On Analytic Multivalent Function Associated with Cosine Hyperbolic Function

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Abstract: In this article, we set up some adequate conditions for analytic functions associated with cosine hyperbolic function. We determine conditions on α such that

$$1+\alpha\frac{\varkappa^{2+\mathcal{P}(j-1)}g'(\varkappa)}{\mathcal{P}g^{j}(\varkappa)}, \text{for each } j=0,1,2,3$$

are subordinated by Janowski function, then $\frac{g(\varkappa)}{\varkappa^{\mathcal{P}}} \prec \cosh(\varkappa)$, $(\varkappa \in \mathfrak{C})$. By choosing specific values of functions g we get some adequate conditions for multivalent starlike function related with cosine hyperbolic.

Keywords: Multivalent functions, Janowski function, Trigonometric functions

1. Introduction and Definations

To understand in a clear way the notions used in our results, we need to add here some basic literature of Geometric function theory. For this we start first with the notation $\mathcal{A}$ which denotes the class of holomorphic or analytic functions in the region $\mathfrak{C}=\{\varkappa\in\mathbb{C}:|\varkappa|<1\}$ and if a function $g\in Å$, then the relations $g(0)=g'(0)-1=0$ must hold. Also, all univalent functions will be in a subfamily $\mathcal{S}$ of Å. Furthermore to done the possibility of subjections between analytic functions g_1 and g_2, indicated by $g_1(\varkappa)\prec g_2(\varkappa)$, as; the functions $g_1, g_2\in Å$, are associated by the connection of subjection, if there exists an analytic function υ with the limitations $\upsilon(0)=0$ and $|\upsilon(\varkappa)|<1$ such that $g_1(\varkappa)=g_2(\upsilon(\varkappa))$. Moreover, if the function $g_2\in\mathcal{S}$ in $\mathfrak{C}$, then we obtain.

$$g_1(\varkappa)\prec g_2(\varkappa)\Leftrightarrow\{g_1(0)=g_2(0)\ \&\ g_1(\mathfrak{C})\subset g_2(\mathfrak{C})\}$$

In 1992, Ma and Minda [16] considered a holomorphic function $\psi(\varkappa)$ normalized by the conditions $\psi(0)=1$ and $\psi'(0)>0$ with $\mathcal{R}e\psi>$ in $\mathfrak{C}$. The function ψ maps the disc $\mathfrak{C}$ onto the region which is star-shaped about 1 and symmetric along the real axis.

In particular, the function $\psi(\varkappa)=(1+Å\varkappa)/(1+\mathcal{B}\varkappa), (-1\le\mathcal{B}<Å\le 1)$ maps $\mathfrak{C}$ onto the disc on the right half plane with center on the real axis and

diameter end points $\frac{1-\mathcal{F}}{1-\mathcal{G}}$ and $\frac{1+\mathcal{F}_{\varkappa}}{1+\mathcal{G}_{\varkappa}}$. This interesting familiar function is named as Janowski function [10]. The image of the function $\psi(\varkappa)=\text{cash}(\varkappa)$ shows that the picture space is limited by the right half of the cosine hyperbolic [25]. The function $\psi(\varkappa)=1+\frac{4}{3}\varkappa+\frac{2}{3}\varkappa^2$ maps $\mathfrak{C}$ into the image set bounded by the cardioid given by $(9d^2+9e^2-8e+5)^2-16(9^2+9e^2-6e+1)=0$, [21] in more studied in [23]. the function $\psi(\varkappa)=1+\sin\varkappa$ was examined by Cho and his coauthors in [3] while $\psi(\varkappa)=e^{\varkappa}$ is of

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late studied in [17] and [24]. Further, by choosing particular ψ, several subclasses of starlike functions have been studied. See the details in [2, 4, 5, 11, 12, 14, 19]. Recently, Ali et al. [1] have acquired adequate conditions on α such that

$$1+\varkappa g'(\varkappa)/g^{i}(\varkappa) \prec \sqrt{1+\varkappa} \Longrightarrow g(\varkappa) \prec \sqrt{1+\varkappa}\text{, for } i=0,1,2.$$

Comparative sort suggestions have been examined in a portion of the new papers by various analysts, for instance see the papers

contributed by Halim and Omar [6] Haq et al [7], Kumar et. al [13, 15] Paprocki and Sokol [18], Raza et al [20] and Sharma et al [22].

In 1994, Hayman [8] studied multivalent ($\mathcal{P}$-valent) functions which is a generalization of univalent functions and is defined as; an analytic function g in an arbitrary domain $\mathfrak{C} \subset \mathbb{C}$ is said to be $\mathcal{P}$-valent, if for every complex number ω, the equation $g(\varkappa)=\omega$ has maximum $\mathcal{P}$ roots in $\mathfrak{C}$ and for a complex number ω_0 the equation $g(\varkappa)=\omega_0$ has exactly $\mathcal{P}$ roots in $\mathfrak{C}$. Let $\mathring{A}_{\mathcal{P}}(\mathcal{P} \in \mathbb{N}=\{1,2,\ldots\ldots.\})$ denote the class of functions, say $g \in \mathring{A}_{\mathcal{P}}$, that are multivalent holomorphic in the unit disc $\mathfrak{C}$ and which have the following series expansion:

$$\mathcal{S}^{*}_{\mathrm{L}_{\mathcal{P}}}=\varkappa^{\mathcal{P}}+\sum_{\mathrm{k}=\mathcal{P}+1}^{\infty}\mathrm{a}^{\mathrm{k}}\varkappa^{\mathrm{k}}, \qquad (\varkappa \in \mathfrak{C}) \tag{1.1}$$

