 doi:10.20944/preprints202103.0789.v1

Article

Fractional Line Integral

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Abstract: This paper proposes a definition of fractional line integral, generalising the concept of fractional definite integral. The proposal replicates the properties of the classic definite integral, namely the fundamental theorem of integral calculus. It is based on the concept of fractional anti-derivative used to generalise the Barrow formula. To define the fractional line integrals the Grünwald-Letnikov and Liouville directional derivatives are introduced and their properties described. The integral is defined first for broken line paths and afterwards to any regular curve.

Keywords: Fractional Integral; Grünwald-Letnikov Fractional Derivative; Fractional Line Integral; Liouville Fractional Derivative

MSC: Primary 26A33; Secondary 26A42

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1. Introduction

It is no use to refer the great evolution that made Fractional Calculus invade many scientifical and technical areas [5,7–9,20]. Advances in various aspects of fractional calculus led to a question: why there are no fractional conterparts of some classic results? In fact and notwithstanding the progresses there are several singular situations. One of them was until recently the non existence of definition for "fractional definite integral". This gap has been filled by Ortigueira and Machado [14]. Here, we will try to fill in another gap, by introducing a definition of fractional line integral. This generalization was motivated by the results presented in [19] where classic theorems of vectorial calculus were introduced but for integrations over rectangular lines. With the integral introduced here, the Green theorem, for example, can be generalised. To do it, we profit the results stated in [17] to propose a fractional line integral. We use directional derivatives. However, as not all the directional derivatives are suitable, according to the considerations done in [17] we opted for the directional derivatives resulting from the generalization of the Grünwald-Letnikov (GL) and Liouville (L). These are introduced and their main properties listed. It is important to refer the presentation of the Liouville directional regularised derivative. We consider the classic definite Riemann integral and list its properties that serve as guide to defining the fractional definite integral, since we require they have the same properties. The fractional definite integral is expressed in terms of the anti-derivative generalising the Barrow formula.
For introducing the fractional line integral (FLI) we start by defining it on a segment of an oriented straight-line. This procedure is enlarged to broken-line paths formed by a sequence of connected

straight-line segments. Using a standard procedure consiting in approximating a curve by a sequence of broken-lines we introduce the integration over any simple rectifiable line. Its main properties are presented.
The paper outlines as follows. In Section 2 we introduce the required background. In Section 3 we describe the GL and L directional derivatives that we will use in definition of the fractional line integral. The corresponding properties are also presented. In Section 4 we introduce the fractional line integral and main properties.

2. Background

2.1. Functional framework

The theory we will develop below is expected to be useful in generalising the classic vectorial theorems suitable for dealing with fractionalisations of important equations of Physics as it is the case of Maxwell equations [12]. Therefore, we will need a framework involving functions $f(t)$, $t \in \mathbb{R}$, that are of exponential order (to have Laplace transform) or absolutely or square integrable (to have Fourier transform). In particular, we will assume also that they

1. are almost everywhere continuous;
2. have bounded variation;
3. verify:
$$|f(x)| < A\frac{1}{|x|^{\gamma+1}},\ \gamma, A \in \mathbb{R}^+, \text{ for } \ x < x_\infty.$$
In particular, we can have $f(x) \equiv 0,\ x < x_\infty$.

Remark 1. *If a given function has bounded support, we will extend it to $\mathbb{R}$, with a null extrapolation. This keeps our framework with the maximum generality.*

2.2. Suitable fractional derivatives

In [14] a discussion on the problem of fractional derivative (FD) definition was initiated and formalised through the introduction of two defining criteria. In [16] it was shown that for applications we require derivatives verifying the *strict sense criterion* that exiges functions defined on $\mathbb{R}$ in order to keep valid some classic relations. In this line of thought, we used such derivatives to be the base of the introduction of a definite fractional integral [17].They are the Grünwald-Letnikov (GL) and (regularised) Liouville (rL) derivatives [6,10,16,21]. The forward Grünwald-Letnikov and Liouville derivatives are given by

$$D^\alpha f(x) = \lim_{h\to 0^+} \frac{\sum\limits_{n=0}^{\infty} (-1)^n \frac{(-\alpha)_n}{n!} f(x-nh)}{h^\alpha}, \tag{1}$$

where $(a)_k$ represents the Pochhammer symbol for the raising factorial, and

$$D^\alpha_L f(x) = \frac{1}{\Gamma(-\alpha)} \int_0^\infty \left[f(x-\tau) - u(\alpha) \sum_{n=0}^{N} \frac{(-1)^n f^{(n)}(x)}{n!} \tau^m \right] \tau^{-\alpha-1} d\tau, \tag{2}$$

