

Article

The Entropy Production of Galaxies

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Abstract: Double-spiral galaxies are common in the Universe. It is known that the logarithmic double spiral is a Maximum Entropy geometry and represents spiral galaxies well. It is also known that the virial mass of such a galaxy can be approximately determined from the entropy of its central supermassive black hole. But over time the black hole must accrete mass, and therefore the overall galactic entropy must increase. From the associated entropic Euler-Lagrange equation (forming the basis of the Principle of Least Exertion, and also enabling the application of Nöther's theorem) we show that the galactic entropy production is a conserved quantity, and we derive an appropriate expression for the relativistic entropic Hamiltonian of an idealised spiral galaxy. We generalise Onsager's celebrated expression for entropy production and demonstrate that galactic entropy production has two parts, one many orders of magnitude larger than the other, and where the smaller is comparable to the Hawking radiation of the central supermassive black hole. We conclude that galaxies cannot be isolated, since even idealised spiral galaxies have non-zero entropy production.

Keywords: Quantitative Geometrical Thermodynamics; info-entropy; galactic evolution; Maximum Entropy Production Principle; accretion

1. Introduction

The logarithmic double-spiral is ubiquitous in Nature, being seen for example in sunflower seed patterns, the shells of Nautilus snails, cyclones, and spiral galaxies. Parker & Jeynes [1] ("PJ2019") have demonstrated that it is a Maximum Entropy (**MaxEnt**) geometry, in the framework of their theory of Quantitative Geometrical Thermodynamics (**QGT**). The idea of a geometry having an entropy is counter-intuitive, but is a consequence of the systematic consideration of the complementary nature of information and entropy that has become accepted since Shannon's seminal introduction of his "information entropy" [2]. We should note that the concept of a "geometric entropy" is already well accepted in quantum gravity contexts (see for example Vacaru *et al.* [3]), but is not usually treated as an intrinsic property of geometrical structure, although perhaps Quevedo's "Geometrothermodynamics" [4] is an exception. We should also note that Wang *et al.* [5] have recently specifically investigated the *entropy production* of certain natural growth processes.

The point is that the stability of a geometry implies that it does not change in time, and if it is also in static equilibrium then no entropy is being produced. PJ2019 considered information specifically as a 3-D integration over time in Minkowski 4-space and showed that *entropy* and *information* are mutually Hodge-duals, where *info-entropy* can have the same geometry as the electromagnetic field in free-space (the double-helix). Both photons and DNA have a double-helix geometry, which PJ2019 proved MaxEnt, and both photons and DNA are very stable. And PJ2019 (their Fig.1 and context) demonstrated that the weird assertion of DNA having a "geometrical entropy" is actually supported in detail by observation.

Now the double-helix (instanced by DNA and photons) is a stable geometry having zero entropy production (being in static equilibrium). The logarithmic double-spiral (of which the double-helix is only a special case) is also MaxEnt, but being instanced by living systems (sunflowers, Nautilus) and violent systems (cyclones, galaxies) demonstrates that being MaxEnt still permits a *dynamic* thermodynamic equilibrium: entropy is certainly being produced in these systems! In this work we develop QGT to display the far-reaching implications of the theory in considering the evolution in time of such MaxEnt systems.

In particular, PJ2019 explicitly showed that spiral galaxies could be idealised as logarithmic double-spirals in a full *entropic* Lagrangian-Hamiltonian treatment (see their Fig.2 and context). Again, this representation is supported in detail by observation: “the galactic *shape*, *aspect ratio* and *structural stability* (which are all highly constrained by the algebra) are consistent with observation” (PJ2019 p8: *italics original*); moreover an eigenvalue of the appropriate entropic Hamiltonian can be related to the galactic *virial mass*, also taking a realistic value.

A few further remarks are needed to set the scene. Variational principles have a fundamental importance in physics: the *Principle of Least Action* also applies (*mutatis mutandis*) in quantum mechanics, as Richard Feynman famously pointed out in 1942 [6]. Edwin Jaynes first formulated the principles of *Maximum Entropy* [7] and *Maximum Caliber* [8] (helpfully reviewed by Pressé *et al.* [9] and further expounded by Dixit *et al.* [10]), and of course we here follow both Willard Gibbs [11] and Lars Onsager [12] who saw the central importance of the variational calculus in the entropic context. Hans Ziegler saw the implications of Onsager’s work in the *Maximum Entropy Production Principle* [13] and subsequently developed it [14], while Martyushev & Seleznev [15] explain that Ilya Prigogine’s apparently contradictory *Minimum Entropy Production Principle* [16] is actually complementary. A helpful context to entropy production is also offered by Walter Grandy in his 2008 textbook [17].

Action is defined as a path integral of the appropriate kinematic Lagrangian. But PJ2019 (Eq.12) isomorphically define the *Exertion* as a path integral of the *entropic* Lagrangian. Action is in the energy-and-time or momentum-and-length domains (with units of Planck’s constant), where exertion is in the entropic-momentum and hyperbolic-space domain (with units of Boltzmann’s constant). Moreover, just as there is a *Principle of Least Action* so there is also (an exactly isomorphic) *Principle of Least Exertion*.