Using the idea of multivalent functions, we now introduce the class $\mathrm{S}^{*}_{\mathrm{L}_{\mathcal{P}}}$ of multivalent starlike functions associated with cosine hyperbolic function and as given below:

$$\mathcal{S}^{*}_{\mathrm{L}_{\mathcal{P}}}=\left\{g(\varkappa)\in \mathring{A}_{\mathcal{P}}:\frac{\varkappa g'(\varkappa)}{\mathcal{P}g(\varkappa)} \prec \cos\hbar(\varkappa),(\varkappa\in\mathfrak{C})\right\}$$

In this paper, we determine conditions on α such that for each

$$1+\alpha\frac{\varkappa^{2+\mathcal{P}(j-1)}g'(\varkappa)}{\mathcal{P}g^{j}(\varkappa)}\text{, for each } j=0,1,2,3$$

are subordinated to Janowski functions implies $\frac{g(\varkappa)}{\varkappa^{\mathcal{P}}} \prec \cosh(\varkappa)$, $(\varkappa\in\mathfrak{C})$, these results are utilized to show that g are in the class $\mathcal{S}^{*}_{\mathrm{L}_{\mathcal{P}}}$.

Lemma 1.1. [9] Let υ be a non-constant analytic function in $\mathfrak{C}$ with $\upsilon(0)=0$. If

$$|\upsilon(\varkappa_0)|=\max\{|\upsilon(\varkappa)|,|\varkappa|\le|\varkappa_0|\},\varkappa\in\mathfrak{C}$$

then there exists a real number $\mathrm{m}(\mathrm{m}\ge 1)$ such that $\varkappa_0\upsilon'(\varkappa_0)=\mathrm{l}\upsilon(\varkappa_0)$.

To evade redundancies, we accept that the accompanying limitations

$$-1\le\mathcal{B}<\mathring{A}\le 1. j\in\mathbb{N}=\{1,2,\ldots..\},\mathrm{k}\in\mathbb{N}_0=\mathbb{N}\cup\{0\}$$

otherwise we will state it where different.

1. Adequate Conditions Associated with Cosh

Theorem 2.1. Let $\mathscr{g} \in \text{Å}_{\mathcal{P}}$ and satisfying

$$1+\frac{\alpha\varkappa^{1-\mathcal{P}}\mathscr{g}'(\varkappa)}{\mathcal{P}} < \frac{1+\mathcal{F}_{\varkappa}}{1+\mathcal{G}_{\varkappa}} \tag{2.1}$$

with the restriction on α is

$$|\alpha| \geq \frac{(\mathcal{F}-\mathcal{G})\mathcal{P}}{(\sin(1)-|\mathrm{B}|\sin\hbar(1))-(1+|\mathrm{B}|\mathcal{P}\cos\hbar(1))} \tag{2.2}$$

Then

$$\frac{\mathscr{g}(\varkappa)}{\varkappa^{\mathcal{P}}} < \cos\hbar(\varkappa)$$

Proof. Let us consider that

$$\hbar(\varkappa) = 1+\frac{\alpha\varkappa^{1-\mathcal{P}}\mathscr{g}'(\varkappa)}{\mathcal{P}} \tag{2.3}$$

Then the function $\hbar$ is holomorphic in $\mathfrak{C}$ with $\hbar(0) = 1$. Additionally, suppose that

$$\upsilon(\varkappa) = \left(\cos\hbar^{-1}\left(\frac{\mathscr{g}(\varkappa)}{\mathcal{P}^{\varkappa}}\right)\right) \tag{2.4}$$

Where we chose the rule parts of the function that are logarithmic and square root. At that point υ is obviously a holomorphic function in $\mathfrak{C}$ with $\upsilon(0) = 0$. likewise, since

$$\cos\hbar^{-1}(\varkappa) = \ln\left[\varkappa+\sqrt{\varkappa^2-1}\right]$$

To finish the evidence of this outcome, we simply need to verification $|\upsilon(\varkappa)| < 1$ in $\mathfrak{C}$. By virtueof (2:4), we have

$$\hbar(\varkappa) = 1+\frac{\alpha\varkappa\upsilon'(\varkappa)\sin\hbar\upsilon(\varkappa)}{\mathcal{P}}+\alpha\cos\hbar\upsilon(\varkappa)$$

Therefore

$$\left|\frac{\hbar(\varkappa)-1}{\mathcal{F}-\mathcal{G}\hbar(\varkappa)}\right| = \left|\frac{\frac{\alpha\varkappa\upsilon'(\varkappa)\sin\hbar\upsilon(\varkappa)}{\mathcal{P}}+\alpha\cos\hbar\upsilon(\varkappa)}{\mathcal{F}-\mathcal{G}\left(1+\frac{\alpha\varkappa\upsilon'(\varkappa)\sin\hbar\upsilon(\varkappa)}{\mathcal{P}}+\alpha\cos\hbar\upsilon(\varkappa)\right)}\right|$$
$$= \left|\frac{\alpha\{\varkappa\upsilon'(\varkappa)\sin\hbar\upsilon(\varkappa)+\mathrm{p}\cos\hbar\upsilon(\varkappa)\}}{(\mathcal{F}-\mathcal{G})\mathcal{P}-\mathrm{B}\alpha(\varkappa\upsilon'(\varkappa)\sin\hbar\upsilon(\varkappa)+\mathcal{P}\cos\hbar\upsilon(\varkappa))}\right|$$

