Article

The structure of n harmonic points and generalizations of Desargues' theorems

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Abstract: In this paper, we consider the relation of more than four harmonic points in a line. For this purpose, starting from the dependence of the harmonic points, Desargues' theorems, and perspectivity, we note that it is necessary to conduct a generalization of the Desargues' theorems for projective complete n-points, which are used to implement the definition of the generalization of harmonic points. We present new findings regarding the uniquely determined and constructed sets of H-points and their structure. The well-known fourth harmonic points represent the special case (n=4) of the sets of H-points of rank 2, which is indicated by P_4^2.

Keywords: Projective transformations, Perspectivity, Harmonic points, Generalized Desargues theorems, Harmonic points, Set of H-points rank k.

1. Introduction

The idea of studying and investigating the possibility of construing more than four harmonic points was originally discussed in the article "The chords of the Non-Ruled quadratic in $PG(3,3)$" (see [1]), where the forty-five chords of an "ellipsoid" in finite space $PG(3,3)$ are described, showing that they may be regarded as the edges of a notable graph that is in the group of automorphism of the symmetric group. The hexastigm is considered to be formed by six points of general positions in any of the four projective spaces. The six vertices of the hexastigm are joined in sets of two, three, or four, and each edge meets the opposite space in a diagonal point (Figure 1).

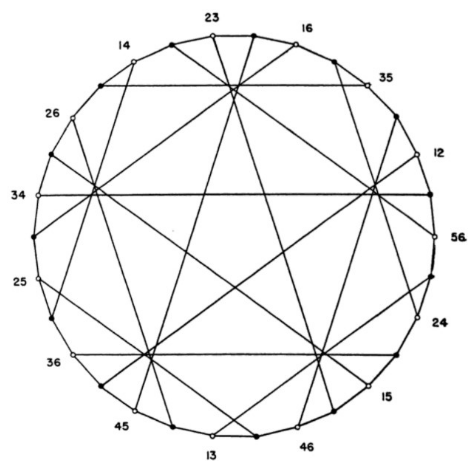

Figure 1. Harmonic conjugate with fifteen diagonal points with respect to the first and second vertices of the hexastigm [1].

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Three edges that together involve all six vertices are met by a unique line called a transversal. The correspondence between edges and transversals is seen in the fact that each transversal meets three edges, and each edge belongs to three transversals, which form a configuration 15_3 of a harmonic conjugate with fifteen diagonal points with respect to the first and second vertices of the hexastigm.

Aiming to generalize the harmony of n-points in the projective geometry, we conducted a study in 1996 titled "Generalization of the Desargues theorems" [2] and more detailed presentation of set of harmonic points in 1998 titled "Generalized Desargues' theorems and its implications" [3].

In 2014, we expanded on our research with the study "generalized Desargues theorem" [4], where an arbitrary number of points on each one of the two distinct planes is considered, allowing the corresponding points on the two planes to coincide and three points on any of the planes to be collinear. In the work [4, p.3] we identified the generalized Desargues' theorem in the following form:

Let p, p' be two distinct planes, (1), (2), . . ., (n), ($n \in$ N), distinct points on p, and (1′), (2′), . . . , (n') distinct points on p', both sets of points in general position.

(A) If all generalized lines (i)(i') go through a common point, then all the intersections of the pairs of lines (i)(j), (i')(j') for $i \neq j$ are nonempty and lie on a common line (the common line of p and p').

(B) If all the intersections of the pairs of lines (i)(j), (i')(j') for $i \neq j$ are nonempty, then all generalized lines (i)(i') go through a common point [4. Pp.3-4].

The proof of the proposition is made by mathematical induction, and we assume that the given points (1), (2), etc. on a plane p; and (1′), (2′), etc., some corresponding points on another plane p', allow for *the possibility of some corresponding points coinciding*. This restricts us from applying the generalization of the harmonic n-points, and we are also not able to prove the unicity of the mentioned generalized Desargues' theorem.

In this paper, we explore the complete n-points (the triangle is considered to have 3-points as special case of n-points and dual-figure n-lines) and all of the cases in which the intersection line p is incident with diagonal points. This allows us to define generalized harmonic points, mainly on the basis of works [3] and [2].

In addition, we present the well-known *four harmonic conjugate points* in the projective space (and in the projective plane).

Four harmonic collinear points: Four collinear points (A, B, C, D) are said to be a harmonic set, which is denoted as $H(AB, CD)$ if there exists a complete quadrangle (complete 4-points) (E, F, G, F) such that two of the points are diagonal points of the complete four points (quadrangle) and the other two points are on the opposite sides, determined by the third diagonal point (Figure 2).

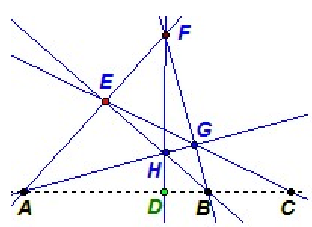

Figure 2. Four Harmonic points

We note that $H(AB, CD)$, $H(BA, CD)$, $H(AB, DC)$, and $H(BA, DC)$ all represent the same harmonic set of points.

Several natural questions arise about harmonic sets of points:

- Does a harmonic set of more points exist?

- Given the determined number of collinear points, what number of a harmonic point sets can be constructed?
- If the answer is yes, is the harmonic set unique or does it depend on the complete n-points defining the points?
- How can we determine when a set of x collinear points is a harmonic set?

In this article, we begin by investigating these questions and present answers for each one.