Exertion is a somewhat elusive concept since hyperbolic-space is different from the conventional Euclidean representation of spacetime (although it obeys the familiar Minkowski metric): Euclidean geometry is a convenient but only a local approximation to hyperbolic-space. Note also that galaxies are enormous and must be treated in hyperbolic-space: they are spread across a large canvas of spacetime where local linear (Euclidean) geometries must fail (but noting that both hyperbolic and Euclidean representations of spacetime are assumed flat [18]).

We have previously constructed the appropriate (non-relativistic) *entropic* Hamiltonian, proving that it satisfies the Euler-Lagrange relations (PJ2019 Eq.11 and context). In this work we derive a fully relativistic entropic Hamiltonian (where the space-like and time-like entropic momentum terms are properly interchangeable), showing also why this is necessary to develop the treatment of *entropy production* needed to analyse the evolution over time of MaxEnt systems far from thermal equilibrium. Idealised spiral galaxies certainly are far from equilibrium (since the central supermassive black hole must grow): QGT methods are probably critical to establishing a full account of such galactic evolution, since QGT treats the logarithmic double-spiral shape as a primary geometrical property rather than as an emergent property of gravitationally-driven density wave dynamics [19].

Onsager’s seminal treatment of thermodynamics has been applied previously to galaxies, as described in the extensive review by Chavanis [20] who points out “*that the statistical mechanics of two-dimensional vortices [citing Onsager’s 1949 treatment [21]] and self-gravitating systems present a deep analogy despite the very different physical nature of these*

systems.” Chavanis & Sommeria [22] have developed this thermodynamical approach for (idealised) elliptical galaxies, in particular looking for Maximum Entropy system responses. In an extended section we generalise Onsager’s original treatment of entropy production [12], presenting a proof of the limit at which our treatment becomes exactly equivalent to his. For a secure understanding of real galaxies we will need a rigorous thermodynamical framework: even if in the present work we have to simplify the treatment for an idealised system, a part of our purpose here is to establish some general results to aid further development.

2. Hyperbolic Space in QGT

To treat the logarithmic double-spiral (whose axis is singled out as the x_3 direction) we use a hyperbolic spacetime, with a space-like dimension $q = R \ln(x/R)$, where x is a normal (Euclidean) space measure and R is a (Euclidean) normalising metric which can often be associated with a radius of the structure under consideration; and with the *entropic* Lagrangian $L_s(q, q', x_3)$ equations based upon the hyperbolic spacetime dimensions q , and their spatial derivatives q' with respect to the x_3 axis. The associated *entropic* Hamiltonian $H_s(q, p, x_3)$ is defined (PJ2019 Appendix B), as is conventional, using the *entropic* momentum p . Parker & Jaynes (PJ2019 Appendix B, Introduction) note that “*in our analysis, the entropic Lagrangian is defined in hyperbolic 3-space q , and its variation is performed with respect to the Euclidean x_3 spatial parameter (thus, $q' \equiv \partial q / \partial x_3$). This is in contrast to the conventional energetic approach, where the kinematic Lagrangian is defined in Euclidean 3-space x , and varied according to the time parameter t* ”. Thus the kinematic (“action”) representation has time-like qualities whereas the entropic (“exertion”) representation is more space-like.

The conjugate variables for the entropic Lagrangian-Hamiltonian analysis are $\{q, p\}$ which are isomorphic to the position-momentum conjugates $\{x, p\}$ of the kinematic Lagrangian (see the list of isomorphisms in PJ2019 Table 1). But in the entropic treatment of QGT, q is the hyperbolic-position with dimension of length (and units: [m]); and p is entropic momentum with dimension of entropy per unit length (and units: [(J/K)/m] or [J/K·m]). Note that the canonical relations hold as usual for the entropic $\{q, p\}$: that is, the equations of state are given as expected by $q' = \partial H_s / \partial p$, and $p' = -\partial H_s / \partial q$ (see PJ2019 Appendix B Eq.B17), where as before the primes indicate differentiation with respect to the space (not the time) dimension ($\partial / \partial x_3$: thus, the hyperbolic velocity q' is dimensionless). In addition, in a separate treatment Parker & Jaynes [23] show how the canonical relations also define an entropic phase space with properties that obey Liouville’s theorem and associated expressions based on the Poisson bracket.

In QGT, because q' is now dimensionless (as is required for taking the logarithms used in hyperbolic space) there is an ambiguity between q' and its inverse $1/q'$. The implications of this are also explored by Parker & Jaynes ([23]: see their Eqs.23 and context), but here we merely observe that the entropic momentum is defined as $p \equiv m_s/q'$ (PJ2019 Eq.9b), but when q' is apparently greater than unity (that is, equivalent to the *phase* hyperbolic velocity q'_ϕ , see the context of Eqs.1 and see [23]) the inverse case applies and we must use $q' \equiv p/m_s$. The entropic mass m_s is given by $m_s \equiv i\kappa_0 k_B$ (where κ_0 is a parameter of the system that looks like a “wavenumber”). That is, the entropic mass m_s is an imaginary quantity ($i^2 = -1$), and it scales with Boltzmann’s constant k_B . The parameter κ_0 is essentially the system’s wavenumber (or the inverse of the holographic wavelength: see [23] for a discussion of the *holographic principle* in this context).