where N is the integer part of α, so that $\alpha - 1 < N \leq \alpha$, $N \in \mathbb{N}_0$. In spite of our work is based in the derivatives GL and rL we include one version of directional derivative for the usual Liouville derivative (L) [6,21]

$$^{RL}D^{\alpha}_{+}f(x) := \frac{1}{\Gamma(m-\alpha)} \frac{d^m}{dx^m} \int_{-\infty}^{x} (x-\xi)^{m-\alpha-1} f(\xi) d\xi, \tag{3}$$

and the Liouville-Caputo derivative (LC) [5]

$$^{LC}D^{\alpha}_{+}f(x) := \frac{1}{\Gamma(m-\alpha)} \int_{-\infty}^{x} (x-\xi)^{m-\alpha-1} \frac{d^m}{d\xi^m} f(\xi) d\xi, \tag{4}$$

where $m - 1 < \alpha \leq m$, $m \in \mathbb{Z}^+$.

The study of the equivalence of the two was done in [13]. It is a simple task to show that they are really equivalent for functions with LT or FT. Without intending to explore existence problems (see [13]) we can say that $f(x)$ must decrease to zero as x goes to $-\infty$, in agreement with our assumption above 2.1.

Remark 2. *The concept of "forward" is tied here to the causality in the sense of "going from past to future." This implies not only an order but also a direction on the real line.*

These derivatives enjoy relevant characteristics namely the index law (29) [10,13]. This means that given a FD of order $\alpha > 0$, there is a FD, of negative order, that we will call "anti-derivative" and verifying

$$D^{\alpha}D^{-\alpha}f(x) = D^{-\alpha}D^{\alpha}f(x) = f(x). \tag{5}$$

It can be shown [14] that the GL, rL, or LC, FD of the constant function are identically null. The L derivative of a constant function does not exist, since the integral is divergent. It must be stressed here that, for negative order (anti-derivative), rL, L, and LC are equal:

$$D^{\alpha}_{L}f(x) = \frac{1}{\Gamma(-\alpha)} \int_{0}^{\infty} f(x-\tau)\tau^{-\alpha-1} d\tau = -\frac{1}{\Gamma(-\alpha+1)} \int_{0}^{\infty} f(x-\tau) d\tau^{-\alpha}, \qquad \alpha < 0. \tag{6}$$

Remark 3. *Everything what will be done here can be replicated for the backward derivatives. We will not do it, since it is not so interesting.*

*2.3. Order **1** definite integral*

Consider a closed interval $[a, b] \subset \mathbb{R}$ where $f(x)$ is continuous. There are several ways of introducing the definite integral [1,2]. Probably, the simplest is through the Riemann sum.

Definition 1. *Divide the interval $[a, b]$ into N small intervals with lengths Δ_i, $i = 1, 2, \cdots, N$. Let ξ_i ($i = 1, 2, \cdots, N$) be any point inside each small interval. We call the define integral of $f(x)$ over $[a, b]$ the limit [1]*

$$S = \lim_{\max(\Delta_i) \to 0} \sum_{i=1}^{N} f(\xi_i)\Delta_i \tag{7}$$

The sum S will be represented by $\int_a^b f(\xi)d\xi$.

Using the time scale approach, we can define a nabla derivative and its inverse [15]. This is given by

$$f^{(-1)}(x) = \lim_{\sup(\Delta_i) \to 0} \sum_{i=0}^{\infty} \Delta_i f(x - h_i)$$

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where $h_i = \sum_{k=0}^{i-1} \Delta_k$. With this anti-derivative we can rewrite (7) as

$$\int_a^b f(x)dx = f^{(-1)}(b) - f^{(-1)}(a) \tag{8}$$

By simplicity, we can use equal length intervals $\Delta_i = \frac{b-a}{N} = h,\ i = 1, 2, \cdots, N,\ h > 0$, and ξ_i uniformly spaced, so that we can set, for example, $\xi_i = a + (i-1)h = b - ih,\ i = 1, 2, \cdots, N$. We have, then

$$\int_a^b f(x)dx = \lim_{h \to 0} h \sum_{i=0}^{N} f(a + ih) = \lim_{h \to 0} h \sum_{i=1}^{N+1} f(b - ih) \tag{9}$$

2.4. Properties of a definite integral

The integral defined in the previous sub-section has several interesting properties that we will require to be verified also by the fractional definite integral. They are

1. Linearity

$$\int_a^b [Af(x) + Bg(x)]\, dx = A \int_a^b f(x)dx + B \int_a^b g(x)dx \tag{10}$$

for any $A, B, \in \mathbb{R}$.