Presently we guess that a point $\varkappa_0 \in \mathfrak{C}$ happens with the end goal that

$$\max_{|\varkappa|\leq|\varkappa_0|} \upsilon(\varkappa) = |\upsilon(\varkappa_0)| = 1$$

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Additionally, by Lemma (1.1), a number $m \geq 1$, be alive with $\varkappa_0 \upsilon'(\varkappa_0) = m\upsilon(\varkappa_0)$. Furthermore, we likewise assume that $\upsilon(\varkappa_0) = e^{i\theta}$ for $\theta \in [-\pi, \pi]$. Then we have

$$\left|\frac{\hbar(\varkappa_0) - 1}{\mathcal{F} - \mathcal{G}\hbar(\varkappa_0)}\right| = \left|\frac{\alpha\big(\varkappa_0 \upsilon'(\varkappa_0)\sinh\hbar\upsilon(\varkappa_0) + \mathcal{P}\mathrm{cas}\hbar\upsilon(\varkappa_0)\big)}{(\mathcal{F} - \mathcal{G})\mathcal{P} - \alpha\mathcal{G}(\varkappa_0\upsilon'(\varkappa_0)\sinh\hbar\upsilon(\varkappa_0) + \mathcal{P}\cosh\hbar\upsilon(\varkappa_0))}\right|$$
$$= \left|\frac{\alpha\big(me^{i\theta}\sin\hbar(e^{i\theta}) + \mathcal{P}\cos\hbar(e^{i\theta})\big)}{(\mathcal{F} - \mathcal{G})\mathcal{P} - \alpha\mathcal{G}(m(e^{i\theta})\sin\hbar(e^{i\theta}) + \mathcal{P}\cos\hbar(e^{i\theta}))}\right|$$
$$= \left|\frac{\alpha\big(m\sin\hbar(e^{i\theta}) + \mathcal{P}\cos\hbar(e^{i\theta})\big)}{(\mathcal{F} - \mathcal{G})\mathcal{P} - \alpha\mathcal{G}(m\sin\hbar(e^{i\theta}) + \mathcal{P}\cos\hbar(e^{i\theta}))}\right|$$

$$\geq \frac{|\alpha|\big(m\left|\sin\hbar(e^{i\theta})\right| - \mathcal{P}\left|\mathrm{cas}\hbar(e^{i\theta})\right|\big)}{(\mathcal{F} - \mathcal{G})\mathcal{P} + |\alpha||\mathcal{G}|(m|\sin\hbar(e^{i\theta})| + \mathcal{P}|\mathrm{cas}\hbar(e^{i\theta})|)} \tag{2.5}$$

If $|\varkappa| = r, -\pi \leq \theta \leq \pi$, at that point basic estimation delineates that

$$\left|\cosh (e^{i\theta})\right|^2 = \cosh^2(\cos\theta)\mathrm{cas}^2(\sin\theta) + \sinh^2(\cos\theta)\sin^2(\sin\theta) = \delta(\theta)$$

$$\left|\sinh (e^{i\theta})\right|^2 = \sinh^2(\cos\theta)\mathrm{cas}^2(\sin\theta) + \cosh^2(\cos\theta)\sin^2(\sin\theta) = \psi(\theta)$$

A standard disentanglement guarantees that $0, \pm\pi, \frac{\pi}{2}$ are the roots of $\delta'(\theta) = 0$ and $\psi'(\theta) = 0$ in$[-\pi, \pi]$. Additionally, since

$$\delta(\theta) = \delta'(-\theta)$$

$$\psi(\theta) = \psi'(-\theta)$$

it is sufficient to reason that $\theta \in [0, \pi]$ and along these lines we accomplish

$$\max\{\delta(\theta)\} = \delta(0) = \delta(\pi) = \cosh^2(1)$$

$$\min\{\delta(\theta)\} = \delta\left(\frac{\pi}{2}\right) = \cos^2(1)$$

$$\max\{\psi(\theta)\} = \psi(0) = \psi(\pi) = \sinh^2(1)$$

$$\min\{\psi(\theta)\} = \psi\left(\frac{\pi}{2}\right) = \sin^2(1)$$

Thus, we have

$$\cos(1) \leq \left|\cosh\big(e^{i\theta}\big)\right| \leq \cos(1) \tag{2.6}$$

$$\sin(1) \leq \left|\sinh\big(e^{i\theta}\big)\right| \leq \sin(1) \tag{2.7}$$

Therefore, using (2.6), (2.7), (2.5), we attain

$$\left|\frac{\hbar(\varkappa_0) - 1}{\mathcal{F} - \mathcal{G}\hbar(\varkappa_0)}\right| \geq \frac{|\alpha|(m\sin(1) - \mathcal{P}\cosh(1))}{(\mathcal{F} - \mathcal{G})\mathcal{P} + |\alpha||\mathcal{G}|(m\sinh(1) + \mathcal{P}\cosh(1))}$$

Now, let

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$$\Xi(\mathrm{m}) = \frac{|\alpha|\big(\mathrm{m}\sin(1) - \mathcal{P}\cos\hbar(1)\big)}{(\mathcal{F}-\mathcal{G})\mathcal{P} + |\alpha||\mathcal{G}|\big(\mathrm{m}\sin\hbar(1) + \mathcal{P}\cos\hbar(1)\big)}$$

Then,

$$\Xi'(\mathrm{m}) = \frac{|\alpha|(\mathcal{F}-\mathcal{G})\mathcal{P}\sin(1) + |\alpha|^2|\mathcal{G}|\mathcal{P}\cos(1)(\sin(1) + \sin\hbar(1))}{(\mathcal{F}-\mathcal{G})\mathcal{P} + |\alpha||\mathcal{G}|\{\mathrm{m}\sin(1) + \mathcal{P}\cos\hbar(1)\}^2} > 0$$

