wp$ 5) Using Definition 9, we get

$$\wp_i \wedge \vartheta = \left\langle \varphi^{*-1}\left(\left(\varphi^*\left(\ell_{\sigma(\wp_i)}\right)\right)^{\vartheta}\right), \sqrt[3]{\left(1-\psi_{\wp_i}^3\right)^{\vartheta} - \left(1-\left(\xi_{\wp_i}^3+\psi_{\wp_i}^3\right)\right)^{\vartheta}}, \sqrt[3]{1-\left(1-\psi_{\wp_i}^3\right)^{\vartheta}} \right\rangle \tag{55}$$

Therefore

$$\begin{aligned}
&\text{FFLWG}\left(\wp_1 \wedge \vartheta, \wp_2 \wedge \vartheta, \ldots, \wp_n \wedge \vartheta\right) \\
&= \left\langle \varphi^{*-1}\left(\prod_{i=1}^n \left(\left(\varphi^*\left(\ell_{\sigma(\wp_i)}\right)\right)^{\vartheta}\right)^{w_i}\right), \sqrt[3]{\prod_{i=1}^n \left(\left(1-\psi_{\wp_i}^3\right)^{\vartheta}\right)^{w_i} - \prod_{i=1}^n \left(\left(1-\left(\xi_{\wp_i}^3+\psi_{\wp_i}^3\right)\right)^{\vartheta}\right)^{w_i}}, \sqrt[3]{1-\prod_{i=1}^n \left(\left(1-\psi_{\wp_i}^3\right)^{\vartheta}\right)^{w_i}} \right\rangle \\
&= \left\langle \varphi^{*-1}\left(\prod_{i=1}^n \left(\left(\varphi^*\left(\ell_{\sigma(\wp_i)}\right)\right)^{w_i}\right)\right)^{\vartheta}, \sqrt[3]{\prod_{i=1}^n \left(\left(1-\psi_{\wp_i}^3\right)^{w_i}\right)^{\vartheta} - \prod_{i=1}^n \left(\left(1-\left(\xi_{\wp_i}^3+\psi_{\wp_i}^3\right)\right)^{w_i}\right)^{\vartheta}}, \sqrt[3]{1-\prod_{i=1}^n \left(\left(1-\psi_{\wp_i}^3\right)^{w_i}\right)^{\vartheta}} \right\rangle \\
&= \left(\text{FFLWFG}\left(\wp_1, \wp_2, \ldots, \wp_n\right)\right) \wedge \vartheta. \qquad \blacksquare
\end{aligned}$$

3.5 *FFL- ordered weighted geometric (FFLOWG) operator*

The ordered weighted geometric (OWG) operator [65] is a common aggregation operator in the field of information fusion. But the existing OWG operator cannot aggregate FFLNs. Now, we define the FFLOWG operator based on the notion of the OWG operator to aggregate FFLNs.

Definition 13: Let $\wp_i = \left\langle \ell_{\sigma(\wp_i)}, \xi_{\wp_i}, \psi_{\wp_i} \right\rangle$, $(i = 1,2,\ldots,n)$ be a collection of FFLNs, the FFLOWG operator of dimension n is a mapping $\text{FFLOWG}:\Omega^n \to \Omega$, that has an associated weight vector $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ such that $\omega_i \in [0,1]$ and $\sum_{i=1}^{n} \omega_i = 1$, then

$$\text{FFLOWG}(\wp_1, \wp_2, \ldots, \wp_n) = \bigotimes_{i=1}^{n} \left(\wp_{\phi(i)} \wedge \omega_i \right), \tag{56}$$

where $\wp_{\phi(i)}$ is the i^{th} largest value of $\wp_i\,(i = 1,2,\ldots,n)$.

Theorem 7: *Let* $\wp_i = \left\langle \ell_{\sigma(\wp_i)}, \xi_{\wp_i}, \psi_{\wp_i} \right\rangle (i = 1,2,\ldots,n)$ *be a collection of FFLNs, then the aggregated value by the FFLOWG operator is also a FFLN and*

$$\text{FFLOWG}(\wp_1, \wp_2, \ldots, \wp_n) = \left\langle \varphi^{*-1}\left(\prod_{i=1}^{n} \left(\varphi^{*}\left(\ell_{\sigma\left(\wp_{\phi(i)}\right)} \right) \right)^{\omega_i} \right), \frac{\sqrt[3]{\prod_{i=1}^{n} \left(1 - \psi_{\wp_{\phi(i)}}^3\right)^{\omega_i} - \prod_{i=1}^{n} \left(1 - \left(\xi_{\wp_{\phi(i)}}^3 + \psi_{\wp_{\phi(i)}}^3\right)\right)^{\omega_i}}}{\sqrt[3]{1 - \prod_{i=1}^{n} \left(1 - \psi_{\wp_{\phi(i)}}^3\right)^{\omega_i}}} \right\rangle. \tag{57}$$

Proof: We can derive the proof similar to Theorem 4, so we omit it here. ■

Moreover, the FFLOWG operator also satisfies properties such as idempotency, monotonicity, boundedness, and commutativity.

3.6 *FFL- hybrid average (FFLHA) operator and FFL- hybrid geometric (FFLHG) operator*

From Definitions 10 to 13, we know that the FFLWA and FFLWG AOs only weight the FFLNs, while the FFLOWA and FFLOWG AOs weight the ordered position of the FFLNs instead of weighting the FFLNs itself. In both cases, the weights address different aspects during the aggregation process of FFLNs. However, the developed aggregation operators for FFLNs consider only one of them. The hybrid averaging (HA) operator [66] is an aggregation operator that uses the weighted average (WA) and the ordered weighted averaging (OWA) operator in the same formulation. In the following, we propose the FFL-hybrid average (FFLHA) operator and the FFL-hybrid geometric (FFLHG) operator.

Definition 14: Let $\wp_i = \left\langle \ell_{\sigma(\wp_i)}, \xi_{\wp_i}, \psi_{\wp_i} \right\rangle (i = 1,2,\ldots,n)$ be a collection of FFLNs, the FFL-hybrid average (FFLHA) operator of dimension n is a mapping $\text{FFLHA}:\Omega^n \to \Omega$, that has an associated weight vector $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ such that $\omega_i \in [0,1]$ and $\sum_{i=1}^{n} \omega_i = 1$, then

$$\text{FFLHA}(\wp_1, \wp_2, \ldots, \wp_n) = \bigoplus_{i=1}^{n} \left(\omega_i \tilde{*} \dot{\wp}_{\phi(i)} \right), \tag{58}$$

where $\dot{\wp}_{\phi(i)}$ is the i^{th} largest number of the weighted FFLNs $\dot{\wp}_i \left(\dot{\wp}_i = (nw_i) \tilde{*} \wp_i,\ i = 1,2,\ldots,n \right)$, $w = (w_1, w_2, \ldots, w_n)^T$ is the weight vector of $\wp_i$ $(i = 1,2,\ldots,n)$ such that $w_i \in [0,1]$, $\sum_{i=1}^{n} w_i = 1$ and n is the balancing coefficient.

