 doi:10.20944/preprints202104.0088.v1

Article

Classical chaos described by a Density matrix

A.M. Kowalski [1,†], A. Plastino [2] and G. Gonzalez Acosta [2]

1 Instituto de Física (IFLP-CCT-Conicet), Fac. de Ciencias Exactas, Universidad Nacional de La Plata, La Plata, Argentina; plastino@fisica.unlp.edu.ar
2 Instituto de Física (IFLP-CCT-Conicet)
* Correspondence: kowalski@fisica.unlp.edu.ar
† Comision de Investigaciones Científicas (CIC)

Abstract: We work with reference to a well-known semiclassical model, in which quantum degrees of freedom interact with classical ones. We show that, in the classical limit, it is possible to represent classical results (e.g., classical chaos) by means a pure-state density matrix.

Keywords: Classical limit, Semiclassical system, Semiclassical Chaos, Clasiccal Chaos, MaxEnt, Density matrix

1. Introduction

The quantum-classical transition is certainly a frontier issues that constitute a transcendental physics topic [1–5]. On the other hand, the use of semi-classical systems to describe problems in physics has a long historical [6–8]. A particularly important case is to be highlighted, in which quantum features in one of the two components of a composite system are negligible in comparison to those in the other. Regarding this scenario as classical simplifies the description and provides deep insight into the combined system dynamics [9]. This methodology is widely used for the interaction of matter with a field. In this effort we will look at these matters through a well-known semi-classical model [10,11]. This model has been analyzed in great detail from a purely dynamic viewpoint [11] and also using statistical quantifiers derived from Information Theory (IT) [12,13]. For this model and in [14], a suitable density matrix was found for describing the system's route on its way to the classical limit. Rather exhaustive numerical results were presented.

The purpose of this work is to analytically determine what happens with the above mentioned mixed density matrix in the exact classical limit. Same interesting insight will ensue.

2. Model

We will consider a Hamiltonian $\hat{H}$ containing classical degrees of freedom (DOF) interacting with strictly quantum DOFs. The dynamical equations for the quantum operators will be the canonical ones [11], i.e., any operator O evolves (in the Heisenberg picture) as

$$\frac{dO}{dt} = \frac{i}{\hbar}[\,H, O\,]\,. \tag{1}$$

The evolution equation for its mean value $\langle O\rangle \equiv \mathrm{Tr}\,[\rho\, O(t)]$ will be $\frac{d\langle O\rangle}{dt} = \frac{i}{\hbar}\langle[\,H, O\,]\rangle$, where the average is taken with respect to a proper quantum density operator ρ. Addi-

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 5 April 2021 doi:10.20944/preprints202104.0088.v1

tionally, the classical variables will obey the Hamilton equations of motion, but where the generator is the mean value of the Hamiltonian, i.e.,

$$\frac{dA}{dt} = \frac{\partial \langle H \rangle}{\partial P_A}, \tag{2a}$$

$$\frac{dP_A}{dt} = -\frac{\partial \langle H \rangle}{\partial A}. \tag{2b}$$

The above equations constitute an autonomous set of coupled differential equations, that allows for a dynamical description in which no quantum rules are violated, e.g., particularly the Principle of Uncertainty is conserved for all times. A plays the role of a time-dependent parameter for the quantum system, and the initial conditions are determined by a proper quantum density operator $\hat{\rho}$.

We consider now a well-known semiclassical system [10,11], whose Hamiltonian is

$$\hat{H} = \frac{1}{2}\left(\frac{\hat{p}^2}{m_q} + \frac{{P_A}^2}{m_{cl}} + m_q \omega^2 \hat{x}^2 \right), \tag{3}$$

where $\hat{x}$ and $\hat{p}$ are quantum operators, while A and P_A are classical variables. The term $\omega^2 = {\omega_q}^2 + e^2 A^2$ is an interaction one introducing nonlinearity in our problem, with ω_q a frequency. m_q and m_{cl} are masses. The Hamiltonian (3) is a particular case of a family of semiclassical ones, quadratic in $\hat{x}$ and $\hat{p}$, without linear terms (see below), This family has as a time-invariant a quantity I that relates to the Uncertainty Principle [11] as

$$I = \langle \hat{x}^2 \rangle \langle \hat{p}^2 \rangle - \frac{\langle \hat{L} \rangle^2}{4} \geq \frac{\hbar^2}{4}. \tag{4}$$