Why use hyperbolic spacetime? The underlying reason is that the laws of thermodynamics are valid at all scales, and it is only in hyperbolic space that the (dimensionless) “velocity” variable q' and its inverse can be considered to be “equivalent” to each other in the sense of yielding entropic properties with reciprocal symmetries (this is developed in [23]). QGT has been demonstrated at the galactic scale (PJ2019), at the molecular scale (for Buckminsterfullerene [24]) and at the nuclear scale [25]. So hyperbolic space clarifies the essentials of systems: both supermassive black holes and alpha

particles are represented in QGT as unitary entities requiring only four scalar quantities for a complete specification: the mass, the charge, the spin, and a scaling parameter (κ_0) which for black holes is related to the Planck length (see the discussion of the Bekenstein-Hawking equation in both PJ2019 and [23]) and for alpha particles is related to the diameter of the proton (see [25]).

3. The Relativistic Entropic Hamiltonian in QGT

In the same way that we can take the Hamiltonian as expressing the *energy* of the system, given by the sum of the kinetic (T) and potential (V) *energy* terms ($H = T + V$), we consider the (non-relativistic) *entropic* Hamiltonian H_s to be the sum of the kinetic (T_s) and potential (V_s) *entropy* terms (PJ2019 Appendix B Eq.B.13b, p.30: note that the non-relativistic Lagrangian is given in PJ2019 Eq.B.40a, and Eq.B.13b is explicitly evaluated in Eq.B.40b):

$$H_s = T_s + V_s = \sum_{n=1}^3 -m_s \ln q'_n + V_s(q_n) \quad (1a)$$

where the subscripts S emphasise that the relevant quantities are *entropic* and the axis of the double-spiral (the x_3 direction) is not symmetrical with the other two directions (x_1 and x_2).

As already discussed, the entropic mass m_s is an imaginary quantity, and the conjugate variables of the entropic Hamiltonian are the vectors in hyperbolic 3-space $\{p, q\}$, where the (dimensionless) hyperbolic velocity q' satisfies the canonical relations. That is, T_s is a function of p (and hence q') alone, and V_s is a function of q alone, as required. However, PJ2019 made no distinction between the group q' and phase q_ϕ' hyperbolic velocities (see [23]), where $q' \leq 1$ and $q_\phi' \geq 1$, with $q' = 1/q_\phi'$. Both hyperbolic velocities yield the same magnitude for the kinetic entropy T_s , but of different sign due to its logarithmic character. Noting that p tends to be greater than m_s (as calculated for the double-helix DNA forms in PJ2019, and also as implied by the analysis below) we assume that the phase hyperbolic velocity is the quantity to be employed in Eq.1a, so that the inverse identity for the group hyperbolic velocity is applicable in this case, $q' = p/m_s$. Setting $V_s = 0$ we can rewrite Eq.1a as:

$$H_s = -\sum_{n=1}^3 m_s (\ln p_n / m_s) = -\sum_{n=1}^3 m_s (\ln p_n - \ln m_s) = m_s \ln m_s - m_s \ln p \quad (1b)$$

We note that the entropic momentum p is a vector in (hyperbolic) Minkowski 3-space: we omit the basis vectors in this space for clarity, but they are explicit in PJ2019 (see Eqs.1 and context, including Appendix A which also has a careful discussion of the appropriate Clifford algebra).

Setting the entropic potential V_s to zero for Eq.1b just follows the usual simplifications used in Special Relativity. In the case considered by PJ2019 (the Milky Way) this simplification is merely formal since they have proved that (in this case) certain approximations are valid, which means that “the hyperbolic accelerations [for the logarithmic double spiral] are therefore all zero indicating the effective absence of any entropic forces or any entropic potentials, $V_s = 0$ ” (see PJ2019 Appendix B after Eq.B.34c). They go on to say that “it is clear that in hyperbolic space the entropic Hamiltonian of a logarithmic double spiral ... is mathematically equivalent to that of a double helix” (for which see also Appendix B after Eq.B.43: “In [the double-helix] case, the entropic field reduces to [a constant] term ... that is ... we can equivalently assume $V_s = 0$ ”). Note that in the general case, the full (unapproximated) entropic Hamiltonian H_s for the logarithmic double-spiral is indeed a constant of the system (see PJ2019 Appendix B and the last line of their Eq.B.40b).

Equation Eq.1b is manifestly non-relativistic, since the two conserved quantities, the entropic Hamiltonian and the entropic momentum, H_s and p , as described by the Euler-Lagrange equations (and using Nöther’s Theorem) are clearly not interchangeable.

Hence, before we can make any progress, we need a credible entropic Hamiltonian that obeys the conventional rules of relativity.