Theorem 8: *Let* $\wp_i = \left\langle \ell_{\sigma(\wp_i)}, \xi_{\wp_i}, \psi_{\wp_i} \right\rangle$, $(i = 1,2,\ldots,n)$ *be a collection of FFLNs, then the aggregated value by using the FFLHA operator is also a FFLN and*

$$\text{FFLHA}\left(\wp_1,\wp_2,\ldots,\wp_n\right)=\left\langle \begin{array}{c}\varphi^{*-1}\left(1-\prod_{i=1}^{n}\left(1-\varphi^{*}\left(\ell_{\sigma\left(\dot{\wp}_{\phi(i)}\right)}\right)\right)^{w_i}\right),\sqrt[3]{1-\prod_{i=1}^{n}\left(1-\xi^3_{\dot{\wp}_{\phi(i)}}\right)^{w_i}},\\ \sqrt[3]{\prod_{i=1}^{n}\left(1-\xi^3_{\dot{\wp}_{\phi(i)}}\right)^{w_i}-\prod_{i=1}^{n}\left(1-\left(\xi^3_{\dot{\wp}_{\phi(i)}}+\psi^3_{\dot{\wp}_{\phi(i)}}\right)\right)^{w_i}}\end{array}\right\rangle. \tag{59}$$

Proof: The proof of this theorem is similar to Theorem 4.

Definition 15: Let $\wp_i=\left\langle \ell_{\sigma(\wp_i)},\xi_{\wp_i},\psi_{\wp_i}\right\rangle,\ (i=1,2,\ldots,n)$ be a collection of n FFLNs, the FFL-hybrid geometric (FFLHG) operator of dimension n is a mapping $\text{FFLHG}:\Omega^n\to\Omega$, that has an associated weight vector $\omega=\left(\omega_1,\omega_2,\ldots,\omega_n\right)^T$ such that $\omega_i\in[0,1]$ and $\sum_{i=1}^{n}\omega_i=1$, then

$$\text{FFLHG}\left(\wp_1,\wp_2,\ldots,\wp_n\right)=\bigotimes_{i=1}^{n}\left(\dot{\wp}_{\phi(i)}\wedge\omega\right), \tag{60}$$

where $\dot{\wp}_{\phi(i)}$ is the i^{th} largest number of the weighted FFLNs $\dot{\wp}_i\left(\dot{\wp}_i=\wp_i\wedge nw_i,\ i=1,2,\ldots,n\right)$, $w=\left(w_1,w_2,\ldots,w_n\right)^T$ is the weight vector of $\wp_i$ $(i=1,2,\ldots,n)$ such that $w_i\in[0,1]$, $\sum_{i=1}^{n}w_i=1$ and n is the balancing coefficient.

Theorem 9: *Let $\wp_i=\left\langle \ell_{\sigma(\wp_i)},\xi_{\wp_i},\psi_{\wp_i}\right\rangle,\ (i=1,2,\ldots,n)$ be a collection of FFLNs, then the aggregated value by using the FFLHG operator is also a FFLN and*

$$\text{FFLHG}\left(\wp_1,\wp_2,\ldots,\wp_n\right)=\left\langle \begin{array}{c}\varphi^{*-1}\left(1-\prod_{i=1}^{n}\left(1-\varphi^{*}\left(\ell_{\sigma\left(\dot{\wp}_{\phi(i)}\right)}\right)\right)^{w_i}\right),\sqrt[3]{\prod_{i=1}^{n}\left(1-\psi^3_{\dot{\wp}_{\phi(i)}}\right)^{w_i}-\prod_{i=1}^{n}\left(1-\left(\xi^3_{\dot{\wp}_{\phi(i)}}+\psi^3_{\dot{\wp}_{\phi(i)}}\right)\right)^{w_i}},\\ \sqrt[3]{1-\prod_{i=1}^{n}\left(1-\psi^3_{\dot{\wp}_{\phi(i)}}\right)^{w_i}}\end{array}\right\rangle. \tag{61}$$

Proof: The proof of this theorem is similar to Theorem 4.

Note that similar to the FFLOWA and the FFLOWG operators, the FFLHA and FFLHG operators follow the idempotent, bounded, monotonic and commutative properties.

Remark 1: If $\omega=\left(\frac{1}{n},\frac{1}{n},\ldots,\frac{1}{n}\right)^T$, then FFLHA and FFLHG operators become the FFLWA operator and FFLWG operator, respectively;

Remark 2: If $w=\left(\frac{1}{n},\frac{1}{n},\ldots,\frac{1}{n}\right)^T$, then the FFLHA and FFLHG operators are reduced into FFLOWA operator and FFLOWG operator, respectively.

In the next section, we formulate a new decision-making method to solve MAGDM problems under the FFL environment. Then, we consider a real-life supplier selection problem to demonstrate the decision-making steps.

4. An Approach to MAGDM Making with FFL Information

4.1*: MAGDM Problem Description*

For a MAGDM problem, let $F=\{F_1,F_2,...,F_m\}$ be a set of alternatives, $A=\{A_1,A_2,...A_n\}$ be an attribute set with the associated weighting vector $\left(w_1,w_2,\ldots,w_n\right)^T$, satisfying $w_j\in[0,1]$ and $\sum_{j=1}^{n}w_j=1$. Assume $E=\{E_1,E_2,...,E_t\}$ is a collection of t experts whose weight vector is $\left(\omega_1,\omega_2,\ldots,\omega_t\right)^T$, satisfying $\omega_q\in[0,1]$ and $\sum_{q=1}^{t}\omega_q=1$. Further, suppose that $\mathfrak{B}^{(q)} = \left(\wp_{ij}{}^{(q)}\right)_{m\times n}$ is a decision matrix, where $\wp_{ij}{}^{(q)} = \left\langle \ell^{(q)}_{\sigma\left(\wp_{ij}\right)}, \xi^{(q)}_{\wp_{ij}}, \psi^{(q)}_{\wp_{ij}} \right\rangle$ represents an attribute evaluation value, given by the expert E_q, for the alternative $F_i \in F$ concerning the attribute $A_j \in A$ such that $0 \le \left(\xi^{(q)}_{\wp_{ij}}\right)^3 + \left(\psi^{(q)}_{\wp_{ij}}\right)^3 \le 1$ and $\ell^{(q)}_{\sigma\left(\wp_{ij}\right)} \in \widehat{L}_{[0,2t]}$, $i = 1,2,\ldots,m$; $j = 1,2,\ldots,n$.Then, the ranking of the alternatives is required to obtain the best alternative(s).

4.2: *Decision Method*

The decision method comprises the following steps.