I describes the deviation of the semiquantum system from the classical one given by $I = 0$. The quantity $\hat{L}$ is defined as $\hat{L} = \hat{x}\hat{p} + \hat{p}\hat{x}$. To investigate the classical limit one needs also to consider the classical analogous of (3), in which all variables are classical. In this case is $L = 2xp$. We analyze in this work the limit $I \to 0$. A well known ODE-theorem establishes uniqueness and a continuous dependence of the ODE-solutions on the initial conditions, if a condition called the Lipschitz one is fulfilled [15]. If the ODE solutions remain bounded as time grows towards infinity, the condition is always satisfied.

Consider semiquantum systems (SS) governed by operators that close a partial Lie algebra with the Hamiltonian. These SS' dynamics will be ruled by closed systems of equations (CSE), involving also the classical variables. These CSE will depend in continuous fashion on the initial conditions. For instance, this happens with the set ($\hat{x}^2$, $\hat{p}^2$, $\hat{L}$) for quadratic (in $\hat{x}$ and $\hat{p}$) Hamiltonians [11]. This fact guarantees the existence of the limit $I \to 0$ [11].

If the Hamiltonian includes lineal terms in $\hat{x}$ and $\hat{p}$, I no longer remains a constant of the motion. In this case one uses, instead of I, $I_\Delta = \Delta^2 x\, \Delta^2 p - \frac{\Delta L^2}{4}$, which is a time-invariant quantity. The pertinent analysis is similar to the one above described.

3. MaxEnt Density operator for the semiquantum problem

We assume

- Complete knowledge about the initial conditions of the classical variables.
- Incomplete knowledge regarding the system's quantum components.
- We only know the initial values of the quantum expectation values of the set of operators $\hat{O}_1 = \hat{x}^2$, $\hat{O}_2 = \hat{p}^2$, $\hat{O}_3 = \hat{L}$.

- This set is the smallest one that carries information regarding the uncertainty principle (via I).

The MaxEnt statistical operator $\hat{\rho}$ is given by [14]

$$\hat{\rho} = \exp - \left(\lambda_0 \hat{\mathcal{I}} + \lambda_1 \hat{x}^2 + \lambda_2 \hat{p}^2 + \lambda_3 \hat{L} \right), \tag{5}$$

where the Lagrange multipliers λ_i are determined so as to fulfill the set of constraints posed by our prior information (i.e., normalization of $\hat{\rho}$ and the supposedly a priori known EV's)

$$\langle \hat{O}_i \rangle = \mathrm{Tr} \left[\, \hat{\rho} \, \hat{O}_i \, \right] , \qquad i = 0, \ldots, 3, \tag{6}$$

($\hat{O}_0 = \hat{I}$ is the identity operator). A simplified way to obtaining the values of the multipliers is that of solving the coupled set of equations [16]

$$\frac{\partial \lambda_0}{\partial \lambda_i} = - \langle \hat{O}_i \rangle, \qquad i = 1, 2, 3, \tag{7}$$

where

$$\lambda_0 = \mathrm{Tr} \left[\exp \left(- \sum_{i=1}^{3} \lambda_i \hat{O}_i \right) \right]. \tag{8}$$

Using Eq. (7), one can determine the "initial" $\hat{\rho}$ given by (5). On the other hand, the statistical operator must evolve in time from (5) according to the Lioville-von Neumann equation

$$i\hbar \frac{d\hat{\rho}}{dt}(t) = [\, \hat{H}, \hat{\rho}(t) \,] \, . \tag{9}$$

As the operators $\hat{O}_i$ close a partial Lie algebra with respect to the Hamiltonian $\hat{H}$ [16,17], we have

$$[\, \hat{H}(t), \hat{O}_i \,] = i\hbar \sum_{j=1}^{3} g_{ji}(t) \hat{O}_j \, , \qquad i = 0, 1, \ldots, 3, \tag{10}$$

the statistical operator depends on the time t according to [17]

$$\hat{\rho}(t) = \exp - \left(\lambda_0 \hat{I} + \lambda_1(t) \hat{x}^2 + \lambda_2(t) \hat{p}^2 + \lambda_3(t) \hat{L} \right), \tag{11}$$

provided that the Lagrange multipliers $\lambda_j(t)$ verify the set of differential equations [17]