The conventional Hamiltonian of kinematics gives the total energy of the system (a conserved quantity): the non-relativistic kinematical Hamiltonian H is frequently used in the Schrödinger equation and given in the non-relativistic approximation (and in the absence of any fields) by $H = p^2/2m$ (where in this case p is the kinematic momentum and m is the inertial mass as usual). In Special Relativity the total energy (ignoring the potential energy terms) is given by $E^2 = c^2p^2 + m^2c^4$ (c is the speed of light as usual), and we have:

$$H \equiv E_0 = (c^2p^2 + m^2c^4)^{1/2} \approx p^2/2m + mc^2 \quad (\text{for } p \ll mc) \quad (2)$$

It is also worth pointing out that the additional constant term mc^2 (the rest mass energy) is a background term which plays no part in the classical Lagrangian calculations since it simply differentiates away and can be ignored. Moreover, the kinematic momentum p is incommensurate with the inertial mass m : they have different units. In kinematics, c is needed to make m commensurate with p .

Considering the similar approximation of Eq.1b (that is, ignoring the potential energy terms; that is, setting $V_s = 0$) let us name the *unapproximated* relativistic entropic Hamiltonian we seek as " p_0 "; the subscript "0" indicating that this is a *time-like* entropic momentum. Then, taking the logarithm and form in Eq.1b as suggestive, we suppose:

$$H_s = m_s \ln(1 + p_0/m_s) \quad (3)$$

such that for $p_0 \ll m_s$ (analogous to $p \ll mc$ in the kinematic case of Eq.2) then $H_s \approx p_0$. Thus:

$$m_s \ln\left(1 + \frac{p_0}{m_s}\right) = m_s \ln p - m_s \ln m_s \quad (4)$$

This can now be straightforwardly rearranged as (remembering that m_s is imaginary):

$$p = p_0 \pm m_s \quad (5a)$$

which are clearly the solutions to the quadratic expression

$$p^2 = p_0^2 - m_s^2 \quad (5b)$$

where the conserved quantities (the entropic Hamiltonian p_0 and the entropic momentum p) are now clearly interchangeable. We have a relativistic entropic Hamiltonian! Writing out the 3-vector p in its components we have $p^2 = p_1^2 + p_2^2 + p_3^2$, and

$$p_0^2 = p_1^2 + p_2^2 + p_3^2 + m_s^2 \quad (6)$$

Subscript numbering in Eq.6 conforms to the conventional numbering for the space-time dimensions γ_μ where $\mu = \{0,1,2,3\}$; that is, γ_0 is the time-like axis and $\gamma_{1,2,3}$ are the space-like axes, using the geometric (Clifford) algebra notation. Note the absence of c in Eq.6. This is because the speed of light c in the kinematic domain is isomorphic to a maximum hyperbolic velocity $q' = 1$ in the entropic domain (recall that q' is dimensionless); moreover, in the entropic domain the entropic momentum is *commensurate* with the entropic mass.

It is clear that just as the kinematic Hamiltonian $H = p^2/2m$ is the non-relativistic approximation to the relativistic energy-momentum expression $E_0^2 = p^2c^2 + m^2c^4$, so the entropic Hamiltonian $H_s = m_s \ln(p/m_s)$ is a non-relativistic approximation to the entropic dispersion relation $p_0^2 = p^2 + m_s^2$. It is also interesting to note that, although they are non-relativistic approximations, both H and H_s still represent conserved quantities (as according to Nöther's theorem) in their respective Hamiltonian-Lagrangian dynamic

equations of state; although the full relativistic conservation laws require that it is E_0 and p_0 that are conserved in their respective kinematic and entropic domains.

In kinematics, considering the special relativity dispersion relation, the energy (that is, the kinematic Hamiltonian H) is equivalent to 'time-like momentum', while the conventional momentum terms are simply the 'space-like momentum' terms. That is why the entropic Hamiltonian H_s is designated as equivalent to p_0 since p is an entropic momentum (denoted as time-like by the subscript "0").

4. Relationship to Onsager's Differo-Integral

We now show how our relativistic entropic Hamiltonian (the positive solution to Eq.5a) is equivalent to Onsager's celebrated differo-integral (over a volume) for the rate of increase of entropy of a system (Eq.5.11b of [12]):

$$\dot{S} + \dot{S}^* - \Phi(J, J) = \int \left[J_n \frac{\partial}{\partial x^n} \left(\frac{1}{T} \right) - \phi(J, J) \right] dV \quad (7)$$

where Einstein's summation convention is in use.

The quantity \dot{S} is known as the *entropy production* (that is, the rate of increase of entropy), and the quantity \dot{S}^* is the entropy given off to the surroundings. In Eq.7 the quantity J_n is a heat flux or heat flow term (in the three space directions), T is the temperature, and V is the volume. The function $\phi(J, J)$ (and its volume integration Φ) is known as the "*dissipation function*", and according to [12] is interpreted as a "*potential*" function for the "*mutual interaction of frictional forces*", where such forces are dissipative (entropy producing). Thus ϕ is positive definite (in accordance with the 2nd Law). Onsager employs the variational principle to demonstrate that the quantity $\dot{S} + \dot{S}^* - \Phi(J, J)$ is a *maximum* for any system, such that one can write:

$$\delta [\dot{S} + \dot{S}^* - \Phi(J, J)] = \delta \int \left[J_n \frac{\partial}{\partial x^n} \left(\frac{1}{T} \right) - \phi(J, J) \right] dV = 0 \quad (8)$$

again using the summation convention.