Step 1: To nullify the effect of the different attributes, transform the decision matrices $\mathfrak{B}^{(q)} = \left(\wp_{ij}{}^{(q)}\right)_{m\times n}$ into the normalized form $\hat{\mathfrak{B}}^{(q)} = \left(\hat{\wp}_{ij}{}^{(q)}\right)_{m\times n} = \left(\left\langle \ell^{(q)}_{\sigma\left(\hat{\wp}_{ij}\right)}, \xi^{(q)}_{\hat{\wp}_{ij}}, \psi^{(q)}_{\hat{\wp}_{ij}} \right\rangle\right)_{m\times n}$. The elements of the normalized decision matrices $\hat{\mathfrak{B}}^{(q)}$ can be obtained as follows:

$$\hat{\wp}^{(q)}_{ij} = \begin{cases} \left\langle \ell^{(q)}_{\sigma\left(\wp_{ij}\right)}, \xi^{(q)}_{\wp_{ij}}, \psi^{(q)}_{\wp_{ij}} \right\rangle, & \text{if } A_j \text{ is benefit type attribute} \\ \left\langle \ell^{(q)}_{2t-\sigma\left(\wp_{ij}\right)}, \psi^{(q)}_{\wp_{ij}}, \xi^{(q)}_{\wp_{ij}} \right\rangle, & \text{if } A_j \text{ is cost type attribute} \end{cases}. \tag{62}$$

Step 2: Aggregate all the $\hat{\mathfrak{B}}^{(q)} = \left(\hat{\wp}_{ij}{}^{(q)}\right)_{m\times n}$ into a collective normalized decision matrix $\breve{\mathfrak{B}} = \left(\breve{\wp}_{ij}\right)_{m\times n} = \left(\left\langle \ell_{\sigma\left(\breve{\wp}_{ij}\right)}, \xi_{\breve{\wp}_{ij}}, \psi_{\breve{\wp}_{ij}} \right\rangle\right)$ by using either FFLOWA operator

$$\breve{\wp}_{ij} = \text{FFLOWA}\left(\hat{\wp}_{ij}{}^{(1)}, \hat{\wp}_{ij}{}^{(2)}, \ldots, \hat{\wp}_{ij}{}^{(t)}\right) = \left\langle \begin{array}{c} \varphi^{*-1}\left(1 - \prod_{q=1}^{t}\left(1 - \varphi^{*}\left(\ell_{\sigma\left(\hat{\wp}^{\phi(q)}_{ij}\right)}\right)\right)^{\omega_q}\right), \sqrt[3]{1 - \prod_{q=1}^{t}\left(1 - \xi^3_{\hat{\wp}^{\phi(q)}_{ij}}\right)^{\omega_q}}, \\ \sqrt[3]{\prod_{q=1}^{t}\left(1 - \xi^3_{\hat{\wp}^{\phi(q)}_{ij}}\right)^{\omega_q} - \prod_{q=1}^{t}\left(1 - \left(\xi^3_{\hat{\wp}^{\phi(q)}_{ij}} + \psi^3_{\hat{\wp}^{\phi(q)}_{ij}}\right)\right)^{\omega_q}} \end{array} \right\rangle, \tag{63}$$

or FFLOWG operator

$$\breve{\wp}_{ij} = \text{FFLOWG}\left(\hat{\wp}_{ij}{}^{(1)}, \hat{\wp}_{ij}{}^{(2)}, \ldots, \hat{\wp}_{ij}{}^{(t)}\right) = \left\langle \begin{array}{c} \varphi^{*-1}\left(\prod_{q=1}^{t}\left(\varphi^{*}\left(\ell_{\sigma\left(\hat{\wp}^{\phi(q)}_{ij}\right)}\right)\right)^{\omega_q}\right), \sqrt[3]{1 - \prod_{q=1}^{t}\left(1 - \psi^3_{\hat{\wp}^{\phi(q)}_{ij}}\right)^{\omega_q} - \prod_{q=1}^{t}\left(1 - \left(\xi^3_{\hat{\wp}^{\phi(q)}_{ij}} + \psi^3_{\hat{\wp}^{\phi(q)}_{ij}}\right)\right)^{\omega_q}}, \\ \sqrt[3]{1 - \prod_{q=1}^{t}\left(1 - \psi^3_{\hat{\wp}^{\phi(q)}_{ij}}\right)^{\omega_q}} \end{array} \right\rangle, \tag{64}$$

where $\hat{\wp}^{\sigma(q)}_{ij} = \left\langle \ell^{\phi(q)}_{\sigma\left(\hat{\wp}_{ij}\right)}, \xi^{\phi(q)}_{\hat{\wp}_{ij}}, \psi^{\phi(q)}_{\hat{\wp}_{ij}} \right\rangle$ is the q^{th} largest value of $\hat{\wp}^{(q)}_{ij}$ and $(\omega_1, \omega_2, \ldots, \omega_t)^T$ represents the associated ordered position weight vector with $\omega_q \in [0,1]$ and $\sum_{q=1}^{t}\omega_q = 1$.

Step 3: Aggregate all the collective preference values $\breve{\wp}_{ij}$ $(j = 1,2,...,n)$ for obtaining the overall assessment values $\tilde{\wp}_i$ $(i = 1,2,...,m)$ corresponding to the alternatives F_i $(i = 1,2,...,n)$, based on either the FFLWA operator

$$\tilde{\wp}_i = \text{FFLWA}\left(\breve{\wp}_{i1}, \breve{\wp}_{i2}, \ldots, \breve{\wp}_{in}\right) = \left\langle \begin{array}{c} \varphi^{*-1}\left(1 - \prod_{j=1}^{n}\left(1 - \varphi^{*}\left(\ell_{\sigma\left(\breve{\wp}_{ij}\right)}\right)\right)^{w_j}\right), \sqrt[3]{1 - \prod_{j=1}^{n}\left(1 - \xi^3_{\breve{\wp}_{ij}}\right)^{w_j}}, \\ \sqrt[3]{\prod_{i=1}^{n}\left(1 - \xi^3_{\breve{\wp}_{ij}}\right)^{w_j} - \prod_{j=1}^{n}\left(1 - \left(\xi^3_{\breve{\wp}_{ij}} + \psi^3_{\breve{\wp}_{ij}}\right)\right)^{w_j}} \end{array} \right\rangle, \tag{65}$$

or FFLWG operator

$$\tilde{\wp}_i = \text{FFLWA}\left(\breve{\tilde{\wp}}_{i1}, \breve{\tilde{\wp}}_{i2}, \ldots, \breve{\tilde{\wp}}_{in}\right) = \left\langle \begin{array}{l} \varphi^{*-1}\left(\prod_{j=1}^{n}\left(\varphi^{*}\left(\ell_{\sigma\left(\breve{\tilde{\wp}}_{ij}\right)}\right)\right)^{w_j}\right), \sqrt[3]{\prod_{j=1}^{n}\left(1-\psi_{\breve{\tilde{\wp}}_{ij}}^{3}\right)^{w_j} - \prod_{j=1}^{n}\left(1-\left(\xi_{\breve{\tilde{\wp}}_{ij}}^{3}+\psi_{\breve{\tilde{\wp}}_{ij}}^{3}\right)\right)^{w_j}}, \\ \sqrt[3]{1-\prod_{j=1}^{n}\left(1-\psi_{\breve{\tilde{\wp}}_{ij}}^{3}\right)^{w_j}} \end{array} \right\rangle, \quad (66)$$

Step 4. According to Definition 8, we get the order of the overall aggregated values $\tilde{\wp}_i$ $(i=1,2,\ldots,n)$.