$$\frac{d\lambda_i}{dt}(t) = \sum_{j=1}^{3} g_{ij} \lambda_j(t) \, , \qquad i = 1, 2, 3, \tag{12}$$

with $\lambda_j(0) = \lambda_j$ from (5). The demonstration of this property can be encountered in the celebrated article given by [17] and is based on the uniqueness of the solutions of the Liouville Equation and the MaxEnt principle, together with the conservation of the Entropy

$$S(\hat{\rho}) = -\mathrm{Tr} \left[\, \hat{\rho} \, \ln \hat{\rho} \, \right] \, = \lambda_0 + \sum_{i=1}^{3} \lambda_i \langle \hat{O}_i \rangle \, , \tag{13}$$

(Boltzmann's constant is set equal to unity), which is maximized by the statistical operator (11).

From now on we will use the fact that $\lambda_j(t) = \lambda_j$ to simplify the notation. In this way, Eqs. (5)–(8) are valid for all t. Additionally, once $\hat{\rho}(t))$ is obtained, we can determine (in the Schrödinger picture), **the temporal evolution of the EV of any operator $\hat{O}$ through**

$$\langle \hat{O} \rangle(t) = \mathrm{Tr}[\hat{\rho}(t) \hat{O}]. \tag{14}$$

Note that in this type of semiclassical problem, the g_{ij} of Eqs. (10) and (12) depend on the

 doi:10.20944/preprints202104.0088.v1

classical variables A and P_A. We use equation (14) (with $\hat{O} = \hat{H}$) in order to obtain $\langle \hat{H} \rangle$ and thus describe, via Eqs. (2), the temporal evolution of A and P_A. The idea is then to regard the set of equations (12), together with the equations (2), as a single autonomous first-order system. Note that the classical equations in turn depend on the mean values. In this case the presence of the term $\langle \hat{x}^2 \rangle$ in the equation for P_A introduces an additional non-linearity (as such a term is a function of the multipliers) through

$$\langle \hat{x}^2 \rangle(t) = \mathrm{Tr}[\hat{\rho}(t)\hat{x}^2], \tag{15}$$

but we will presently see that this non-linearity can be easily handled.

4. Some convenient mathematical results

It is necessary to calculate λ_0 to relate the initial values of the multipliers and their respective EV's, using Eq. (7). We begin by performing a change of representation, made by recourse to the unitary transformation [14]

$$\hat{x} = \frac{\sqrt{2}}{2}\left(\frac{\lambda_2}{\lambda_1}\right)^{1/4}\left(\left(\frac{\lambda_T}{\lambda_V}\right)^{1/4}\hat{X} + \left(\frac{\lambda_V}{\lambda_T}\right)^{1/4}\hat{P}\right), \tag{16a}$$

$$\hat{p} = \frac{\sqrt{2}}{2}\left(\frac{\lambda_1}{\lambda_2}\right)^{1/4}\left(-\left(\frac{\lambda_T}{\lambda_V}\right)^{1/4}\hat{X} + \left(\frac{\lambda_V}{\lambda_T}\right)^{1/4}\hat{P}\right), \tag{16b}$$

where $\lambda_V = \sqrt{\lambda_1\lambda_2} + \lambda_3$ and $\lambda_T = \sqrt{\lambda_1\lambda_2} - \lambda_3$. For reasons of convergence, λ_1, λ_2, and $\lambda_1\lambda_2 - {\lambda_3}^2$ must be positive. Then, λ_V and λ_T become positive too and I_λ in (18) is well defined. Of course, the transformation (16) preserves commutation relations. Thus, I is also preserved. These new operators are not dimensionless ones [they are expressed in units of the square root of an action and do not depend on $\hbar$, which is a convenient fact at the time of going over to thhe classical limit]. Further, $\hat{X}$ and $\hat{P}$, via the λ's that appear as coefficients in their definition, are explicitly time-dependent and contain all the relevant information regarding the classical degrees of freedom. Now $\rho(t)$ becomes [14]

$$\hat{\rho}(t) = \exp(-\lambda_0)\exp\left[-I_\lambda\left(\hat{X}^2 + \hat{P}^2\right)\right]. \tag{17}$$

The quantity I_λ defined as

$$I_\lambda = \left(\lambda_1\lambda_2 - {\lambda_3}^2\right)^{1/2}, \tag{18}$$

a constant of the motion [14]. This invariant is the equivalent of the one in Eq. (4), expressed in terms of the λ's.