2. Limit reversion

$$\int_b^a f(x)dx = -\int_a^b f(x)dx \tag{11}$$

3. Domain partition
Let $a < c < b$. Then

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx = f^{(-1)}(c) - f^{(-1)}(a) + f^{(-1)}(b) - f^{(-1)}(c) \tag{12}$$

4. *Fundamental theorem of integral calculus* (FTIC)

Theorem 1. *Define $F(x)$ by*

$$F(x) = \int_a^x f(t)dt, \quad x \in [a, b] \tag{13}$$

The function $F(x)$ has three important properties: continuity on $[a, b]$, differentiability on (a, b), and

$$F'(x) = DF(x) = f(x), \quad x \in (a, b). \tag{14}$$

This leads to the well-known Barrow formula

$$\int_a^b f(x)dx = F(b) - F(a). \tag{15}$$

This result shows that the function $F(x)$ is aside an additive constant the anti-derivative of $f(x)$. This establishes the connection between anti-derivative and primitive. The anti-derivative is the left and right inverse of the derivative, while the primitive is only right inverse (see [18]).

 doi:10.20944/preprints202103.0789.v1

2.5. Fractional definite integral

A generalisation of the concept of definite integral must be conform with the properties described in the previous sub-section. The first approach to obtain a fractional definite integral was done in [17]. Here, we are going to present a slightly different definition.

Definition 2. *We define α-order fractional integral (FI) of $f(x)$ ($f(-\infty) = 0$) over the interval $(-\infty, a)$ by*

$$I^{\alpha} f(-\infty, a) = \frac{1}{\Gamma(\alpha)} \int_0^{\infty} f(a-\tau)\tau^{\alpha-1} d\tau = \frac{1}{\Gamma(\alpha+1)} \int_0^{\infty} f(a-\tau) \mathrm{d}\tau^{\alpha} = f^{(-\alpha)}(a), \tag{16}$$

For simplification, we will use the notation

$$I^{\alpha} f(-\infty, a) = \int_{-\infty}^{a} f(x) \mathrm{d}x^{\alpha}$$

Corollary 1. *If $b > a$, then:*

$$I^{\alpha} f(a, b) = \int_a^b f(x) dx^{\alpha} = f^{(-\alpha)}(b) - f^{(-\alpha)}(a), \tag{17}$$

In fact, this relation must be valid, in order to keep valid the formula (12). The relation (17) is nothing else than the fractional Barrow formula.

The definition of fractional definite integral (17) is consistent with *fractional fundamental theorem of integral calculus* (integer order):

Theorem 2.

$$I^{\alpha} D^{\alpha} g(a, x) = D^{-\alpha} D^{\alpha} g(x) - D^{-\alpha} D^{\alpha} g(a) = g(x) - g(a). \tag{18}$$

and

$$D^{\alpha} \left[I^{\alpha} g(a, x) \right] = g(x) \tag{19}$$

These results come immediately from the properties of the rL (or GL) derivative. In particular, the derivative of a constant is zero.

3. The Grünwald-Letnikov and Liouville directional derivatives

The usefulness, advantages, and properties of the GL and rL derivatives, introduced above, were studied in [12,13,16,18]. Here, we are going to present their directional formulations.

Definition 3. *Consider a function $f(\mathbf{x})$, $\mathbf{x} \in \mathbb{R}^n$, and let $\mathbf{v} \in \mathbb{R}^n$ be a unitary vector defining the direction of derivative computation and the half straight-line*

$$\left\{ \boldsymbol{\xi} : \boldsymbol{\xi} = (\mathbf{x} - kh\mathbf{v}), \; h \in \mathbb{R}^{+}, \mathbf{x} \in \mathbb{R}^n \; k \in \mathbb{N}_0 \right\}. \tag{20}$$

Consider a continuous function, $f(\mathbf{x})$, such that $|f(\mathbf{x} - k\mathbf{v})|$ decreases at least as $\frac{1}{k^{|\alpha|+1}}$, when $k \to \infty$ [12]. We define the GL directional derivative as

$$D_{\mathbf{v}}^{\alpha} f(\mathbf{x}) = \lim_{h \to 0} h^{-\alpha} \sum_{k=0}^{\infty} \frac{(-\alpha)_k}{k!} f(\mathbf{x} - kh\mathbf{v}), \tag{21}$$

where again $(a)_k$ represents the Pochhammer symbol for the raising factorial.

The relation of the GL and Liouville derivatives studied in [13] leads us to introduce similar definition for the directional case, considering the general regularised case, not presented elsewhere.

Definition 4. *We define the Liouville directional integral (anti-derivative) by*

$$D_{\mathbf{v}}^{\alpha} f(\mathbf{x}) = \frac{1}{\Gamma(-\alpha)} \int_0^{\infty} v^{-\alpha-1} f(\mathbf{x} - v\mathbf{v})\, dv \tag{22}$$

where $\alpha < 0$.