Step 5. Rank all the alternatives F_i $(i=1,2,\ldots,n)$ and hence select the most desirable one (s).

4.3 *Numerical Example*

In order to illustrate the application of the developed approach in practice, we consider a real-life decision problem about searching the global supplier with FFL information.

Example 5: Supplier selection is one of the most important processes to accomplish an effective supply chain because a supplier comprehensively contributes to the overall supply chain performance. Due to the involvement of a group of persons and many factors, supplier selection is typically considered a MAGDM problem. In the last few years, the supplier selection problem has been received a considerable amount of attention by research from academics and industries.

A Chilean company specializing in commercialized computer and office materials wants to select a suitable material supplier to assign the raw materials' optimum order. After preliminary screening, five potential global suppliers $\{F_1, F_2, F_3, F_4.F_5\}$ were shortlisted for further evaluation. The company invites four experts $\{E_1, E_2, E_3, E_4\}$ to evaluate the shortlisted suppliers concerning five attributes (i) overall cost of the product A_1 (ii) service performance of the supplier A_2 (iii) reputation of the supplier A_3 (iv) quality of the product A_4 (v) delivery time of the product A_5. The attribute weight vector is given as $w=(0.20,0.15,0.25,0.25,0.15)^T$. The experts provide their evaluation information corresponding to each attribute in terms of FFLNs based on the following LTS:

$$L = \left\{ \begin{array}{l} \ell_0 = \text{EP(extremly poor)}, \ell_1 = \text{VP}(\text{very poor}), \ell_2 = \text{P}(\text{ poor}), \ell_3 = \text{SP}(\text{slightly poor}), \ell_4 = \text{F}(\text{fair}), \\ \ell_5 = \text{SG}(\text{slightly good}), \ell_6 = \text{G}(\text{good}), \ell_7 = \text{VG}(\text{very good}), \ell_8 = E\text{G}(\text{extremly good}) \end{array} \right\}.$$

The experts provide the following FFL decision matrices $\mathfrak{B}^{(q)} = \left(\wp_{ij}^{(q)}\right)_{5\times5} (q=1,2,3,4)$, as listed in Tables 3-6, respectively.

Table 3: Decision matrix $\mathfrak{B}^{(1)}$

	A_1	A_2	A_3	A_4	A_5
F_1	$\langle\ell_3,0.8,0.3\rangle$	$\langle\ell_1,0.5,0.5\rangle$	$\langle\ell_4,0.6,0.1\rangle$	$\langle\ell_1,0.2,0.3\rangle$	$\langle\ell_5,0.4,0.6\rangle$
F_2	$\langle\ell_5,0.7,0.2\rangle$	$\langle\ell_4,0.6,0.4\rangle$	$\langle\ell_7,0.7,0.3\rangle$	$\langle\ell_6,0.8,0.1\rangle$	$\langle\ell_4,0.5,0.7\rangle$
F_3	$\langle\ell_4,0.4,0.7\rangle$	$\langle\ell_2,0.2,0.8\rangle$	$\langle\ell_3,0.4,0.6\rangle$	$\langle\ell_2,0.6,0.6\rangle$	$\langle\ell_5,0.5,0.1\rangle$
F_4	$\langle\ell_1,0.7,0.5\rangle$	$\langle\ell_3,0.4,0.5\rangle$	$\langle\ell_4,0.3,0.4\rangle$	$\langle\ell_4,0.2,0.1\rangle$	$\langle\ell_2,0.6,0.2\rangle$
F_4	$\langle\ell_3,0.3,0.1\rangle$	$\langle\ell_4,0.7,0.1\rangle$	$\langle\ell_1,0.8,0.5\rangle$	$\langle\ell_5,0.5,0.8\rangle$	$\langle\ell_4,0.9,0.1\rangle$

Table 4: Decision matrix $\mathfrak{B}^{(2)}$

	A_1	A_2	A_3	A_4	A_5
F_1	$\langle \ell_2, 0.5, 0.2 \rangle$	$\langle \ell_3, 0.7, 0.6 \rangle$	$\langle \ell_5, 0.2, 0.4 \rangle$	$\langle \ell_5, 0.3, 0.9 \rangle$	$\langle \ell_7, 0.4, 0.3 \rangle$
F_2	$\langle \ell_4, 0.6, 0.1 \rangle$	$\langle \ell_7, 0.9, 0.5 \rangle$	$\langle \ell_6, 0.8, 0.4 \rangle$	$\langle \ell_8, 0.9, 0.3 \rangle$	$\langle \ell_3, 0.8, 0.3 \rangle$
F_3	$\langle \ell_5, 0.9, 0.3 \rangle$	$\langle \ell_4, 0.5, 0.2 \rangle$	$\langle \ell_3, 0.4, 0.2 \rangle$	$\langle \ell_2, 0.4, 0.5 \rangle$	$\langle \ell_1, 0.7, 0.1 \rangle$
F_4	$\langle \ell_7, 0.5, 0.4 \rangle$	$\langle \ell_2, 0.5, 0.1 \rangle$	$\langle \ell_4, 0.6, 0.7 \rangle$	$\langle \ell_5, 0.2, 0.8 \rangle$	$\langle \ell_5, 0.5, 0.3 \rangle$
F_4	$\langle \ell_5, 0.2, 0.5 \rangle$	$\langle \ell_1, 0.6, 0.8 \rangle$	$\langle \ell_1, 0.9, 0.6 \rangle$	$\langle \ell_7, 0.1, 0.7 \rangle$	$\langle \ell_2, 0.2, 0.2 \rangle$