Despite the characteristics assigned to $\hat{X}$ and $\hat{P}$, the operator $\hat{X}^2 + \hat{P}^2$ has a discrete spectrum, one resembling that of a the Harmonic Oscillator, because the commutation relations are preserved for all time. After a little algebra, it is easy to see from (8) that

$$\lambda_0 = -\ln[\exp(\hbar\, I_\lambda) - \exp(-\hbar\, I_\lambda)], \tag{19}$$

and using Eq. (7) (or Eq. 14), the particular EV's can be cast in the fashion [14]

$$\langle \hat{x}^2 \rangle = \frac{T(I_\lambda)}{I_\lambda}\lambda_2, \tag{20a}$$

$$\langle \hat{p}^2 \rangle = \frac{T(I_\lambda)}{I_\lambda}\lambda_1, \tag{20b}$$

$$\langle \hat{L} \rangle = -2\frac{T(I_\lambda)}{I_\lambda}\lambda_3, \tag{20c}$$

with $T(I_\lambda)$ given by [14]

$$T(I_\lambda) = \frac{\hbar}{2}\left(\frac{\exp(2\hbar\, I_\lambda)+1}{\exp(2\hbar\, I_\lambda)-1}\right). \tag{21}$$

Further, we deduce from (20) that

$$T(I_\lambda) \;=\; \sqrt{I}\,. \tag{22}$$

Now, by recourse to the Eqs. (2), (12), and (20a), we are in position to write down our dynamical system of equations as a closed one in both multipliers and classical variables. We have [14]

$$\frac{d\lambda_1}{dt} = 2m_q\omega^2\lambda_3, \tag{23a}$$

$$\frac{d\lambda_2}{dt} = -\frac{2}{m_q}\lambda_3, \tag{23b}$$

$$\frac{d\lambda_3}{dt} = -\frac{1}{m_q}\lambda_1 + m_q\omega^2\lambda_2, \tag{23c}$$

$$\frac{dA}{dt} = \frac{P_A}{m_{cl}}, \tag{23d}$$

$$\frac{dP_A}{dt} = -e^2 m_q\, A\frac{T(I_\lambda)}{I_\lambda}\lambda_2. \tag{23e}$$

This system associates **a kind of phase-space** to the density operator (11), determined by classical variables and Lagrange multipliers. The system (23) depends in nonlinear fashion upon the classical variable A, via ω^2, but the non-linear term $T(I_\lambda)$ in (23e) is easily tractable as a function of I, using (22). This non-linearity is thus replaced by a dependence upon I plus the initial conditions. This last dependence emerges via the invariant I_λ (which in turn is fixed by $\hat\rho(0)$, i.e. by the initial values of the Lagrange multipliers).

5. Useful previous results

In [14], we investigated the dynamics described by the density operator (11) as a function of the relative energy E_r, defined as $E_r = \frac{|E|}{I^{1/2}\omega_q}$. The classical limit obtains for $E_r \to \infty$ (a particular case is $I \to 0$, which we will study below).

In [14], we also showed that, by augmenting E_r (for example decreasing I), the physical system passes through three regions: a quasiclassical one, a transitional one, and a classical one (see Figs. 1 and 2 of [14]). As E_r grows, complexity augments and, eventually, chaos emerges. This is a phenomenon of a semi-classical nature, since the classical dynamics-stage has, obviously, not yet been reached. Remark on the coexistence of the Uncertainty Principle with chaos and also on that, having $\hat\rho(t)$, one can know the time dependence of any expectation value via Eq. (14).
Also, from Eqs. (21) and (22) we found in [14] that

$$I_\lambda \;=\; \frac{1}{2\hbar}\ln\left(\frac{\sqrt{I}+\frac{\hbar}{2}}{\sqrt{I}-\frac{\hbar}{2}}\right), \tag{24}$$

relating I_λ to I. Note here that as I decreases, I_λ augments. If I approaches $\hbar^2/4$, then $I_\lambda \to \infty$, since $X^2 + \hat{P}^2$ approaches the ground state. Even then $I \neq 0$. Thus, we do not reach the classical limit yet. We need to take the limit $\hbar \to 0$ and still $I_\lambda \to \infty$ holds [14].