This conforms, that the function Ξ is increasing and therefore $\Xi(\mathrm{m}) \geq \Xi(1)$ for $\mathrm{m} \geq 1$, so

$$\left|\frac{\hbar(\varkappa_0) - 1}{\mathcal{F} - \mathcal{G}\hbar(\varkappa_0)}\right| \geq \frac{|\alpha|(\sin(1) - \mathcal{P}\cos\hbar(1))}{(\mathcal{F}-\mathcal{G})\mathcal{P} + |\alpha||\mathcal{G}|(\sin\hbar(1) + \mathcal{P}\cos\hbar(1))}$$

Now, utilizing (2.2), we obtain

$$\left|\frac{\hbar(\varkappa_0) - 1}{\mathcal{F} - \mathcal{G}\hbar(\varkappa_0)}\right| \geq 1$$

which contradicts the fact that $\hbar(\varkappa) \prec \frac{1+\mathcal{F}_\varkappa}{1+\mathcal{G}_\varkappa}$. Thus $|\upsilon(\varkappa)| < 1$ and so we get the desired proof.

If we replace $g(\varkappa) = \frac{\varkappa^{\mathcal{P}+1}\mathrm{q}'(\varkappa)}{\mathcal{P}\mathrm{q}(\varkappa)}$ in (2.1), we acquire the underneath result.

Corollary 2.2. Let $\mathrm{q} \in \mathring{\mathrm{A}}_{\mathcal{P}}$ and justifying

$$1 + \frac{\alpha\varkappa\mathrm{q}'(\varkappa)}{\mathcal{P}^2\mathrm{q}(\varkappa)}\left(\mathcal{P} + 1 + \frac{\varkappa\mathrm{q}''(\varkappa)}{\mathrm{q}'} - \frac{\varkappa\mathrm{q}'(\varkappa)}{\mathrm{q}(\varkappa)}\right) \prec \frac{1+\mathcal{F}_\varkappa}{1+\mathcal{G}_\varkappa} \tag{2.8}$$

with the condition on α is

$$|\alpha| \geq \frac{(\mathcal{F}-\mathcal{G})\mathcal{P}}{(\sin(1) - |\mathrm{B}|\sin\hbar(1)) - (1+|\mathrm{B}|)\mathcal{P}\cos\hbar(1)}$$

Then $\mathrm{q} \in \mathrm{S}^*_{\mathrm{L}_\mathcal{P}}$.

Theorem 2.3. Let $g \in \mathring{\mathrm{A}}_{\mathcal{P}}$

$$1 + \frac{\alpha\varkappa g'(\varkappa)}{\mathcal{P}g(\varkappa)} \prec \frac{1+\mathcal{F}_\varkappa}{1+\mathcal{G}_\varkappa} \tag{2.9}$$

With

$$|\alpha| \geq \frac{(\mathcal{F}-\mathcal{G})\mathcal{P}\cos\hbar(1)}{(\sin(1) - |\mathrm{B}|\sin\hbar(1)) - (1+|\mathrm{B}|)\mathcal{P}\cos\hbar(1)} \tag{2.10}$$

Then

$$\frac{g(\varkappa)}{\varkappa^{\mathcal{P}}} \prec \cos\hbar(\varkappa)$$

Proof. let us dene a function

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$$\hbar(\varkappa) = 1 + \frac{\alpha\varkappa g'(\varkappa)}{\mathcal{P}g(\varkappa)}$$

Then the function $\hbar$ is holomorphic in $\mathfrak{C}$ with $\hbar(0) = 1$. Inserting (2.4), we get

$$\hbar(\varkappa) = 1 + \alpha\frac{\varkappa\upsilon'(\varkappa)\sin\hbar(\upsilon(\varkappa))}{\mathcal{P}\cos\hbar\upsilon(\varkappa)} + \alpha$$

and so

$$\left|\frac{\hbar(\varkappa)-1}{\mathcal{F}-\mathcal{G}\hbar(\varkappa)}\right| = \left|\frac{\frac{\alpha\varkappa\upsilon'(\varkappa)\sin\hbar\upsilon(\varkappa)}{\mathcal{P}\cos\hbar\upsilon(\varkappa)} + \alpha}{\mathcal{F}-\mathcal{G}\left(1+\frac{\alpha\varkappa\upsilon'(\varkappa)\sin\hbar\upsilon(\varkappa)}{\mathcal{P}\cos\hbar\upsilon(\varkappa)} + \alpha\right)}\right|$$

$$= \left|\frac{\alpha(\varkappa\upsilon'(\varkappa)\sin\hbar\upsilon(\varkappa) + \mathrm{p}\cos\hbar\upsilon(\varkappa))}{(\mathcal{F}-\mathcal{G})\mathcal{P}\cos\hbar\upsilon(\varkappa) - \alpha\mathcal{G}(\varkappa\upsilon'(\varkappa)\sin\hbar\upsilon(\varkappa) + \mathcal{P}\cos\hbar\upsilon(\varkappa))}\right|$$