Table 5: Decision matrix $\mathfrak{B}^{(3)}$

	A_1	A_2	A_3	A_4	A_5
F_1	$\langle \ell_1, 0.7, 0.2 \rangle$	$\langle \ell_4, 0.6, 0.1 \rangle$	$\langle \ell_3, 0.8, 0.3 \rangle$	$\langle \ell_1, 0.6, 0.5 \rangle$	$\langle \ell_2, 0.8, 0.0 \rangle$
F_2	$\langle \ell_2, 0.6, 0.4 \rangle$	$\langle \ell_8, 0.9, 0.2 \rangle$	$\langle \ell_6, 0.6, 0.1 \rangle$	$\langle \ell_7, 0.9, 0.6 \rangle$	$\langle \ell_3, 0.7, 0.0 \rangle$
F_3	$\langle \ell_4, 0.9, 0.6 \rangle$	$\langle \ell_4, 0.6, 0.7 \rangle$	$\langle \ell_2, 0.4, 0.4 \rangle$	$\langle \ell_4, 0.1, 0.8 \rangle$	$\langle \ell_5, 0.6, 0.2 \rangle$
F_4	$\langle \ell_3, 0.4, 0.4 \rangle$	$\langle \ell_5, 0.7, 0.0 \rangle$	$\langle \ell_6, 0.2, 0.5 \rangle$	$\langle \ell_5, 0.6, 0.8 \rangle$	$\langle \ell_7, 0.1, 0.4 \rangle$
F_4	$\langle \ell_7, 0.3, 0.9 \rangle$	$\langle \ell_3, 0.8, 0.3 \rangle$	$\langle \ell_1, 0.7, 0.6 \rangle$	$\langle \ell_6, 0.3, 0.5 \rangle$	$\langle \ell_1, 0.8, 0.6 \rangle$

Table 6: Decision matrix $\mathfrak{B}^{(4)}$

	A_1	A_2	A_3	A_4	A_5
F_1	$\langle \ell_2, 0.7, 0.1 \rangle$	$\langle \ell_4, 0.7, 0.0 \rangle$	$\langle \ell_3, 0.6, 0.6 \rangle$	$\langle \ell_4, 0.8, 0.2 \rangle$	$\langle \ell_5, 0.2, 0.9 \rangle$
F_2	$\langle \ell_3, 0.6, 0.9 \rangle$	$\langle \ell_6, 0.5, 0.3 \rangle$	$\langle \ell_7, 0.9, 0.2 \rangle$	$\langle \ell_8, 0.9, 0.4 \rangle$	$\langle \ell_5, 0.7, 0.1 \rangle$
F_3	$\langle \ell_4, 0.3, 0.4 \rangle$	$\langle \ell_3, 0.6, 0.9 \rangle$	$\langle \ell_4, 0.2, 0.5 \rangle$	$\langle \ell_5, 0.4, 0.7 \rangle$	$\langle \ell_4, 0.2, 0.6 \rangle$
F_4	$\langle \ell_3, 0.3, 0.5 \rangle$	$\langle \ell_1, 0.3, 0.4 \rangle$	$\langle \ell_3, 0.7, 0.4 \rangle$	$\langle \ell_3, 0.8, 0.3 \rangle$	$\langle \ell_2, 0.6, 0.7 \rangle$
F_4	$\langle \ell_5, 0.6, 0.4 \rangle$	$\langle \ell_4, 0.4, 0.8 \rangle$	$\langle \ell_1, 0.5, 0.5 \rangle$	$\langle \ell_2, 0.4, 0.7 \rangle$	$\langle \ell_4, 0.5, 0.6 \rangle$

Step 1: Since A_1 is a cost-type attribute while A_2, A_3, A_4 and A_5 are benefit-type attributes, so the normalized decision matrices $\hat{\mathfrak{B}}^{(q)}\,(q=1,2,3,4)$ are obtained using Eq. (62) as follows (see Tables 7-10)

Table 7: Normalized decision matrix $\hat{\mathfrak{B}}^{(1)}$

	A_1	A_2	A_3	A_4	A_5
F_1	$\langle \ell_5, 0.3, 0.8 \rangle$	$\langle \ell_1, 0.5, 0.5 \rangle$	$\langle \ell_4, 0.6, 0.1 \rangle$	$\langle \ell_1, 0.2, 0.3 \rangle$	$\langle \ell_5, 0.4, 0.6 \rangle$
F_2	$\langle \ell_3, 0.2, 0.7 \rangle$	$\langle \ell_4, 0.6, 0.4 \rangle$	$\langle \ell_7, 0.7, 0.3 \rangle$	$\langle \ell_6, 0.8, 0.1 \rangle$	$\langle \ell_4, 0.5, 0.7 \rangle$
F_3	$\langle \ell_4, 0.7, 0.4 \rangle$	$\langle \ell_2, 0.2, 0.8 \rangle$	$\langle \ell_3, 0.4, 0.6 \rangle$	$\langle \ell_2, 0.6, 0.6 \rangle$	$\langle \ell_5, 0.5, 0.1 \rangle$
F_4	$\langle \ell_7, 0.5, 0.7 \rangle$	$\langle \ell_3, 0.4, 0.5 \rangle$	$\langle \ell_4, 0.3, 0.4 \rangle$	$\langle \ell_4, 0.2, 0.1 \rangle$	$\langle \ell_2, 0.6, 0.2 \rangle$
F_5	$\langle \ell_5, 0.1, 0.3 \rangle$	$\langle \ell_4, 0.7, 0.1 \rangle$	$\langle \ell_1, 0.8, 0.5 \rangle$	$\langle \ell_5, 0.5, 0.8 \rangle$	$\langle \ell_4, 0.9, 0.1 \rangle$

Table 8: Normalized decision matrix $\hat{\mathfrak{B}}^{(2)}$

	A_1	A_2	A_3	A_4	A_5
F_1	$\langle \ell_6, 0.2, 0.5 \rangle$	$\langle \ell_3, 0.7, 0.6 \rangle$	$\langle \ell_5, 0.2, 0.4 \rangle$	$\langle \ell_5, 0.3, 0.9 \rangle$	$\langle \ell_7, 0.4, 0.3 \rangle$
F_2	$\langle \ell_4, 0.1, 0.6 \rangle$	$\langle \ell_7, 0.9, 0.5 \rangle$	$\langle \ell_6, 0.8, 0.4 \rangle$	$\langle \ell_8, 0.9, 0.3 \rangle$	$\langle \ell_3, 0.8, 0.3 \rangle$
F_3	$\langle \ell_3, 0.3, 0.9 \rangle$	$\langle \ell_4, 0.5, 0.2 \rangle$	$\langle \ell_3, 0.4, 0.2 \rangle$	$\langle \ell_2, 0.4, 0.4 \rangle$	$\langle \ell_1, 0.7, 0.1 \rangle$
F_4	$\langle \ell_1, 0.4, 0.5 \rangle$	$\langle \ell_2, 0.5, 0.1 \rangle$	$\langle \ell_4, 0.6, 0.7 \rangle$	$\langle \ell_5, 0.2, 0.8 \rangle$	$\langle \ell_5, 0.5, 0.3 \rangle$
F_5	$\langle \ell_3, 0.5, 0.2 \rangle$	$\langle \ell_1, 0.6, 0.8 \rangle$	$\langle \ell_1, 0.9, 0.6 \rangle$	$\langle \ell_7, 0.1, 0.7 \rangle$	$\langle \ell_2, 0.2, 0.2 \rangle$