6. Present results regarding the classical limit (CL)

Our present elaborations begin at this point. We are going to analytically study the limit $I \to 0$ of the density operator (17). Speaking of a CL entails that both $\hbar$ and $I \to 0$. In going to this limit we must always respect the restriction (4). Two roads are open to us

1. Take first $\hbar \to 0$ (and then $I \to 0$). Classical statistics and quantum one are both compatible with (4), for any $\hbar > 0$. Additionally in the classical case may be $\hbar = 0$. In the limit $\hbar \to 0$, the density matrix (17) adopts the form

$$\rho = \frac{\mathcal{I}}{Tr[\mathcal{I}]}, \tag{25}$$

with $\mathcal{I}$ the identity matrix. One has

$$\lim_{\hbar \to 0} I_\lambda = \frac{1}{2\sqrt{I}}, \tag{26}$$

as a result of

$$\lim_{\hbar \to 0} \hbar\, I_\lambda = 0, \tag{27}$$

where we employed Eq. (24). (25) is the maximally mixed density matrix of diagonal elements $1/n$, $n \,\varepsilon\, \mathcal{N}$, with $n \to \infty$. Such matrix should arise out of a decoherence process. We have obtained a **statistical quantum limit**. The limit $I \to 0$ would entail classicality and can not be taken now. To better understand this issue an analysis made with classical statistic is added in the Appendix.

2. Proceed to effect $\lim_{\hbar \to 0} \lim_{I \to \hbar^2/4} \Lambda$, Λ referring here to any of our quantities of interest. This second choice of venue respects the restriction (4) and would constitute the correct way to go. According to (24), we have

$$\lim_{\hbar \to 0} \left(\lim_{I \to \hbar^2/4} I_\lambda \right) = \infty, \tag{28a}$$

$$\lim_{\hbar \to 0} \left(\lim_{I \to \hbar^2/4} \hbar\, I_\lambda \right) = \infty, \tag{28b}$$

$$\lim_{\hbar \to 0} \left(\lim_{I \to \hbar^2/4} \lambda_i \right) = \infty, \qquad i = 0, 1, 2, \tag{28c}$$

$$\lim_{\hbar \to 0} \left(\lim_{I \to \hbar^2/4} |\lambda_3| \right) = \infty. \tag{28d}$$

Note that in the second instance, when I tends to its minimum possible value $\hbar^2/4$, ρ (17) tends to its ground state. Thus, considering the pseudo *generalized temperature* $1/I_\lambda$, we ascertain that $1/I_\lambda \to 0$. Remark that I_λ depends on both the classical variables and the initial conditions for the EVs. Our results holds also for $\hbar \to 0$. Lo and behold, we have found that the classical limit is represented by a pure-state density matrix!.

Looking at the asymptotic behavior of λ_0 en (19), we see that $\exp(-\lambda_0) \sim \exp(\hbar\, I_\lambda)$, entailing that the asymptotic eigenvalues of ρ become $\exp\left[-n\hbar I_\lambda\right]$, $n = 0, 1, 2, \ldots$. Thus, ρ (17) (or (5)), asymptotically, in its eigen-basis has the associate density matrix $\mathcal{R}(t)$

$$\mathcal{R}(t) = \begin{pmatrix} 1 & 0 & 0 & \ldots \\ 0 & 0 & 0 & \ldots \\ 0 & 0 & 0 & \ldots \\ & \vdots & & \end{pmatrix}. \tag{29}$$

This is a rather surprising. Not only the classical features of the semiclassical evolution depicted in Figs. 1 and 2 of [14] are represented by a mixed quantum density matrix, but purely classical results with $I = 0$, are masked by a pure-state density matrix. In the first case semi-classical chaos is obtained. **In the second case, totally classical Chaos,** because the classical system is chaotic [14].

The expectation values $\langle \hat{X}^n \hat{P}^m \rangle$ will be null at all times, thus being trivially classic. Additionally, the EVs of the set $(\hat{x}^2, \hat{p}^2, \hat{L})$ will evolve asymptotically with the classical equations corresponding to the classical counterpart of our quantum Hamiltonian (3). Any other asymptotic value of a given EV can be calculated using equations (14) and (16).