As well known, when $\alpha > 0$, the above integral is singular. However, it can be regularised through the procedure followed in [11,13] to get

$$D_{\mathbf{v}}^{\alpha} f(\mathbf{x}) = \frac{1}{\Gamma(-\alpha)} \int_0^{\infty} \left(f(\mathbf{x} - v\mathbf{v}) - \sum_{m=0}^{N} \frac{1}{m!} \frac{d^m f(\mathbf{x} - v\mathbf{v})}{dv^m}\bigg|_{v=0} v^m \right) v^{-\alpha-1}\, dv \tag{23}$$

where integer part of α: $N = \lfloor \alpha \rfloor$.

Definition 5. *We define the Liouville directional derivative (L) by*

$${}^{RL}D_{\mathbf{v}}^{\alpha} f(\mathbf{x}) = \frac{1}{\Gamma(m-\alpha)} D_{\mathbf{v}}^{m} \int_0^{\infty} v^{m-\alpha-1} f(\mathbf{x} - v\mathbf{v})\, dv, \tag{24}$$

where $D_{\mathbf{v}}^{m}$ means to apply m-times the usual directional derivative in the direction $\mathbf{v}$, $m-1 < \alpha \leq m$ and $m \in \mathbb{Z}^+$.

Definition 6. *We define the Liouville-Caputo derivative (LC) by*

$${}^{LC}D_{\mathbf{v}}^{\alpha} f(\mathbf{x}) = \frac{(-1)^m}{\Gamma(m-\alpha)} \int_0^{\infty} v^{m-\alpha-1} \frac{d^m}{dv^m} f(\mathbf{x} - v\mathbf{v})\, dv, \tag{25}$$

where $m-1 < \alpha \leq m$ and $m \in \mathbb{Z}^+$.

To test the coherence of the result, take the exponential $f(\mathbf{x}) = e^{\mathbf{s}\cdot\mathbf{x}}$, $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{s} \in \mathbb{C}^n$. Inserting $f(\mathbf{x})$ into (21) and (25), we obtain [21]

$$D_{\mathbf{v}}^{\alpha} e^{\mathbf{s}\cdot\mathbf{x}} = (\mathbf{s}\cdot\mathbf{v})^{\alpha}\, e^{\mathbf{s}\cdot\mathbf{x}}, \qquad Re(\mathbf{s}) > \mathbf{0}. \tag{26}$$

where the general integral representation for the Gamma function [4] was used. The expression $Re(\mathbf{s}) > \mathbf{0}$ means that each component $s_j, j = 1, 2, \cdots, n$ of $\mathbf{s}$ have positive real parts. Relation (26) is in agreement with similar results in [18,21] and means that

1. The exponential is the eigenfunction of introduced derivatives,
2. The (bilateral) Laplace transforms, $F(\mathbf{s})$ of the derivatives of $f(\mathbf{x})$ are given by

$$\mathcal{L}\left[D_{\mathbf{v}}^{\alpha} f(\mathbf{x})\right] = (\mathbf{s}\cdot\mathbf{v})^{\alpha}\, F(\mathbf{s}), \qquad Re(\mathbf{s}) > \mathbf{0}. \tag{27}$$

where $F(\mathbf{s}) = \mathcal{L}\left[f(\mathbf{x})\right]$. With $\mathbf{s} = i\omega$, we obtain the same result for the Fourier transform.

The main properties of the above derivatives are easily deduced from (22) to (25), see, for example, [10].