Table 9: Normalized decision matrix $\hat{\mathfrak{B}}^{(3)}$

	A_1	A_2	A_3	A_4	A_5
F_1	$\langle \ell_7, 0.2, 0.7 \rangle$	$\langle \ell_4, 0.6, 0.1 \rangle$	$\langle \ell_3, 0.8, 0.3 \rangle$	$\langle \ell_1, 0.6, 0.5 \rangle$	$\langle \ell_2, 0.8, 0.0 \rangle$
F_5	$\langle \ell_6, 0.4, 0.6 \rangle$	$\langle \ell_8, 0.9, 0.2 \rangle$	$\langle \ell_6, 0.6, 0.1 \rangle$	$\langle \ell_7, 0.9, 0.6 \rangle$	$\langle \ell_3, 0.7, 0.0 \rangle$
F_3	$\langle \ell_4, 0.6, 0.9 \rangle$	$\langle \ell_4, 0.6, 0.7 \rangle$	$\langle \ell_2, 0.4, 0.4 \rangle$	$\langle \ell_4, 0.1, 0.8 \rangle$	$\langle \ell_5, 0.6, 0.2 \rangle$
F_4	$\langle \ell_5, 0.4, 0.4 \rangle$	$\langle \ell_5, 0.7, 0.0 \rangle$	$\langle \ell_6, 0.2, 0.5 \rangle$	$\langle \ell_5, 0.6, 0.8 \rangle$	$\langle \ell_7, 0.1, 0.4 \rangle$
F_4	$\langle \ell_1, 0.9, 0.3 \rangle$	$\langle \ell_3, 0.8, 0.3 \rangle$	$\langle \ell_1, 0.7, 0.6 \rangle$	$\langle \ell_6, 0.3, 0.5 \rangle$	$\langle \ell_1, 0.8, 0.6 \rangle$

Table 10: Normalized decision matrix $\hat{\mathfrak{B}}^{(4)}$

	A_1	A_2	A_3	A_4	A_5
F_1	$\langle \ell_6, 0.1, 0.7 \rangle$	$\langle \ell_4, 0.7, 0.0 \rangle$	$\langle \ell_3, 0.6, 0.6 \rangle$	$\langle \ell_4, 0.8, 0.2 \rangle$	$\langle \ell_5, 0.2, 0.9 \rangle$
F_2	$\langle \ell_5, 0.9, 0.6 \rangle$	$\langle \ell_6, 0.5, 0.3 \rangle$	$\langle \ell_7, 0.9, 0.2 \rangle$	$\langle \ell_8, 0.9, 0.4 \rangle$	$\langle \ell_5, 0.7, 0.1 \rangle$
F_3	$\langle \ell_4, 0.4, 0.3 \rangle$	$\langle \ell_3, 0.6, 0.9 \rangle$	$\langle \ell_4, 0.2, 0.5 \rangle$	$\langle \ell_5, 0.4, 0.7 \rangle$	$\langle \ell_4, 0.2, 0.6 \rangle$
F_4	$\langle \ell_5, 0.5, 0.5 \rangle$	$\langle \ell_1, 0.3, 0.4 \rangle$	$\langle \ell_3, 0.7, 0.4 \rangle$	$\langle \ell_3, 0.8, 0.3 \rangle$	$\langle \ell_2, 0.6, 0.7 \rangle$
F_5	$\langle \ell_3, 0.4, 0.6 \rangle$	$\langle \ell_4, 0.4, 0.8 \rangle$	$\langle \ell_1, 0.5, 0.5 \rangle$	$\langle \ell_2, 0.4, 0.7 \rangle$	$\langle \ell_4, 0.5, 0.6 \rangle$

Step 2: First, we calculate the experts' weighting vector $\omega = (0.1550, 0.3450, 0.3450, 0.1550)^T$ based on the normal distribution method [67]. Then, utilizing the FFLOWA operator mentioned in Eq. (63)) (without loss of generality, we have taken the linguistic scaling function $\varphi^* = \varphi_2^*$, $\theta = 1.4$) to obtain the collective normalized decision matrix $\breve{\hat{\mathfrak{B}}} = \left(\breve{\tilde{\wp}}_{ij}\right)_{5\times5}$. The collective normalized decision matrix is summarized in Table 11.

Table 11: Collective normalized decision matrix $\breve{\hat{\mathfrak{B}}}$ based on FFLOWA operator

	A_1	A_2	A_3	A_4	A_5
F_1	$\langle \ell_{6.3636}, 0.2046, 0.7016 \rangle$	$\langle \ell_{3.0842}, 0.6463, 0.4778 \rangle$	$\langle \ell_{4.0947}, 0.5943, 0.3966 \rangle$	$\langle \ell_{1.9102}, 0.6272, 0.5882 \rangle$	$\langle \ell_{4.8442}, 0.6262, 0.5809 \rangle$
F_2	$\langle \ell_{4.9662}, 0.5876, 0.6750 \rangle$	$\langle \ell_{8.0000}, 0.8048, 0.4686 \rangle$	$\langle \ell_{6.6072}, 0.3266, 0.3266 \rangle$	$\langle \ell_{8.000}, 0.8892, 0.5234 \rangle$	$\langle \ell_{3.4637}, 0.7244, 0.3855 \rangle$
F_3	$\langle \ell_{3.8309}, 0.4735, 0.8315 \rangle$	$\langle \ell_{3.2911}, 0.5571, 0.8364 \rangle$	$\langle \ell_{3.1444}, 0.2951, 0.5097 \rangle$	$\langle \ell_{2.6951}, 0.3791, 0.6367 \rangle$	$\langle \ell_{4.0135}, 0.4683, 0.4152 \rangle$
F_4	$\langle \ell_{5.1312}, 0.3942, 0.5286 \rangle$	$\langle \ell_{2.5714}, 0.5054, 0.3709 \rangle$	$\langle \ell_{4.1499}, 0.5602, 0.5041 \rangle$	$\langle \ell_{4.4243}, 0.5652, 0.6810 \rangle$	$\langle \ell_{4.4155}, 0.5367, 0.4497 \rangle$
F_5	$\langle \ell_{2.9330}, 0.6195, 0.4349 \rangle$	$\langle \ell_{3.0842}, 0.6778, 0.6322 \rangle$	$\langle \ell_{0.9998}, 0.7717, 0.6001 \rangle$	$\langle \ell_{5.9625}, 0.3895, 0.7325 \rangle$	$\langle \ell_{2.7125}, 0.6724, 0.4952 \rangle$

Step 3: Utilize the FFLWA operator (Eq. (65)) with $w = (0.20, 0.15, 0.25, 0.25, 0.15)^T$ to derive the overall FFL preference values $\tilde{\wp}_i$ $(i = 1,2,3,4,5)$ corresponding to each alternative F_i $(i = 1,2,3,4,5)$. Table 12 presents the results.