As a proof of the correctness of our results, it is easy to see that I calculated with $\rho(t)$ given by (29) vanishes. Denoting the ground state by $|0>$, we have $< 0|\hat{X}^2|0 >=< 0|\hat{P}^2|0 >= \lim_{\hbar\to 0} \hbar/2$ and $< 0|\hat{L}|0 >= 0$, so that $I = 0$. Moreover, via (13), we obtain for the entropy

$$S = -\lambda_0 - 2\, I_\lambda \sqrt{I}, \tag{30}$$

which is a decreasing monotonic function of I, with asymptotic value $S = 0$, as expected for a pure state. In this way, the Density Operator smoothly y becomes less and less mixed, as I tends to zero, until it is gets represented by a pure-state density matrix.

7. Results and Conclusions

In this work we have exhaustively investigated the classical limit of a density operator ρ associated to a well-known non-linear semi-classical system that possesses both classical and quantum interacting degrees of freedom. This ρ was presented previously in [14], in a context of *incomplete* prior information.

In [14] its authors detected three well delimited and different regions in traversing the road towards the classical limit. These zones were characterized by the parameter $E_r = \frac{|E|}{I^{1/2}\omega_q}$, con $E_r \to \infty$, with E the total energy and I a dynamical invariant intimately linked to the uncertainty principle.

One had a quasiclassical region, a transitional one, and a classical zone. As E_r grows, complexity augments and, eventually, chaos emerges. This was a phenomenon of a semi-classical nature. On the other hand, the analogous classic system is chaotic.

It is article focused attention specifically on the classical limit per se, not on the road to it as in [14].

A purely analytical treatment was effected, for $I \to 0$. Two possible paths were contemplated to perform our study. The first was to research the $\hbar \to 0$ calculation. Some difficulties were encountered in such instance, that were discussed in the text.

The second path turned to be both correct and coherent. It consist in taking first $\lim I \to \hbar^2/4$, approaching the minimum $I-$value that quantum mechanics permits. A posteriori one deals with the limit $\hbar \to 0$. In quite a counter-intuitive fashion, we stumbled on an asymptotic density matrix $\mathcal{R}$ corresponding to a pure state (29). $\mathcal{R}$ adequately describes classical features.
Indeed, the EVs of the set $(\hat{x}^2, \hat{p}^2, \hat{L})$ will evolve asymptotically with the classical equations corresponding to the classical counterpart of our Hamiltonian. In particular, we conclusively showed that $\mathcal{R}$ competently describes classical chaos.

8. Acknowledgments

A. M. K. acknowledges support from the Comisión de Investigaciones Científicas de la Provincia de Buenos Aires (CICPBA) of Argentina.

Appendix A The pertinent classical statistical limit treatment

For completeness, let us consider the concomitant classical statistical procedure. This analysis could shed some clarity on oyr proceedings. The inequality (4) is satisfied both for the pure quantum case and for both a quantum and a classical statistics. To avoid notation problems, we rewrite (4)) for the classical case as

$$I_{cl} = \langle x^2\rangle\langle p^2\rangle - \frac{\langle L\rangle^2}{4} \geq \frac{k^2}{4}, \tag{A1}$$

where I_{cl} is the classical version of (4) and x^2, p^2 and $L = 2xp$ are simple functions. We have introduced the constant k for obvious convenience. k is any number that verifies $k \geq 0$ and plays the role of $\hbar$ here. **Obviously, taking the limit $\hbar \to 0$ in (4), is equivalent to taking the limit** $k \to 0$. In other words, this limit is compatible with both statistics and the result does not express certainty in any case. To solve this situation in the quantum case, it is clear that the second path of the previous section must be used. Let us see now how to proceed in classic case. Let us consider the equivalent classical statistical case. The pertinent MaxEnt Probability Density Function corresponding to (11) is

$$\rho(x, p, t) = \exp -\Big(\lambda_{0cl} + \lambda_{1cl}x^2 + \lambda_{2cl}p^2 + \lambda_{3cl}L\Big). \tag{A2}$$