By virtue of Lemma 1.1 along with (2.7) and (2.6), we have

$$\left|\frac{\hbar(\varkappa_0)-1}{\mathcal{F}-\mathcal{G}\hbar(\varkappa_0)}\right| = \left|\frac{\alpha\big(\varkappa_0\upsilon'(\varkappa_0)\sin\hbar\,\upsilon(\varkappa_0) + \mathcal{P}\mathrm{cas}\hbar\upsilon(\varkappa_0)\big)}{(\mathcal{F}-\mathcal{G})\mathcal{P}\cos\hbar\upsilon(\varkappa_0) - \alpha\mathcal{G}\big(\varkappa_0\upsilon'(\varkappa_0)\sin\hbar\upsilon(\varkappa_0) + \mathcal{P}\cos\hbar\upsilon(\varkappa_0)\big)}\right|$$

$$= \left|\frac{\alpha\left(\mathrm{m}\sin\hbar(e^{i\theta}) + \mathcal{P}\cos\hbar(e^{i\theta})\right)}{(\mathcal{F}-\mathcal{G})\mathcal{P}\cos\hbar(e^{i\theta}) - \alpha\mathcal{G}\big(\mathrm{m}\sin\hbar(e^{i\theta}) + \mathcal{P}\cos\hbar(e^{i\theta})\big)}\right|$$

$$\geq \left|\frac{|\alpha|\big(\mathrm{m}|\sin\hbar(e^{i\theta})| - \mathrm{p}|\cos\hbar(e^{i\theta})|\big)}{(\mathcal{F}-\mathcal{G})\mathcal{P}|\cos\hbar(e^{i\theta})| + |\alpha||\mathcal{G}|(\mathrm{m}|\sin\hbar(e^{i\theta})| + \mathcal{P}|\cos\hbar(e^{i\theta})|)}\right|$$

$$\geq \frac{|\alpha|(\mathrm{m}\sin(1) - \mathcal{P}\cos\hbar(1))}{(\mathcal{F}-\mathcal{G})\mathcal{P}\cos\hbar(1) + |\alpha||\mathcal{G}|(\mathrm{m}\sin\hbar(1) + \mathcal{P}\mathrm{cas}\hbar(1))}$$

Now, let

$$\Xi_1(\mathrm{m}) = \frac{|\alpha|\big(\mathrm{m}\sin(1) - \mathcal{P}\cos\hbar(1)\big)}{(\mathcal{F}-\mathcal{G})\mathcal{P}\cos\hbar(1) + |\alpha||\mathcal{G}|\big(\mathrm{m}\sin\hbar(1) + \mathcal{P}\cos\hbar(1)\big)}$$

Then

$$\Xi_1'(\mathrm{m}) = \frac{|\alpha|(\mathcal{F}-\mathcal{G})\mathcal{P}\cos\hbar(1)\sin(1) + |\alpha|^2|\mathcal{G}|\mathcal{P}\cos\hbar(1)(\sin(1) + \sin\hbar(1))}{\{(\mathcal{F}-\mathcal{G})\mathcal{P}\cos\hbar(1) + |\alpha||\mathcal{G}|(\mathrm{m}\sin\hbar(1) + \mathcal{P}\cos\hbar(1))\}^2} > 0$$

Which illustrates that the function $\Xi_1(\mathrm{m})$ is increasing and hence $\Xi_1(\mathrm{m}) \geq \Xi_1(1)$ for $\mathrm{m} \geq 1$

So

$$\left|\frac{\hbar(\varkappa_0)-1}{\mathcal{F}-\mathcal{G}\hbar(\varkappa_0)}\right| \geq \frac{|\alpha|(\sin(1) - \mathcal{P}\cos\hbar(1))}{(\mathcal{F}-\mathcal{G})\mathcal{P}\cos\hbar(1) + |\alpha||\mathcal{G}|(\sin\hbar(1) + \mathcal{P}\cos\hbar(1))}$$

Applying (2.10), we have

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$$\left|\frac{\hbar(\varkappa_0)-1}{\mathcal{F}-\mathcal{G}\hbar(\varkappa_0)}\right| \geq 1$$

A contradicts (2.9), this $|\upsilon(\varkappa)| < 1$ and so the desired proof is completed.

If we replace $\mathscr{g}(\varkappa) = \frac{\varkappa^{\mathcal{P}+1}\mathrm{q}'(\varkappa)}{\mathcal{P}\mathrm{q}(\varkappa)}$ in last theorem, we obtain the below corollary

Corollary 2.4. Let $\mathrm{q} \in \text{Å}_{\mathrm{p}}$ and justifying

$$1+\frac{\alpha}{\mathcal{P}}\left(\mathcal{P}+1+\frac{\varkappa\mathrm{q}''(\varkappa)}{\mathrm{q}'(\varkappa)}-\frac{\varkappa\mathrm{q}'(\varkappa)}{\mathrm{q}(\varkappa)}\right) < \frac{1+\mathcal{F}_{\varkappa}}{1+\mathcal{G}_{\varkappa}} \tag{2.11}$$

with

$$|\alpha| \geq \frac{(\mathcal{F}-\mathcal{G})\mathcal{P}\cos\hbar(1)}{(\sin(1)-|\mathrm{B}|\sin\hbar(1))-(1+|\mathrm{B}|)\mathcal{P}\cos\hbar(1)}$$

Then $\mathrm{q} \in \mathcal{S}^*_{\mathrm{L}_{\mathcal{P}}}$.