Table 12: The overall FFL preference values $\tilde{\wp}_i$ based on the FFLWA operator

F_1	F_2	F_3	F_4	F_5
$\langle \ell_{4.3017}, 0.5831, 0.5647 \rangle$	$\langle \ell_{8.0000}, 0.7473, 0.5436 \rangle$	$\langle \ell_{3.2991}, 0.4379, 0.7336 \rangle$	$\langle \ell_{4.2602}, 0.5254, 0.5559 \rangle$	$\langle \ell_{3.2791}, 0.6571, 0.6108 \rangle$

Step 4: According to Definition 8, we have

$$\mathfrak{S}(\tilde{\wp}_1)=0.2641,\ \mathfrak{S}(\tilde{\wp}_2)=0.6284,\ \mathfrak{S}(\tilde{\wp}_3)=0.1562,\ \mathfrak{S}(\tilde{\wp}_4)=0.2511,\ \mathfrak{S}(\tilde{\wp}_5)=0.2385.$$

Step 5. The final ranking order of the suppliers following the score values $\mathfrak{S}(\tilde{\wp}_i)$ is $F_2 \succ F_1 \succ F_4 \succ F_5 \succ F_3$, thus, the most desirable supplier is F_2.

Additional, if we use the FFLOWG operator in Step 2 and the FFLWG operator in Step 3 instead of FFLOWA and FFLWA operators, respectively, in the developed method, then the procedure steps are as follows:

Step 1: Same as above.

Step 2: Utilizing the FFLOWG operator to aggregate all the normalized decision matrices $\hat{\mathfrak{B}}^{(q)}\ (q=1,2,3,4)$, the obtained results corresponding to each alternative is shown in Table 13.

Table 13: Aggregated normalized decision matrix $\breve{\hat{\mathfrak{B}}}$ based on FFLOWG operator

	A_1	A_2	A_3	A_4	A_5
F_1	$\langle \ell_{6.2263}, 0.2137, 0.7007 \rangle$	$\langle \ell_{2.7656}, 0.6538, 0.4637 \rangle$	$\langle \ell_{3.9865}, 0.5937, 0.3980 \rangle$	$\langle \ell_{1.4768}, 0.5857, 0.6112 \rangle$	$\langle \ell_{3.9815}, 0.5743, 0.6318 \rangle$
F_2	$\langle \ell_{4.7577}, 0.6489, 0.6190 \rangle$	$\langle \ell_{6.4552}, 0.8239, 0.4017 \rangle$	$\langle \ell_{6.5188}, 0.3211, 0.3211 \rangle$	$\langle \ell_{7.3788}, 0.9050, 0.4721 \rangle$	$\langle \ell_{3.3857}, 0.7149, 0.4160 \rangle$
F_3	$\langle \ell_{3.8103}, 0.6273, 0.7881 \rangle$	$\langle \ell_{3.1929}, 0.6306, 0.7975 \rangle$	$\langle \ell_{3.0632}, 0.3571, 0.5091 \rangle$	$\langle \ell_{2.4951}, 0.4715, 0.6414 \rangle$	$\langle \ell_{3.4022}, 0.5068, 0.4347 \rangle$
F_4	$\langle \ell_{4.1466}, 0.4034, 0.5233 \rangle$	$\langle \ell_{2.2839}, 0.5001, 0.3805 \rangle$	$\langle \ell_{3.9398}, 0.5654, 0.5041 \rangle$	$\langle \ell_{4.3441}, 0.5767, 0.6729 \rangle$	$\langle \ell_{3.3720}, 0.5447, 0.4377 \rangle$
F_5	$\langle \ell_{2.6262}, 0.6119, 0.4495 \rangle$	$\langle \ell_{2.7656}, 0.6350, 0.6754 \rangle$	$\langle \ell_{0.9998}, 0.7958, 0.5559 \rangle$	$\langle \ell_{5.3982}, 0.4159, 0.7244 \rangle$	$\langle \ell_{2.3979}, 0.6756, 0.4893 \rangle$

Step 3: The overall FFL preference values $\tilde{\wp}_i\ (i=1,2,3,4,5)$ of each alternative $F_i\ (i=1,2,3,4,5)$ using the FFLWG operator are summarized in Table 14.

Table 14: The overall FFL preference values $\tilde{\wp}_i$ based on the FFLWG operator

F_1	F_2	F_3	F_4	F_5
$\langle \ell_{3.1520}, 0.5552, 0.5861 \rangle$	$\langle \ell_{6.0409}, 0.7785, 0.4727 \rangle$	$\langle \ell_{3.0867}, 0.5716, 0.6739 \rangle$	$\langle \ell_{3.6215}, 0.5375, 0.5460 \rangle$	$\langle \ell_{2.3249}, 0.6565, 0.6115 \rangle$

Step 4: The score values $\mathfrak{S}(\tilde{\wp}_i)$ of the overall FFL preference values obtained during Step 3 are calculated as

$$\mathfrak{S}(\tilde{\wp}_1)=0.2143,\ \mathfrak{S}(\tilde{\wp}_2)=0.4602,\ \mathfrak{S}(\tilde{\wp}_3)=0.1923,\ \mathfrak{S}(\tilde{\wp}_4)=0.2363,\ \mathfrak{S}(\tilde{\wp}_5)=0.1934.$$

Step 5. Since $\mathfrak{S}(\tilde{\wp}_2) \succ \mathfrak{S}(\tilde{\wp}_4) \succ \mathfrak{S}(\tilde{\wp}_1) \succ \mathfrak{S}(\tilde{\wp}_5) \succ \mathfrak{S}(\tilde{\wp}_3)$, therefore we obtain the final ranking order of the suppliers as $F_2 \succ F_4 \succ F_1 \succ F_5 \succ F_3$. Hence, F_2 is the most desirable supplier.

It is worth noting that a decision-maker can choose the appropriate aggregation operator based on his/her behavior towards the aggregation procedure. If a decision-maker has optimistic behavior towards the aggregation of experts' preference information and the pessimistic behavior towards the final decision, then he/she use FFLOWA and FFLWG operator in Step 2 and Step 3, respectively, of the developed approach. A complete analysis has been conducted to analyze the effect of the decision-makers behavioral attitude on final ranking, and obtained results are summarized in Table 15 along with the suppliers' ranking order. The results shown in Table 15 indicate that when we use the FFLOWA (or the FFLOWG) operator in Step 2 and the FFLWA (or FFLHA) operator in Step 3 then the ranking order of the alternatives is always $F_2 \succ F_1 \succ F_4 \succ F_5 \succ F_3$. On the other hand, if we use the FFLOWH (or the FFLOWG) operator in Step 2 and the FFLWG (or FFLHG) operator in Step 3 then the ranking order of the alternatives is obtained as $F_2 \succ F_4 \succ F_1 \succ F_5 \succ F_3$, which is slightly different from previous ones. It shows the effect of the nature of varying aggregation operators on the final ranking order of the alternatives.