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1. Linearity
$$D_{\mathbf{v}}^{\alpha}\left[f(\mathbf{x})+g(\mathbf{x})\right]=D_{\mathbf{v}}^{\alpha}f(\mathbf{x})+D_{\mathbf{v}}^{\alpha}g(\mathbf{x}). \tag{28}$$
2. Commutativity and additivity of the orders
If $\alpha,\beta\in\mathbf{R}$
$$D_{\mathbf{v}}^{\alpha}\left[D_{\mathbf{v}}^{\beta}f(\mathbf{x})\right]=D_{\mathbf{v}}^{\beta}\left[D_{\mathbf{v}}^{\alpha}f(\mathbf{x})\right]=D_{\mathbf{v}}^{\alpha+\beta}f(\mathbf{x}). \tag{29}$$
3. Neutral and inverse elements
In particular, $\alpha+\beta=0$; so the inverse derivative exists and can be obtained by using the same formula.
$$D_{\mathbf{v}}^{\alpha}\left[D_{\mathbf{v}}^{-\alpha}f(\mathbf{x})\right]=D_{\mathbf{v}}^{0}f(\mathbf{x})=f(\mathbf{x}). \tag{30}$$
4. Rotation
Suppose that exists a matrix A, invertible, such that we can perform the variable change $A\mathbf{x}$ for $\mathbf{x}$. Then
$$\begin{aligned}D_{\mathbf{v}}^{\alpha}f(A\mathbf{x})&=\frac{1}{\Gamma(-\alpha)}\int_0^{\infty}u^{-\alpha-1}f\left(A\left[\mathbf{x}-u\mathbf{v}\right]\right)\,du\\&=\frac{1}{\Gamma(-\alpha)}\int_0^{\infty}u^{-\alpha-1}f\left(A\mathbf{x}-uA\mathbf{v}\right)\,du.\end{aligned} \tag{31}$$
As $uA\mathbf{v}=u\left\|A\mathbf{v}\right\|\frac{A\mathbf{v}}{\|A\mathbf{v}\|}$, we introduce $\text{¸}=\|A\mathbf{v}\|\,u$ and $\mathbf{w}=\frac{A\mathbf{v}}{\|A\mathbf{v}\|}$ to get
$$D_{\mathbf{v}}^{\alpha}f(A\mathbf{x})=\frac{\|A\mathbf{v}\|^{\alpha}}{\Gamma(-\alpha)}\int_0^{\infty}w^{-\alpha-1}f\left(A\mathbf{x}-\text{¸}\mathbf{w}\right))\,d\text{¸}, \tag{32}$$
which leads to
$$D_{\mathbf{v}}^{\alpha}f(A\mathbf{x})=\|A\mathbf{v}\|^{\alpha}\,D_{\mathbf{w}}^{\alpha}f(\mathbf{x}')\big|_{\mathbf{x}'=A\mathbf{x}}, \tag{33}$$
that is also a generalisation of a classic result.

4. On the fractional line integrals

The introduction of the notion of fractional definite integral was done in [14] and reformulated above 2.5. Here, we reproduce such definition, but with a vectorial representation.

Let $f(\mathbf{x})$, $\mathbf{x}\in\mathbb{R}^n$, such that exist its directional derivatives of any order. We will asume the canonical base $\mathbf{e_j}$, $j=1,2,\cdots,n$. Let us denote their anti-derivatives along the x_1 axis by $f_{\mathbf{e_1}}^{(-\alpha)}(\mathbf{x})$.

Definition 7. *Let $\alpha>0$. We define α-order fractional integral of $f(\mathbf{x})$ over the interval $(-\infty,a)$ on the $x\mathbf{e_1}$ axis through*
$$I_{\mathbf{e_1}}^{\alpha}f_{\mathbf{x}}(-\infty,\mathbf{a})=\int_{-\infty}^{a}f(x\mathbf{e_1})dx^{\alpha}=f_{\mathbf{e_1}}^{(-\alpha)}(\mathbf{a}), \tag{34}$$
and
$$I_{\mathbf{e_1}}^{\alpha}f_{\mathbf{x}}(\mathbf{a},\mathbf{b})=\int_a^b f(x\mathbf{e_1})dx^{\alpha}=f_{\mathbf{e_1}}^{(-\alpha)}(\mathbf{b})-f_{\mathbf{e_1}}^{(-\alpha)}(\mathbf{a}), \tag{35}$$
where $(\mathbf{a},\mathbf{b})=(a\mathbf{e_1},b\mathbf{e_1})$.

Using the expression of the Liouville anti-derivative we can write
$$I_{\mathbf{e_1}}^{\alpha}f_{\mathbf{x}}(\mathbf{a},\mathbf{b})=\frac{1}{\Gamma(\alpha)}\int_0^{\infty}\left[f(b\mathbf{e_1}-x\mathbf{e_1})-f(a\mathbf{e_1}-x\mathbf{e_1})\right]dx^{\alpha}. \tag{36}$$

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From the standard (integer order) Barrow formula $\int_a^b f'(x)dx = f(b) - f(a)$ we obtain the expression

$$I^{\alpha}_{\mathbf{e_1}} f_{\mathbf{x}}(\mathbf{a},\mathbf{b}) = \frac{1}{\Gamma(\alpha)} \int_0^{\infty} \int_a^b D^1_{\mathbf{e_1}} f(y\mathbf{e_1} - x\mathbf{e_1}) dy dx^{\alpha}. \tag{37}$$

Using (29) it comes

$$I^{\alpha}_{\mathbf{e_1}} f_{\mathbf{x}}(\mathbf{a},\mathbf{b}) = \int_a^b f^{(-\alpha+1)}_{\mathbf{e_1}}(x\mathbf{e_1}) dx. \tag{38}$$

If the integration path is, instead of one base axis, any straight line, defined by a vector $\mathbf{v}$, we need to generalize the above procedure. Consider a scalar field $f : \mathbb{D} \subset \mathbb{R}^n \to \mathbb{R}$ and two points $\mathbf{a}, \mathbf{b} \in \mathbb{D}$, defining the vector $\mathbf{v} = \frac{\mathbf{b}-\mathbf{a}}{\|\mathbf{b}-\mathbf{a}\|}$.