Equations 7&8 can be simplified if the system under consideration is isolated, such that no heat flows across its boundary; that is, we can assume $\dot{S}^* = 0$. It is noteworthy that Onsager wrote his paper in 1931, long before Shannon expounded his probabilistic approach to entropy in his famous 1948 paper [2]; we have essentially used the Shannon definition of entropy to derive the variational equation for our entropic Hamiltonian H_s (PJ2019, Table 1):

$$\delta S = \delta \int H_s dx_3 = 0 \quad (9)$$

A key difference between Eq.8 and Eq.9 is that whereas Eq.8 is a volume integral, Eq.9 is a line integral. This is because Eq.9 assumes a holomorphism requiring an axis of rotational symmetry in the x_3 direction. Thus we first need to invoke a suitable cross-sectional area A , to make Eqs.8,9 mutually commensurate. A is constant with respect to x_3 since in hyperbolic space the logarithmic double-spiral behaves as a double-helix (see the context of Eq.1b).

Thus we assume the heat flux J_3 is across a cross-sectional area A in the x_3 direction, and for our entropic geometries of interest (in this case the 'cylindrical' geometry of the double helix) we assume that $J_1 = J_2 = 0$. The double helix geometry also implies $\partial/\partial x_1 = \partial/\partial x_2 = 0$. In addition, Onsager's equation is intrinsically based upon temporal derivatives of entropy (and energy), which are not present in Eq.9.

We therefore redeploy the relativistic entropic momentum term p_0 in place of the original (non-relativistic) entropic Hamiltonian H_s :

$$\begin{aligned}\delta S &= \delta \int H_S dx_3 = \delta \int p_0 dx_3 \\ \Rightarrow \delta \dot{S} &= \delta \left(\frac{\partial S}{\partial t} \right) = \delta \left(\frac{\partial}{\partial t} \int p_0 dx_3 \right) = \delta \left(\int p_0 d \frac{\partial x_3}{\partial t} \right) = \delta \left(\int p_0 dc \right) = \delta (cp_0)\end{aligned}\quad (10)$$

In Eq.10 we have assumed that p_0 is time-independent; that is, we assume that the entropic momenta (both time-like as well as space-like) of the stable spatial geometries of interest do not change over time (that is, $\delta \dot{S} = 0$). This is true for our maximum entropy systems, and the same is true in conventional kinematics (without dissipation and without the presence of potential fields) where the total energy E_0 is a constant of the system. We have also assumed that the velocity quantity $\partial x_3 / \partial t = c$ is simply the speed of light, which is also a relativistically-invariant universal constant.

Eq.10 shows that the time-like entropic momentum term p_0 is equivalent to the entropy production \dot{S} when made commensurate by the normalising constant c . This is important for the physical interpretation of p_0 . Thus:

$$\dot{S} \equiv cp_0 \equiv cH_S \quad (11)$$

Since the Hamiltonian can be offset by a constant factor we can rewrite the first line of Eq.10 with a (positive-definite) offset Φ/c , where we can interpret Φ as the volume-integrated dissipation function of Eq.7:

$$\begin{aligned}\delta S &= \delta \int (p_0 + \Phi/c) dx_3 \\ \Rightarrow \delta \dot{S} &= \delta (cp_0 + \Phi) \\ \Rightarrow \dot{S} - \Phi &= cp_0\end{aligned}\quad (12)$$

To prove complete consistency with Onsager's variational differo-integral of Eq.8 we now employ the positive solution of Eq.5a, $p \equiv p_n \gamma^n = p_0 + m_S$ (summation convention: as before, γ^n is the set of basis vectors describing the three spatial coordinates of Minkowski 4-space – see PJ2019 Eqs.1), which is the key expression linking Onsager's equation (Eq.7) to our geometric entropy analysis. We substitute in Eqs.10 & 12:

$$\begin{aligned}\delta (\dot{S} - \Phi) &= \delta \frac{\partial}{\partial t} \int p_0 dx_3 = \delta \frac{\partial}{\partial t} \int (p_n \gamma^n - m_S) dx_3 \\ &= \delta \int \frac{1}{A} \frac{\partial}{\partial t} (p_n \gamma^n - m_S) A dx_3 \\ &= \delta \int \frac{1}{A} \frac{\partial}{\partial t} \left(m_S \frac{\partial x_3}{\partial q_n} \gamma^n - m_S \right) dV\end{aligned}\quad (13)$$

(summation convention). In Eq.13 we have introduced a notional cross-sectional area A , so that the infinitesimal volume is given by $dV = A dx_3$, and where we also use the relation for the entropic momentum terms (PJ2019 Eq.9b): $p_n = m_S (\partial q_n / \partial x_3)^{-1} = m_S / q'_n$.

We emphasise that Onsager's analysis is based on a temperature T , whereas our analysis is not temperature-based and is resolutely purely *entropic* in nature; no notion of "temperature" is ever employed in our entropic analysis. This is because temperature is the coupling coefficient linking energy and entropy ($T = dE/dS$) enabling translation between the "entropic" and "energetic" domains. The isomorphism between the entropic and kinematic descriptions is illustrated in some detail in PJ2019 Table 1.