Theorem 2.5. Let $\mathscr{g} \in \text{Å}_{\mathcal{P}}$ and satisfy the subordination relation

$$1+\frac{\alpha\varkappa^{1-\mathcal{P}}\mathscr{g}'(\varkappa)}{\mathcal{P}(\mathscr{g}(\varkappa))^2} < \frac{1+\mathcal{F}_{\varkappa}}{1+\mathcal{G}_{\varkappa}} \tag{2.12}$$

with the condition on α

$$|\alpha| \geq \frac{(\mathcal{F}-\mathcal{G})\mathcal{P}(\cos\hbar(1))^2}{(\sin(1)-|\mathrm{B}|\sin\hbar(1))-(1+|\mathrm{B}|)\mathcal{P}\cos\hbar(1)} \tag{2.13}$$

is true, then

$$\frac{\mathscr{g}(\varkappa)}{\varkappa^{\mathcal{P}}} < \cos\hbar(\varkappa)$$

Proof. let us dene a function

$$\hbar(\varkappa) = 1+\frac{\alpha\varkappa^{1-\mathcal{P}}\mathscr{g}'(\varkappa)}{\mathcal{P}(\mathscr{g}(\varkappa))^2}$$

Then the function $\hbar$ is holomorphic in $\mathfrak{C}$ with $\hbar(0) = 1$. Applying some straightforward calculation, we acquire

$$\hbar(\varkappa) = 1+\frac{\alpha\varkappa\upsilon'(\varkappa)\sin\hbar\,(\upsilon(\varkappa))}{\mathcal{P}(\cos\hbar\upsilon(\varkappa))^2}+\frac{\alpha}{\cos\hbar\upsilon(\varkappa)}$$

and so

$$\left|\frac{\hbar(\varkappa)-1}{\mathcal{F}-\mathcal{G}\hbar(\varkappa)}\right| = \left|\frac{\frac{\alpha\varkappa\upsilon'(\varkappa)\sin\hbar\upsilon(\varkappa)}{\mathcal{P}(\cos\hbar\upsilon(\varkappa))^2}+\frac{\alpha}{\cos\hbar\upsilon(\varkappa)}}{\mathcal{F}-\mathcal{G}\left(1+\frac{\alpha\varkappa\upsilon'(\varkappa)\sin\hbar\upsilon(\varkappa)}{\mathcal{P}(\cos\hbar\upsilon(\varkappa))^2}+\frac{\alpha}{\cos\hbar\upsilon(\varkappa)}\right)}\right|$$
$$= \left|\frac{\alpha(\varkappa\upsilon'(\varkappa)\sin\hbar\upsilon(\varkappa)+\mathrm{p}\cos\hbar\upsilon(\varkappa))}{(\mathcal{F}-\mathcal{G})\mathcal{P}(\cos\hbar\upsilon(\varkappa))^2-\alpha\mathcal{G}(\varkappa\upsilon'(\varkappa)\sin\hbar\upsilon(\varkappa)+\mathcal{P}\cos\hbar\upsilon(\varkappa))}\right|$$

By virtue of Lemma 1.1 along with (2.7) and (2.6), we have

$$\left|\frac{\hbar(\varkappa_0)-1}{\mathcal{F}-\mathcal{G}\hbar(\varkappa_0)}\right| = \left|\frac{\alpha(\varkappa_0\upsilon'(\varkappa_0)\sin\hbar\upsilon(\varkappa_0)+\mathcal{P}\cos\hbar\upsilon(\varkappa_0))}{(\mathcal{F}-\mathcal{G})\mathcal{P}(\cos\hbar\upsilon(\varkappa_0))^2-\alpha\mathcal{G}(\varkappa_0\upsilon'(\varkappa_0)\sin\hbar\upsilon(\varkappa_0)+\mathcal{P}\cos\hbar\upsilon(\varkappa_0))}\right|$$

$$= \left|\frac{\alpha\left(\mathrm{m}\sin\hbar(e^{i\theta})+\mathrm{p}\cos\hbar(e^{i\theta})\right)}{(\mathcal{F}-\mathcal{G})\mathcal{P}(\cos\hbar(e^{i\theta}))^2-\alpha\mathcal{G}\left(\mathrm{m}\sin\hbar(e^{i\theta})+\mathcal{P}\cos\hbar(e^{i\theta})\right)}\right|$$

$$\geq \left|\frac{|\alpha|\left(\mathrm{m}\left|\sin\hbar(e^{i\theta})\right|-\mathcal{P}\left|\cos\hbar(e^{i\theta})\right|\right)}{(\mathcal{F}-\mathcal{G})\mathcal{P}(|\cos\hbar(e^{i\theta})|)^2+|\alpha||\mathcal{G}|(\mathrm{m}|\sin\hbar(e^{i\theta})|+\mathcal{P}|\cos\hbar(e^{i\theta})|)}\right|$$

$$\geq \frac{|\alpha|(\mathrm{m}\sin(1)-\mathcal{P}\cos\hbar(1))}{(\mathcal{F}-\mathcal{G})\mathcal{P}(\cos\hbar(1))^2+|\alpha||\mathcal{G}|(\mathrm{m}|\sin\hbar(1)|+\mathcal{P}|\mathrm{cas}\hbar(1)|)}$$

Now, let

$$\Xi_2(\mathrm{m}) = \frac{|\alpha|\left(\mathrm{m}\sin(1)-\mathcal{P}\cos\hbar(1)\right)}{(\mathcal{F}-\mathcal{G})\mathcal{P}(\cos\hbar(1))^2+|\alpha||\mathcal{G}|\left(\mathrm{m}\sin\hbar(1)+\mathcal{P}\cos\hbar(1)\right)}$$

Then

$$\Xi_2'(\mathrm{m}) = \frac{|\alpha|(\mathcal{F}-\mathcal{G})\mathcal{P}(\cos\hbar(1))^2\sin(1)+|\alpha|^2|\mathcal{G}|\mathcal{P}\cos\hbar(1)(\sin(1)+\sin\hbar\,(1))}{\{(\mathcal{F}-\mathcal{G})\mathcal{P}(\cos\hbar(1))^2+|\alpha||\mathcal{G}|(\mathrm{m}\sin\hbar(1)+\mathcal{P}\cos\hbar(1))\}^2} > 0$$