Definition 8. *We define the fractional integral over the straight line segment from* $\mathbf{a}$ *to* $\mathbf{b}$ *by*

$$I^{\alpha}_{\mathbf{v}} f_{\mathbf{x}}(\mathbf{a},\mathbf{b}) = \int_a^b f(x\mathbf{v}) dx^{\alpha} = f^{(-\alpha)}_{\mathbf{v}}(\mathbf{b}) - f^{(-\alpha)}_{\mathbf{v}}(\mathbf{a}). \tag{39}$$

Theorem 3. *Suppose that the integration path* γ *is a sequence of* N *connected straight line segments* $\mathbf{fl}_k$, $k = 0, 1, 2, \cdots, N-1$ *with initial and final points* $\mathbf{a}_k$ *and* $\mathbf{a}_{k+1} = \mathbf{b}_k$*, respectively* ($\mathbf{a}_0 = \mathbf{a}$, $\mathbf{a}_N = \mathbf{b}$)*. Then*

$$I^{\alpha}_{\mathbf{fl}} f = \sum_{k=0}^{N-1} \int_{\mathbf{a}_k}^{\mathbf{a}_{k+1}} f(x\mathbf{v}_k) dx^{\alpha} = \sum_{k=0}^{N-1} \left[f^{(-\alpha)}_{\mathbf{v}_k}(\mathbf{a}_{k+1}) - f^{(-\alpha)}_{\mathbf{v}_k}(\mathbf{a}_k) \right], \tag{40}$$

where $\mathbf{v}_k = \frac{\mathbf{a}_{k+1}-\mathbf{a}_k}{\|\mathbf{a}_{k+1}-\mathbf{a}_k\|}$.

Proof. We only need to follow the Definition 8. □

In Figure 1 we illustrate the way how the directional derivative computations are done.

Remark 4. *We could be more general by letting the order vary from segment to segment.*

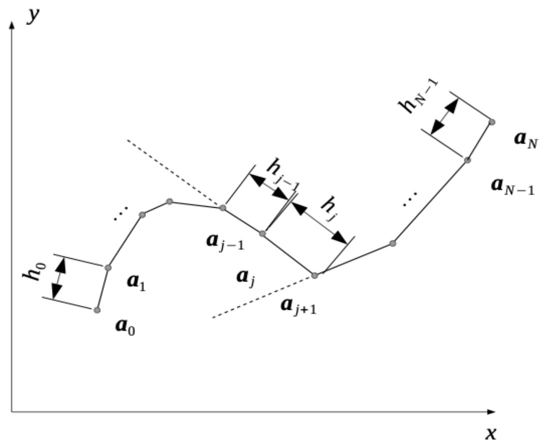

Figure 1. Broken line integration path.

Theorem 4. *Let γ be an rectifiable curve. Suppose that $f^{(-\alpha)}(\mathbf{x})$ is differentiable in a domain $\mathbb{D} \subset \mathbf{R}^n$ with $\gamma \subset \mathbb{D}$. Then*

$$I^{\alpha}_{\mathbf{fl}} f = \int_{\gamma} f(x\mathbf{v}) dx^{\alpha} = \lim_{N\to\infty} \sum_{k=0}^{N-1} f^{(-\alpha+1)}_{\mathbf{v}_k}(\mathbf{a}_k) h_k = \int_{\gamma} f^{(-\alpha+1)}_{\mathbf{v}}(\mathbf{fl}) \|d\gamma\| . \tag{41}$$

Proof. We construct a sequence of straight line segments $\bar{\gamma}$ approximating the curve γ, such that the initial and final points coincide (see Figure 2). For each segment, set $\mathbf{b}_k = \mathbf{a}_{k+1} = \mathbf{a}_k + h_k\mathbf{v}_k$, where $h_k > 0$ is the length of kth segment. If we consider several possible approximants, $\bar{\gamma}_n$, $n = 1, 2, \cdots$ for the curve and let h_{max} the maximum length in each approximant, then we obtain the expression for fractional line integral

$$\begin{aligned} I^{\alpha}_{\mathbf{fl}} f = \int_{\gamma} f(x\mathbf{v}) dx^{\alpha} &= \lim_{h_{max}\to 0} \sum_{k=0}^{N-1} \left[f^{(-\alpha)}_{\mathbf{v}_k}(\mathbf{a}_k + h_k\mathbf{v}_k) - f^{(-\alpha)}_{\mathbf{v}_k}(\mathbf{a}_k) \right] \\ &= \lim_{h_{max}\to 0} \sum_{k=0}^{N-1} \left[f^{(-\alpha)}_{\mathbf{v}_k}(\mathbf{b}_k) - f^{(-\alpha)}_{\mathbf{v}_k}(\mathbf{b}_k - h_k\mathbf{v}_k) \right] . \end{aligned} \tag{42}$$