To demonstrate the equivalence of our approach to Onsager's, we insert a nominal (spatially uniform in the x_1 - x_2 directions) temperature term T into Eq.13. We also explicitly expand the entropic mass term m_S ($m_S \equiv i\kappa k_B$), also forcing it to be positive-definite as is required for a dissipation function term. We also take advantage of the (Fourier) identity relationship $d/dx_3 \equiv i\kappa$ (PJ2019 Eq.15) so that we can write:

$$\begin{aligned}
\delta(\dot{S} - \Phi) &= \delta \int \frac{1}{A} \frac{\partial}{\partial t} \left(m_s T \frac{\partial(x_3/T)}{\partial q_n} \gamma^n - |m_s| \right) dV \\
&= \delta \int \frac{\partial}{\partial t} \left(\frac{i \kappa k_B T}{A} \frac{\partial(x_3/T)}{\partial q_n} \gamma^n - \frac{\kappa k_B}{A} \right) dV \\
&= \delta \int \frac{\partial}{\partial t} \left(\frac{k_B T}{A} \frac{\partial}{\partial x_3} \frac{\partial(1/T)}{\partial q_n} \gamma^n - \frac{\kappa k_B}{A} \right) dV \\
&= \delta \int \left(\frac{\partial}{\partial t} \left(\frac{k_B T}{A} \right) \frac{\partial(1/T)}{\partial q_n} \gamma^n - \frac{\partial}{\partial t} \left(\frac{\kappa k_B}{A} \right) \right) dV
\end{aligned} \tag{14}$$

(summation convention). We note that the resulting $k_B T$ quantity is clearly an energy term, with the resulting quantity $\partial/\partial t(k_B T/A)$ therefore representing an energy flux term. This allows us to identify the following equivalent relations between our entropic geometry and Onsager's entropy equation:

$$J_3 \equiv \frac{\partial(k_B T/A)}{\partial t} \tag{15a}$$

$$\frac{\partial}{\partial x_n} \left(\frac{1}{T} \right) \equiv \frac{\partial}{\partial q_n} \left(\frac{1}{T} \right) \tag{15b}$$

$$\phi(J, J) \equiv \frac{\partial(\kappa k_B/A)}{\partial t} = \sqrt{|J_3|^2} \frac{\kappa}{T} \tag{15c}$$

Eq.15b makes clear the distinction between Euclidean and hyperbolic geometries: Onsager's formalism employs a Euclidean geometry where we employ a hyperbolic one. As we have already mentioned (and as discussed by PJ2019 – see their Eq.9a and context), when the entropic system is being considered close to its holographic boundary we find that hyperbolic position q closely approximates its Euclidian counterpart x . Thus, at least for the particular entropic geometries that we are interested in, where $\partial/\partial x_1 = \partial/\partial x_2 = 0$ and $J_1 = J_2 = 0$, we find that our relativistic entropic Hamiltonian formalism is therefore exactly equivalent to Onsager's variational approach.

It is worth noting that the dissipation functions in Eq.7 and Eq.12 are closely identified with the 'entropic mass' m_s term that we have previously defined; that is, the term ϕ can be understood to be the entropic mass-density flow, whereas Φ is the volume-integrated entropic mass-density flow. Being mass-like in character, it is also clear that all of m_s , ϕ and Φ must therefore also always be positive-definite.

Whereas Onsager considers the entropy production \dot{S} and Φ to be at a maximal extremum (in accordance with the 2nd Law), we have considered the entropy S in our variational analysis and have proved that it is also at a maximal extremum (MaxEnt) for our geometric structures of interest. Our analysis has also identified the quantity p_0 (the time-like entropic momentum which could be considered analogous to an 'entropic energy'), which is equivalent to the entropic Hamiltonian, and which we now find is also directly proportional to the entropy production \dot{S} according to Eq.11 and is therefore also positive-definite in accord with the 2nd Law.

It is now clear that we can therefore also generalise Onsager's differo-integral for the hyperbolic-space (q geometry) case (with the summation convention):

$$\dot{S} + \dot{S}^* - \Phi(J, J) = \int \left[J_n \frac{\partial}{\partial q^n} \left(\frac{1}{T} \right) - \phi(J, J) \right] dV_q \tag{16}$$

where the volume V_q is assumed to be in the appropriate hyperbolic space.

5. Conservation of Entropy Production

The entropy production \dot{S} is a conserved quantity (according to Nöther's theorem and the physics of Special Relativity) since the entropic Euler-Lagrange equations are satisfied and given by (PJ2019, Eq.13b):

$$\frac{d}{dx_3} \frac{\partial L_S}{\partial q'_n} - \frac{\partial L_S}{\partial q_n} = 0 \quad (n \in \{1, 2, 3\}) \quad (17)$$

where the entropic Lagrangian L_S is related to the entropic Hamiltonian via the Legendre transformation (PJ2019, Eq.11) $L_S = 3m_S - p_0$.