Table 15: The score values $\mathfrak{S}(\tilde{\wp}_i)$ and ranking order of the suppliers

The operator used in Step 2	The operator used in Step 3	Score values					Ranking of the suppliers
		$\mathfrak{S}(\tilde{\wp}_1)$	$\mathfrak{S}(\tilde{\wp}_2)$	$\mathfrak{S}(\tilde{\wp}_3)$	$\mathfrak{S}(\tilde{\wp}_4)$	$\mathfrak{S}(\tilde{\wp}_5)$	
FFLOWA	FFLWA	0.2641	0.6284	0.1562	0.2511	0.2385	$F_2 \succ F_1 \succ F_4 \succ F_5 \succ F_3$
	FFLWG	0.2407	0.4832	0.1679	0.2501	0.2030	$F_2 \succ F_4 \succ F_1 \succ F_5 \succ F_3$
	FFLHA	0.2618	0.6245	0.1568	0.2513	0.2235	$F_2 \succ F_1 \succ F_4 \succ F_5 \succ F_3$
	FFLHG	0.2427	0.5334	0.1704	0.2484	0.2165	$F_2 \succ F_4 \succ F_1 \succ F_5 \succ F_3$
FFLOWG	FFLWA	0.2408	0.4812	0.1765	0.2369	0.2228	$F_2 \succ F_1 \succ F_4 \succ F_5 \succ F_3$
	FFLWG	0.2143	0.4602	0.1923	0.2363	0.1934	$F_2 \succ F_4 \succ F_1 \succ F_5 \succ F_3$
	FFLHA	0.2385	0.4665	0.1804	0.2366	0.2151	$F_2 \succ F_1 \succ F_4 \succ F_5 \succ F_3$
	FFLHG	0.2125	0.4236	0.1873	0.2282	0.1899	$F_2 \succ F_4 \succ F_1 \succ F_5 \succ F_3$

Apart from the above analysis, to examine the influence of the different LSFs on the alternatives' ranking order, we have been employed different LSFs in the calculation process of the developed decision-making approach. Then, after applying the steps, the corresponding results are summarized in Table 16.

Table 16: The score values $\mathfrak{S}(\tilde{\wp}_i)$ and ranking order of the suppliers based on different LSFs

LSF	The operator used in Step 2	The operator used in Step 3	Score values					Ranking of the suppliers
			$\mathfrak{S}(\tilde{\wp}_1)$	$\mathfrak{S}(\tilde{\wp}_2)$	$\mathfrak{S}(\tilde{\wp}_3)$	$\mathfrak{S}(\tilde{\wp}_4)$	$\mathfrak{S}(\tilde{\wp}_5)$	
$\varphi^* = \varphi_1^*$	FFLOWA	FFLWA	0.2813	0.6283	0.1610	0.2616	0.2414	$F_2 \succ F_1 \succ F_4 \succ F_5 \succ F_3$
		FFLWG	0.2531	0.4785	0.1702	0.2564	0.1797	$F_2 \succ F_4 \succ F_1 \succ F_5 \succ F_3$
		FFLHA	0.2905	0.6120	0.1602	0.3027	0.2336	$F_2 \succ F_4 \succ F_1 \succ F_5 \succ F_3$
		FFLHG	0.2523	0.4719	0.1686	0.2516	0.1813	$F_2 \succ F_1 \succ F_4 \succ F_5 \succ F_3$
	FFLOWG	FFLWA	0.2368	0.5157	0.1629	0.2309	0.2074	$F_2 \succ F_1 \succ F_4 \succ F_5 \succ F_3$
		FFLWG	0.2002	0.4793	0.1752	0.2265	0.1629	$F_2 \succ F_4 \succ F_1 \succ F_3 \succ F_5$
		FFLHA	0.2373	0.5013	0.1638	0.2500	0.1929	$F_2 \succ F_4 \succ F_1 \succ F_5 \succ F_3$
		FFLHG	0.2000	0.4386	0.1722	0.2373	0.1560	$F_2 \succ F_4 \succ F_1 \succ F_3 \succ F_5$
$\varphi^* = \varphi_3^*$ $(\rho = \tau = 0.8)$	FFLOWA	FFLWA	0.2913	0.6283	0.1591	0.3061	0.2448	$F_2 \succ F_4 \succ F_1 \succ F_5 \succ F_3$
		FFLWG	0.2554	0.5060	0.1669	0.2891	0.1695	$F_2 \succ F_4 \succ F_1 \succ F_5 \succ F_3$
		FFLHA	0.2905	0.6120	0.1602	0.3027	0.2336	$F_2 \succ F_4 \succ F_1 \succ F_5 \succ F_3$
		FFLHG	0.2569	0.4772	0.1661	0.2779	0.1718	$F_2 \succ F_4 \succ F_1 \succ F_5 \succ F_3$
	FFLOWG	FFLWA	0.2398	0.5270	0.1537	0.2319	0.2009	$F_2 \succ F_1 \succ F_4 \succ F_5 \succ F_3$
		FFLWG	0.1931	0.4864	0.1650	0.2274	0.1901	$F_2 \succ F_4 \succ F_1 \succ F_5 \succ F_3$
		FFLHA	0.2412	0.5124	0.1554	0.2337	0.1907	$F_2 \succ F_1 \succ F_4 \succ F_5 \succ F_3$
		FFLHG	0.1942	0.4455	0.1640	0.2232	0.1481	$F_2 \succ F_4 \succ F_1 \succ F_3 \succ F_5$

From Table 16, it has been observed that although the score values of the alternatives are entirely different when we use different LSFs, the best alternative is always F_2 for the considered problem. Note that the use of different LSFs shows an influence on the final ranking order of the alternatives. It is also worth noting that, in other real-life decision problems, the best alternative may be different depending on the use of different aggregation operators. Our developed method provides an ability to the DMs for choosing the appropriate LSF according to his/her personal choice and the actual semantic environment.

5. Conclusions

In this paper, we have studied the MAGDM problems in which the attributes evaluation values are given in FFLNs. First, the paper has defined some new algebraic operational laws for FFLNs based on LSF to overcome the shortcomings of the existing operational laws for FFLNs. Besides, a number of mathematical properties have been studied on them. Based on the proposed operational laws, we have defined several AOs, including the FFLWA operator, the FFLWG operator, the FFLOWA operator, the FFLOWG operator, the FFLHA operator, and the FFLHG operator to aggregate different FFLNs.

Furthermore, the work has been studied many important properties of the proposed AOs, such as idempotency, monotonicity, commutativity, and boundedness. Using these AOs, we have developed a new decision-making approach to solve MAGDM problems with FFL information. Finally, a real-life supplier selection problem has been considered to illustrate the steps of the proposed method.

We shall develop some new aggregation operators such as Bonferroni mean operator, Heronian mean operator, Hamy mean operator to aggregate the correlative FFL information in future work. We shall also explore the application of the proposed AOs in other fields such as image processing, medical diagnosis, and personnel selection.

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Compliance with Ethical Standards

Conflict of interest: The author declares that he has no conflict of interest.

Ethical Approval: This article does not contain any studies with human participants or animals performed by any of the authors

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