As it is evident from the picture, the shorter the straight line segments the better is the approximation of the curve γ. In such situation, $h_i \approx h_j$ and $\mathbf{v}_k$ is approximately tangent to γ. Define $d\mathbf{fl}$ as the tangent vector to γ at $\mathbf{a}_k$, such that $\|d\mathbf{fl}\| = h_k$. When the length of straight line segments tends to zero we obtain at each point of the curve a unitary tangent vector $\mathbf{v} = d\gamma / \|d\gamma\|$, that assumes the role of $\mathbf{v}_k$ in (42).

Because $f^{(-\alpha)}$ is differentiable in the curve γ, we obtain

$$\left[f^{(-\alpha)}_{\mathbf{v}_k}(\mathbf{b}_k) - f^{(-\alpha)}_{\mathbf{v}_k}(\mathbf{b}_k - h_k\mathbf{v}_k) \right] = f^{(-\alpha+1)}_{\mathbf{v}_k}(\mathbf{b}_k) h_k + \eta(h_k),$$

with $\lim_{h_{max}\to 0} \eta(h_k) = 0$. It follows the desired result. □

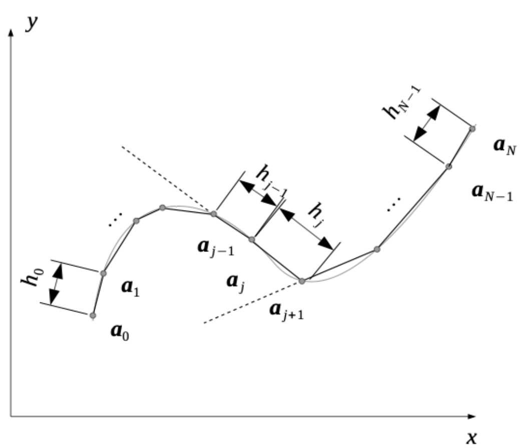

Figure 2. Rectifiable integration path.

Corollary 2. *Let the curve γ be parametrized by $\mathbf{r}(u)$, $a \le u \le b$, then*

$$I^{\alpha}_{\mathbf{fl}} f = \int_a^b f^{(-\alpha+1)}_{\mathbf{v}}(\mathbf{r}(u)) \|\mathbf{r}'(u)\| du. \tag{43}$$

where $\mathbf{v} = \mathbf{r}'(u) / \|\mathbf{r}'(u)\|$ and the fractional derivative, $f^{(-\alpha+1)}_{\mathbf{v}}(\mathbf{r}(u))$, is taken in order to $\mathbf{r}$, not to u.

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Example 1. *In the case when γ is a circumference of radius R, $\mathbf{r}(u) = (R\cos u, R\sin u)$, $0 \le u \le 2\pi$, (43) acquires the form*

$$I_{\text{fl}}^{\alpha} f = R\int_0^{2\pi} f_{\mathbf{v}}^{(-\alpha+1)}(R\cos u, R\sin u)du, \tag{44}$$

with $\mathbf{v} = (-\sin u, \cos u)$. By the definition of directional derivative we obtain that

$$\begin{aligned} I_{\text{fl}}^{\alpha} f &= \frac{R}{\Gamma(\alpha-1)}\int_0^{2\pi}\int_0^{\infty} v^{\alpha} f(R\cos u + v\sin u, R\sin u - v\cos u)dvdu \\ &= \frac{R}{\Gamma(\alpha-1)}\int_0^{\infty} v^{\alpha}\int_0^{2\pi} f(R\cos u + v\sin u, R\sin u - v\cos u)dudv \end{aligned} \tag{45}$$

If we define $\zeta(u) = (R\cos u + v\sin u, R\sin u - v\cos u), 0 \le u \le 2\pi$, then

$$\begin{aligned} ||\zeta'(u)|| &= ||(-R\sin u + v\cos u, R\cos u + v\sin u)|| \\ &= \sqrt{(-R\sin u + v\cos u)^2 + (R\cos u + v\sin u)^2} \\ &= \sqrt{R^2 + v^2} \end{aligned} \tag{46}$$

So

$$I_{\text{fl}}^{\alpha} f = \frac{R}{\Gamma(\alpha-1)}\int_0^{\infty}\frac{v^{\alpha}}{\sqrt{R^2+v^2}}\left[\int_0^{2\pi} f(\zeta(u))||\zeta'(u)||du\right]dv. \tag{47}$$