The mean value of any general $F(x, p, t)$, for all t, is given via $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(x, p, t)\,\rho(x, p, t)\,dxdy$. Using a transformation equivalent to (16), but for classical variables, we obtain the classical version of (17), with $\lambda_{0cl} = \ln(\pi/I_{\lambda cl})$. After some manipulation we are led to

$$\langle x^2\rangle = \frac{\sqrt{I_{cl}}}{I_{\lambda cl}}\lambda_{2cl}, \tag{A3a}$$

$$\langle p^2\rangle = \frac{\sqrt{I_{cl}}}{I_{\lambda cl}}\lambda_{1cl}, \tag{A3b}$$

$$\langle L\rangle = -\frac{2\sqrt{I_{cl}}}{I_{\lambda cl}}\lambda_{3cl}, \tag{A3c}$$

where

$$I_{\lambda cl} = \Big(\lambda_{1cl}\lambda_{2cl} - \lambda_{3cl}{}^2\Big)^{1/2}, \tag{A4}$$

is a time-invariant quantity, since the λ_{icl} obey the same system of equations used in the quantum treatment (Eqs. 23). Moreover, Eqs. (A3) coincide with Eqs. (20), together with (22). However, in this instance the dependence of $I_{\lambda cl}$ with I_{cl} is not given by (24), since

$$I_{\lambda cl} = \frac{1}{2\sqrt{I_{cl}}}, \tag{A5}$$

but will coincide with Eq. (26), as one may expect. Obviously, to complete the present analysis, the limit given by $I_{cl} \to 0$ (or $I_{\lambda cl} \to \infty$) is demanded. The probability density function (A2) will read

$$\lim_{I_{cl}\to 0}\rho(x, p, t) = \delta(X)\delta(P), \tag{A6}$$

being a Dirac delta function of X and P, as one should expect. In the limit (A6), also $\langle\hat{X}^n\hat{P}^m\rangle = 0$ at all times and all results with total certainty are obtained via (16).

References

1. Zeh, H. D. Why Bohms quantum theory?. *Found. Phys. Lett.* **1999**, *10*, 197-200.
2. Zurek, W. H. Pointer basis of quantum apparatus: Into what mixture does the wave packet collapse? *Phys. Rev. D* **1981**, *24*, 1516.
3. Zurek, W. H. Decoherence, einselection, and the quantum origins of the classical. *Rev. Mod. Phys.* **2003**, *75*, 715.

4. Halliwell, J. J.; Yearslay, J. M. Arrival times, complex potentials, and decoherent histories. *Phys. Rev. A* **2009**, *79*, 062101.
5. Everitt, M.J.; Munro, W.J.; Spiller, T.P. Quantum-classical crossover of a field mode. *Phys. Rev. A***2009**, *79*, 032328.
6. Bloch, F. Nuclear Induction. *Phys. Rev.* **1946**, *70*, 460.
7. Milonni, P.; Shih, M.; Ackerhalt, J.R. Chaos in Laser-Matter Interactions. World Scientific Publishing Co., Singapore, 1987.
8. Ring, P., Schuck, P. The Nuclear Many-Body Problem, Springer-Verlag: Berlin, Germany, 1980.
9. Kowalski, A.M.; Rossignoli, R. Nonlinear dynamics of a semiquantum Hamiltonian in the vicinity of quantum unstable regimes *Chaos, Solitons and Fractals* **2018**, *109*, pp. 140-145.
10. Cooper, F.; Dawson, J.; Habib, S.; Ryne, R.D. Chaos in time-dependent variational approximations to quantum dynamics.*Phys. Rev. E.* **1998**, *57*, 1489.
11. Kowalski, A.M.; Plastino, A.; Proto, A.N. Classical limits. *Phys. Lett. A.* **2002**, *297*, pp. 162-172.
12. Kowalski, A.M.; Martín, M.T.; Plastino, A.; Rosso, O.A. Bandt-Pompe approach to the classical-quantum transition. *Phys. D.* **2007**, *233*, 21.
13. Kowalski, A.M.; Martin, M.T.; Plastino, A.; Judge, G. Kullback-Leibler Approach to Chaotic Time Series. *SOP Transactions on Theoretical Physics 1* **2014**, *3*, pp. 40-49.
14. Kowalski, A.M.; Plastino, A. Chaotic density matrix in the classical limit. *Phys. Lett. A.* **2020**, *384*, 126450.
15. Coddington, E.; Levinson, N. Theory of Ordinary Differential Equations, McGraw-Hill Publishing, New York, 1976.
16. Katz, A. Principles of statistical mechanics, Freeman, San Francisco, 1967.
17. Alhassid, Y.; Levine, R.D. Connection between the maximal entropy and the scattering theoretic analyses of collision processes. *Phys. Rev. A.* **1978**, *18*, 89.