Eq.17 indicates that the space-like entropic momentum components $p_n \equiv \partial L_S / \partial q'_n$ (PJ2019, Appendix B, Eq.B.16c) must be conserved as required by Nöther's theorem. Taken with the relativistic entropic momentum dispersion relation (Eq.6) it is reasonable to assume that the entropic Hamiltonian p_0 is also conserved along with the entropic mass m_S – using the same mathematical and physical reasoning that makes energy, kinematic momentum and inertial mass conserved quantities in Special Relativity. The fact that the entropy production is simply the product of the entropic Hamiltonian with the speed of light c (a universal constant) means that the entropy production $\dot{S} \equiv cp_0$ must have equivalent properties to the entropic Hamiltonian, and consequently should also be a conserved (constant) feature of any entropic system under consideration (provided the usual conditions hold, such as no external potential fields).

6. The Entropy Production of an Idealised Spiral Galaxy

Here (as in PJ2019) we have considered a zeroth-order model of an isolated logarithmic double-spiral galaxy, consisting of a pair of spiral arms with a supermassive black hole at its centre. Such a geometry has maximum entropy and is therefore stable; but is not thereby necessarily in static equilibrium. In fact, since black holes necessarily grow the galactic entropy must be increasing. This is a boundary condition: the central black hole in the Milky Way has a Hawking temperature of 15 fK where the cosmic microwave background (CMB) temperature is about 3 K: photons at least must flow into the black hole.

We are here considering a simpler case: our model galaxy is isolated (we do not consider the CMB). We wish to explore the properties of this highly simplified model. In particular, we will demonstrate the remarkable fact that the entropy production for the isolated logarithmic double-spiral geometry *does not vanish*.

Of course, real supermassive black holes in the centres of real galaxies vigorously accrete hadronic matter in extraordinarily energetic processes, processes that are currently the subject of intense interest [26]. Undoubtedly QGT will provide valuable insights into the real situation, but at present we must start with our zeroth-order model.

Combining Eq.5a (the positive solution) and Eq.12 we consider the entropy production of the galaxy to consist of two parts on the RHS:

$$\dot{S} - \Phi \equiv cp_0 = cp - cm_S \quad (18)$$

We will show (Eqs.23,24) that the first component, cp (that is, the entropy production due to the entropic momentum p) is closely related to the Hawking radiation. It is the second component (related to the entropic mass m_S of the galactic structure) that is the dominant part of the entropy production.

We first consider the entropic-momentum component aspect cp , using the basic equation (PJ2019, Appendix B, Eq.B.7) for the entropic momentum in the vicinity of the black hole event horizon of Schwarzschild radius (r_{BH}) where the radius R of the local structure is given by $R = r_{BH}$:

$$p = \frac{k_B}{r_{BH}} \quad (19)$$

The black hole Schwarzschild radius for a black hole of mass M_{BH} is given by:

$$r_{BH} = \frac{2GM_{BH}}{c^2} \quad (20)$$

where G is the gravitational constant. We can then write the associated entropy production as:

$$\dot{S} = cp = \frac{ck_B}{r_{BH}} = \frac{c^3 k_B}{2GM_{BH}} \quad (21)$$

Multiplying by the black hole temperature T_{BH} , which is given by:

$$T_{BH} = \frac{\hbar c^3}{8\pi GM_{BH} k_B} \quad (22)$$

(where \hbar is the reduced Planck constant as usual) we then find that the associated power [J/s] associated with that entropy production is given by:

$$P = cpT_{BH} = \frac{c^3 k_B}{2GM_{BH}} \frac{\hbar c^3}{8\pi GM_{BH} k_B} = \frac{\hbar c^6}{16\pi G^2 M_{BH}^2} \quad (23)$$

which is similar to the Hawking radiation, given by [27]:

$$P_{Hawking} = \frac{\hbar c^6}{15360\pi G^2 M_{BH}^2} \quad (24)$$

Using the parameters of the Milky Way ($M_{BH} = 4.3 \times 10^6 M_\odot$) the Hawking radiation of the central supermassive black hole of the Milky Way is equivalent to a (negative) entropy production of 3.3×10^{-28} J/K.s (Eq.24), and the entropic momentum component of the entropy production of the idealised Milky Way galaxy is similar: 3.2×10^{-25} J/K.s (Eq.23).

Next, we consider the component of the overall galactic entropy production due to the intrinsic entropic mass m_s of the galactic structure. The entropic mass at the black hole's Schwarzschild radius is given by (ref.[1]):

$$m_s \equiv i\kappa_{BH} k_B = i \frac{2\pi}{l_P} k_B \quad (25)$$

where l_P is the Planck length given by:

$$l_P = \sqrt{\frac{\hbar G}{c^3}} \quad (26)$$

Since entropy production is a positive-definite quantity (as required by Eq.14) we can write the second component of the galactic entropy production as:

$$\Phi = c|m_s| = \frac{2\pi}{l_P} ck_B = 2\pi \sqrt{\frac{c^5}{\hbar G}} k_B \quad (27)$$

For the Milky Way in the vicinity of its supermassive black hole, and with $l_P = 1.616 \times 10^{-35}$ m, and $c = 3 \times 10^8$ m/s, we obtain the numerical value for the entropy production (dissipation function) of the Milky Way: 1.6×10^{21} J/K.s. That is, the *entropic mass* component of the entropy production of the idealised Milky Way galaxy is 46 orders

of magnitude larger than the *entropic momentum* component (3.2×10^{-25} J/K·s, Eq.23; comparable to the Hawking radiation, Eq.24).