The integral between brackets is known as integral of line relative to arc length [3]. If f represents the mass of a thin wire $\zeta(u)$ per unit length, then $\int_0^{2\pi} f(\zeta(u))||\zeta'(u)||du$ is the total mass M_v of the wire. So that

$$I_{\text{fl}}^{\alpha} f = \frac{R}{\Gamma(\alpha-1)}\int_0^{\infty}\frac{v^{\alpha}M_v}{\sqrt{R^2+v^2}}dv. \tag{48}$$

Suppose that the parametrization $\mathbf{r}(u)$ of γ can be written in terms of two distint parameters u and τ and that $u_0 \le u \le u_1$ and $\tau_0 \le \tau \le \tau_1$. Hence

$$\int_{u_0}^{u_1} f_{\mathbf{v}}^{(-\alpha+1)}(\mathbf{r}(u))\left\|\mathbf{r}'(u)\right\|du = \int_{\tau_0}^{\tau_1} f_{\mathbf{v}}^{(-\alpha+1)}(\mathbf{r}(\tau))\left\|\mathbf{r}'(\tau)\right\|d\tau. \tag{49}$$

It follows that the fractional line integral (43) does not depend of the parametric representation of γ.

The line integral (49) has some interesting properties, easily deduced:

1. Linearity

$$I_{\gamma}^{\alpha}(cf + dg) = cI_{\gamma}^{\alpha} f + dI_{\gamma}^{\alpha} g, \tag{50}$$

with c and d are constants.

2. Additivity
Let γ_1 and γ_2 be two disjoint lines. If $\gamma = \gamma_1 \bigcup \gamma_2$, then $I_{\gamma}^{\alpha} f = I_{\gamma_1}^{\alpha} f + I_{\gamma_2}^{\alpha} f$.
3. Orientation
Let γ be the curve $\mathbf{r}(u), a \le u \le b$. The change in the orientation is obtained in the fractional derivative computation by reversing the tangent vector and the integration limits. Hence

$$I_{-\gamma}^{\alpha} f = \int_b^a f_{-\mathbf{v}}^{(-\alpha+1)}(\mathbf{r}(u))\left\|\mathbf{r}'(u)\right\|du. \tag{51}$$

While in the $\alpha = 1$ case, $I_{-\gamma}^{\alpha} f = -I_{\gamma}^{\alpha} f$ this may not happen in the fractional case, since the direct and reverse fractional anti-derivatives may not be equal.

Next, we present an illustrative example.

Example 2. *Assume a two-dimensional problem where* $f(\mathbf{x}) = \|\mathbf{x}\|^{-2}$, $\mathbf{x} \in \mathbb{R}^2$ *and* $\mathbf{r}(u)$ *a circle with radius* ρ. *Suppose that* $\alpha \in \mathbb{R}$. *In this case,*

$$f(\mathbf{r}(u)) = \rho^{-2},$$

$$\mathbf{r}(u) = \rho\cos(u)\mathbf{e}_1 + \rho\sin(u)\mathbf{e}_2, \quad 0 \le u \le 2\pi,$$

and

$$\|\mathbf{r}'(u)\| = \rho.$$

Now, $\mathbf{v} = (-\sin u, \cos u)$ *and*

$$f_{\mathbf{v}}^{(-\alpha+1)}(\mathbf{r}(u)) = \frac{\rho^{\alpha-3}}{2\Gamma(\alpha-1)} B\left(\frac{1}{2}\alpha - \frac{1}{2}, -\frac{1}{2}\alpha + \frac{3}{2}\right), \quad 1 < \alpha < 3, \tag{52}$$

where B *is the beta function. Finally*

$$\begin{aligned} I_{\mathbf{r}}^{\alpha} f &= \frac{\pi\rho^{\alpha-3}}{\Gamma(\alpha-1)} B\left(\frac{1}{2}\alpha - \frac{1}{2}, -\frac{1}{2}\alpha + \frac{3}{2}\right) \\ &= \frac{\pi\rho^{\alpha-3}}{\Gamma(\alpha-1)} \Gamma\left(\frac{1}{2}\alpha - \frac{1}{2}\right) \Gamma\left(-\frac{1}{2}\alpha + \frac{3}{2}\right) \\ &= \frac{\pi^2\rho^{\alpha-3}}{\Gamma(\alpha-1)\sin\left(\pi\left(\frac{1}{2}\alpha - \frac{1}{2}\right)\right)}, \end{aligned} \tag{53}$$

with $1 < \alpha < 3$.

Funding: This work was partially funded by Portuguese National Funds through the FCT-Foundation for Science and Technology within the scope of the CTS Research Unit - Center of Technology and Systems / UNINOVA / FCT / NOVA, under the reference UIDB / 00066/2020. The first author was supported by Autonomous University of Mexico City (UACM) under the project PI-CCyT-2019-15

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