Which exhibits that the function $\Xi_2(\mathrm{m})$ is increasing and $\Xi_2(\mathrm{m}) \geq \Xi_2(1)$ for $\mathrm{m} \geq 1$, hence

$$\left|\frac{\hbar(\varkappa_0)-1}{\mathcal{F}-\mathcal{G}\hbar(\varkappa_0)}\right| \geq \frac{|\alpha|\sin(1)-\mathcal{P}\cos\hbar(1)}{(\mathcal{F}-\mathcal{G})\mathcal{P}(\cos\hbar(1))^2+|\alpha||\mathcal{G}|(\mathrm{m}\sin\hbar(1)+\mathcal{P}\cos\hbar(1))}$$

Applying (2.13), we have easily achieved

$$\left|\frac{\hbar(\varkappa_0)-1}{\mathcal{F}-\mathcal{G}\hbar(\varkappa_0)}\right| \geq 1$$

A contradicts (2.12), this $|\upsilon(\varkappa)| < 1$ and so the desired proof is completed.

If we set $g(\varkappa) = \frac{\varkappa^{\mathcal{P}+1}\mathrm{q}'(\varkappa)}{\mathcal{P}\mathrm{q}(\varkappa)}$ in last theorem, we easily have the following corollary.

Corollary 2.6. Let $\mathrm{q} \in \mathring{A}_{\mathcal{P}}$ and justifying

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$$1+\frac{\alpha q(\varkappa)}{\varkappa^{2\mathcal{P}+1}q'(\varkappa)}\left(\mathcal{P}+1+\frac{\varkappa q''(\varkappa)}{q'(\varkappa)}-\frac{\varkappa q'(\varkappa)}{q(\varkappa)}\right)<\frac{1+\mathcal{F}_\varkappa}{1+\mathcal{G}_\varkappa} \tag{2.14}$$

With

$$|\alpha|\geq\frac{(\mathcal{F}-\mathcal{G})\mathcal{P}(\cosh(1))^2}{(\sin(1)-|B|\sinh(1))-(1+|B|)\mathcal{P}\cosh(1)}$$

Then $q\in\mathcal{S}^*_{L_{\mathcal{P}}}$.

Theorem 2.7. Let $\mathcal{g}\in Å_{\mathcal{P}}$ and satisfy the subordination relation

$$1+\frac{\alpha\varkappa^{1-2\mathcal{P}}\mathcal{g}'(\varkappa)}{\mathcal{P}(\mathcal{g}(\varkappa))^3}<\frac{1+\mathcal{F}_\varkappa}{1+\mathcal{G}_\varkappa} \quad (2.15$$

With

$$|\alpha|\geq\frac{(\mathcal{F}-\mathcal{G})\mathcal{P}(\cosh(1))^3}{(\sin(1)-|B|\sinh(1))-(1+|B|)\mathcal{P}\cosh(1)} \tag{2.16}$$

Then

$$\frac{\mathcal{g}(\varkappa)}{\varkappa^{\mathcal{P}}}<\cosh(\varkappa)$$

Proof. let us dene a function

$$\hbar(\varkappa)=1+\frac{\alpha\varkappa^{1-2\mathcal{P}}\mathcal{g}'(\varkappa)}{\mathcal{P}(\mathcal{g}(\varkappa))^3}$$

So, the function $\hbar$ is holomorphic in $\mathfrak{C}$ with $\hbar(0)=1$. Applying some straightforward calculation, we get

$$\hbar(\varkappa)=1+\frac{\alpha\varkappa\upsilon'(\varkappa)\sinh(\upsilon(\varkappa))}{\mathcal{P}(\cosh\upsilon(\varkappa))^3}+\frac{\alpha}{(\cosh\upsilon(\varkappa))^2}$$

and so

$$\left|\frac{\hbar(\varkappa)-1}{\mathcal{F}-\mathcal{G}\hbar(\varkappa)}\right|=\left|\frac{\frac{\alpha\varkappa\upsilon'(\varkappa)\sinh\upsilon(\varkappa)}{\mathcal{P}(\cosh\upsilon(\varkappa))^3}+\frac{\alpha}{(\cosh\upsilon(\varkappa))^2}}{\mathcal{F}-\mathcal{G}\left(1+\frac{\alpha\varkappa\upsilon'(\varkappa)\sinh\upsilon(\varkappa)}{\mathcal{P}(\cosh\upsilon(\varkappa))^3}+\frac{\alpha}{(\cosh\upsilon(\varkappa))^2}\right)}\right|$$
$$=\left|\frac{\alpha(\varkappa\upsilon'(\varkappa)\sinh\upsilon(\varkappa)+\mathcal{P}\cosh\upsilon(\varkappa))}{(\mathcal{F}-\mathcal{G})\mathcal{P}(\cosh\upsilon(\varkappa))^3-\alpha\mathcal{G}(\varkappa\upsilon'(\varkappa)\sinh\upsilon(\varkappa)+\mathcal{P}\cosh\upsilon(\varkappa))}\right|$$