7. Discussion

Entropy production in galaxies is key to galactic evolution, since it is a conserved quantity in the idealised representation: even in an evolving MaxEnt system it must be a constant. That is, to a very good approximation it specifies how fast the supermassive black hole at the galactic centre is growing – and this quantity is a *constant* (in the absence of any additional entropic potential fields)! It turns out that ideal galaxies are “simple” systems because the logarithmic double-spiral can be treated analytically, and because galaxies can be treated quite well as isolated systems – which sunflowers and cyclones certainly are not!

The present treatment is of an *idealised* spiral galaxy: that is, the energy (and entropy) flow from the CMB is neglected since the galaxy is considered as isolated from the environment. Also, the effects of stellar material are ignored: this is a reasonable “zeroth-order” approximation since it is well-known that so-called “dark matter” (and not the sensible hadronic matter) dominates the virial mass of galaxies. QGT is new, unfamiliar, and consequently conceptually rather challenging. We regard the present analysis as an essential first step in a realistic treatment of the thermodynamics of spiral galaxies, even though the calculated entropy production Φ for the (idealised) “Milky Way” is apparently too small.

Of course, in real galaxies the central supermassive black hole grows essentially by highly energetic accretion processes (such as Eddington or so-called super-Eddington accretion [26]), which involve entropy production orders of magnitude larger again than the rates we have calculated for the idealised galaxy; proper consideration of other relevant physical phenomena and parameters (accretion disk temperatures, stellar interactions and the effects of inter-stellar gas, gravitational density waves etc.) might account for the additional entropy production over and above the quantity predicted by our zeroth-order model.

It is worth commenting upon the *sign* of the component of the entropy production (Eq.23) that is comparable to the Hawking radiation even though it is almost insignificant when compared to the magnitude of the larger component (Eq.27). Clearly, the Hawking radiation represents mass (and therefore entropy) *loss* in contrast to the larger component associated with the accretion of mass by the black hole (and a positive entropy production for it). In our analysis, although \dot{S} is assumed positive definite, yet even Onsager’s equation (Eq.7) allows for a negative sign in front of the dissipation function Φ , which we showed is closely related to our entropic mass term m_s (Eqs.14 & 15c). Indeed, it is also noteworthy that in the context of minimum entropy production Ilya Prigogine suggests that “... [entropy] *production expresses a kind of ‘inertial’ property of nonequilibrium systems*” [28], thus alluding to the fact that there is a ‘mass-like’ aspect to entropy production, which we see here being expressed by the entropic mass m_s . Indeed, Eq.5a indicates that the two commensurate components (the entropic momentum p , and the entropic mass m_s) might intrinsically exhibit different signs and therefore physical behaviours. On the one hand, this is because Eq.5a admits both positive and negative (conjugate) solutions for the entropic mass component m_s ; while on the other hand m_s is explicitly imaginary with respect to p . Thus this indicates an intrinsically different origin and nature to these entropic phenomena: respectively more time-like (m_s) and more space-like (p) in their origin and behaviours. From this perspective, it is perhaps reasonable to assign the time-like (cms) entropy production term to the “dissipation function” Φ , and the space-like (cp) entropy production term to \dot{S} of Eqs.7 & 16 – where of course the signs of these terms must be consistent with the frame of reference: either that of the surrounding galaxy or that of the black hole (beyond the event horizon). That is, either side of the hyperbolic boundary, entropy production terms must change sign.

8. Conclusions

Using QGT we have for the first time calculated the entropy production of an idealised spiral galaxy which is defined only by its central supermassive black hole (that is, it is isolated from the Universe and ignores the presence of stellar material).

We have extended QGT to generalise Onsager's differo-integral formalism for the entropy production of systems in hyperbolic (Minkowski) 4-space: this shows that the relativistic entropic Hamiltonian directly determines the entropy production of the idealised galaxy; and also that both the entropic Hamiltonian and the entropy production are conserved quantities, in the absence of any entropic fields or forces. Applying this to a spiral galaxy (representing a maximum entropy unitary structure) we have demonstrated that the galactic rate of entropy increase remains constant even as the galaxy evolves. We find that the galactic entropy production consists of two components: an extremely small aspect that is related to the Hawking radiation of the central (supermassive) black hole, and a much larger contribution due to the entropic mass of the galaxy.

Extending the analysis to non-isolated systems (such as living entities, as well as weather-like natural phenomena) suggests the existence of a new system invariant (the entropy production) based on the principle of least exertion, that offers a new means to analyse and understand the system under consideration that is complementary to the conventional kinematical analyses based on the principle of least action.

Such a new QGT-based methodology to understand entropy-driven phenomena (such as black holes, galactic evolution, plant and animal life forms, meteorological manifestations, and dynamic chemical equilibria, to name but a few) offers both a powerful new analytical tool to evince new insights into their operation as well as the possibility for new methods to engineer their performance and efficiency.

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