By virtue of Lemma 1.1 along with (2.7) and (2.6), we have

$$\left|\frac{\hbar(\varkappa_0)-1}{\mathcal{F}-\mathcal{G}\hbar(\varkappa_0)}\right|=\left|\frac{\alpha\left(\varkappa_0\upsilon'(\varkappa_0)\sinh\upsilon(\varkappa_0)+\mathcal{P}\cosh\upsilon(\varkappa_0)\right)}{(\mathcal{F}-\mathcal{G})\mathcal{P}\left(\cosh\upsilon(\varkappa_0)\right)^3-\alpha\mathcal{G}\left(\varkappa_0\upsilon'(\varkappa_0)\sinh\upsilon(\varkappa_0)+\mathcal{P}\cosh\upsilon(\varkappa_0)\right)}\right|$$

$$=\left|\frac{\alpha\left(m\sinh(e^{i\theta})+p\cosh(e^{i\theta})\right)}{(\mathcal{F}-\mathcal{G})\mathcal{P}(\cosh(e^{i\theta}))^3-\alpha\mathcal{G}\left(m\sinh(e^{i\theta})+\mathcal{P}\cosh(e^{i\theta})\right)}\right|$$

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$$\geq \left|\frac{|\alpha|\left(m\left|\sin\hbar\left(e^{i\theta}\right)\right| - \mathcal{P}\left|\cos\hbar\left(e^{i\theta}\right)\right|\right)}{(\mathcal{F}-\mathcal{G})\mathcal{P}(|\cos\hbar(e^{i\theta})|)^3 + |\alpha||\mathcal{G}|(m|\sin\hbar(e^{i\theta})| + \mathcal{P}|\cos\hbar(e^{i\theta})|)}\right|$$

$$\geq \frac{|\alpha|(m\sin(1) - \mathcal{P}\cos\hbar(1))}{(\mathcal{F}-\mathcal{G})\mathcal{P}(\cos\hbar(1))^3 + |\alpha||\mathcal{G}|(m|\sin\hbar(1)| + \mathcal{P}|\cas\hbar(1)|)}$$

Now let,

$$\Xi_3(m) = \frac{|\alpha|\left(m\sin(1) - \mathcal{P}\cos\hbar(1)\right)}{(\mathcal{F}-\mathcal{G})\mathcal{P}(\cos\hbar(1))^3 + |\alpha||\mathcal{G}|\left(m\sin\hbar(1) + \mathcal{P}\cos\hbar(1)\right)}$$

Then

$$\Xi_3'(m) = \frac{|\alpha|(\mathcal{F}-\mathcal{G})\mathcal{P}(\cos\hbar(1))^3\sin(1) + |\alpha|^2|\mathcal{G}|\mathcal{P}\cos\hbar(1)(\sin(1) + \sin\hbar\,(1))}{\{(\mathcal{F}-\mathcal{G})\mathcal{P}(\cos\hbar(1))^3 + |\alpha||\mathcal{G}|(m\sin(1) + \mathcal{P}\cos\hbar(1))\}^2} > 0$$

Which exhibits that the function $\Xi_3(m)$ is increasing and $\Xi_3(m) \geq \Xi_3(1)$ from $m \geq 1$, hence

$$\left|\frac{\hbar(\varkappa_0) - 1}{\mathcal{F} - \mathcal{G}\hbar(\varkappa_0)}\right| \geq \frac{|\alpha|\sin(1) - \mathcal{P}\cos\hbar(1)}{(\mathcal{F}-\mathcal{G})\mathcal{P}(\cos\hbar(1))^3 + |\alpha||\mathcal{G}|(\sin(1) + \mathcal{P}\cos\hbar(1))}$$

Applying (2.16), we have

$$\left|\frac{\hbar(\varkappa_0) - 1}{\mathcal{F} - \mathcal{G}\hbar(\varkappa_0)}\right| \geq 1$$

A contradicts (2.15), this $|\upsilon(\varkappa)| < 1$ and so the desired proof is completed.

If we set $g(\varkappa) = \frac{\varkappa^{\mathcal{P}+1}q'(\varkappa)}{\mathcal{P}q(\varkappa)}$ in last theorem, we easily have the following corollary.

Corollary 2.8. Let $q \in \mathring{A}_{\mathcal{P}}$ and justifying

$$1 + \alpha\frac{\mathcal{P}(q(\varkappa))^2}{\varkappa^{3\mathcal{P}+2}(q'(\varkappa))^2}\left(\mathcal{P} + 1 + \frac{\varkappa q''(\varkappa)}{q'(\varkappa)} - \frac{\varkappa q'(\varkappa)}{q(\varkappa)}\right) \prec \frac{1+\mathcal{F}_\varkappa}{1+\mathcal{G}_\varkappa} \tag{2.17}$$

With

$$|\alpha| \geq \frac{(\mathcal{F}-\mathcal{G})\mathcal{P}(\cos\hbar(1))^3}{(\sin(1) - |B|\sin\hbar(1)) - (1+|B|)\mathcal{P}\cos\hbar(1)}$$

Then

2. **Conclusions.**

In the current exploration paper, we inspected a few fascinating traits of starlike functions related with the cosine hyperbolic function. We decide conditions on to such an extent that

$$1 + \alpha\frac{\varkappa^{2+\mathcal{P}(j-1)}g'(\varkappa)^{j}}{\mathcal{P}g^{j}(\varkappa)} \prec \frac{1+\mathcal{F}_\varkappa}{1+\mathcal{G}_\varkappa} \quad \Rightarrow \frac{g(\varkappa)}{\varkappa^{\mathcal{P}}} \prec \cos\hbar\,(\varkappa), (\varkappa \in \mathfrak{C})$$

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are subordinated by janowski function. Further, choosing some specie values of functions these results are utilized to prove that the function g, we get some sufficient criteria for multivalent starlike functions connected with cosine hyperbolic.

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