In an exercise on basic theorems, we produced the generalized Desargues' theorem for two complete n-points and the existence of a *complete n-point [5]*; thus, from these results, it follows that a harmonic set of x collinear points exists and can be constructed.

2. Generalization of the Desargues' theorems

Desargues theorem, the converse of Desargues theorem, and the theorem of the perspective quadrangle are the principal theorems in projective geometry [5, pp. 31-32].

A brief description of three principal theorems is presented below:

Theorem 1. (Desargues theorem). *If two triangles are perspective from a point, they are perspective from a line.*

Theorem 2. (the converse of Desargues theorem). *If two triangles are perspective from a line, they are perspective from a point.*

Theorem 3. (perspective quadrangles). *If two complete quadrangles are in (1-1) correspondence and so situated that five pairs of homologous sides intersect in points of the same straight line, then the point of intersection of the sixth pair of the homologous sides is also on this line, and the quadrangles are perspective to one another from a point and from a line.*

It is important to note that the proof of Theorem 3 is based on the Desargues theorem and the converse of Desargues' theorem.

In addition, the objects called points and lines satisfy at least one of the following three axioms:

(i) Any two distinct points are incident with exactly one line.

(ii) Any two distinct lines are incident with exactly one point.

(iii) There exist four points of which no three are collinear and where incidence is used as a neutral word meaning that the point belongs or lies on the line and meaning that the line passes through or contains the point.

Collinearity has the usual meaning. We know that for projective spaces of at list three dimension, are defined as the projective planes either by a set of incidence axioms or by algebraic constructions, Desargues' theorems are always true [5].

In this study, we use a property of the projective geometry, *the principle of duality,* which proposes that once a theorem is proved, by the principal of duality (in the plane and in the space), the duality of the theorem is also valid [5].

Additionally, we bear in mind that Desargues' theorem and its duality presented a relationship between the vertices and sides of the perspective triangles (three points).

We next consider the relationships between certain points and sides determined from complete n-points and/or its dual figure complete n-lines; we present all of the proof of the propositions related to the complete n-points.

The simple question derived from Theorem 3 is *when are two complete plane n-points perspective from a point and from a line?*

First, let us define the n-points:

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A complete n-point is a set of n points (vertices) of a plane, no three of which are collinear, and the $\frac{n(n-1)}{2}$ line (side) joins every pair of these points. We denote the complete plane n-points $A_1A_2 \dots A_n$, where n is a natural number and $n > 2$.

All of the sides A_iA_j correspond to the order number $r(A_iA_j)$ defined by

$$r\left(A_iA_j\right)_n = j + (i-1)\cdot n - \frac{i\,(i+1)}{2}, \quad i,j = 1,2,3,\dots,n. \qquad (*)$$

for example, the side A_1A_2 corresponds to the number $r(A_1A_2) = 1$; the side A_1A_3 corresponds to the number $r(A_1A_3) = 2$; the side $r(A_{n-1}A_n) = \frac{n\,(n-1)}{2}$; etc.

Now, we define the perspectivity for two complete plane n-points [5]:

Two complete planes n-points are in the perspective position if they are in a $(1-1)$ correspondence such that pairs of homologous vertices are joined by lines concurrent at one point.

Based on the principle of duality in projective geometry, we accept and consider definitions for the perspectivity of the two complete plane n-lines.

Is it important to determine when and what the conditions are when the two complete n-points are perspective?

We find the answer in the following theorem [2]:

Let there be two given coplanar (or noncoplanar) complete plane n-points, $A_1A_2 \dots A_n$ and $B_1B_2 \dots B_n$, and let there be $X_{r(i,j)n} = A_iA_j \cdot B_iA_j$ points of intersections of the correspondent sides, where $r\left(A_iA_j\right)_n = j + (i-1)\cdot n - \frac{i\,(i+1)}{2}$; $i,j = 1,2,3,\dots,n$, and $i < j$.

GCD Theorem (Generalization of the Converse of Desargues' theorem): *If two complete plane n-points are in $(1-1)$ correspondence and so situated that $(2n-3)$ intersections' points $X_{r(i,j)n}$ are collinear points, then the remaining $\frac{(n-2)(n-3)}{2}$ intersections' points of homologous sides of the two complete n-points are collinear with the same line, and the two n-points are perspective from a point and from a line.*

Proof of GCD Theorem. We prove this theorem via mathematical induction.

Let there be two given coplanar (or noncoplanar) complete plane n-points, $A_1A_2 \dots A_n$ and $B_1B_2 \dots B_n$. (Figure 3).

i) For $n = 3$, the GCD theorem is equivalent to the Theorem 2 (the converse of Desargues' theorem).

For $n = 4$, the GCD theorem is equivalent to Theorem 3 (perspective quadrangles).

ii) We suppose that the GCD theorem is true for $n = k-1$.

We must prove that the GCD theorem is true for $n = k$.

We consider $X_{r(i,j)k} = A_iA_j \cdot B_iB_j$ and intersections of the correspondent sides, where

$$r\left(A_iA_j\right)_k = j + (i-1)\cdot k - \frac{i\,(i+1)}{2};\; i = 1,2,3,\dots,k-2, \text{ and } j = 2,3,\dots,k-1.$$

Thus, by the hypothesis of this theorem, the points $X_{r(i,j)k}$ where $i,j = 1,2,3,\dots k$; $i < j$ are colinear points with p, and the straight lines A_iB_j are concurrent lines with the point O.

The complete plane k-points $A_1A_2 \dots A_k$ and $B_1B_2 \dots B_k$ are in $(1-1)$ correspondence, and $(2k-3)$ sides pass through A_1 and A_2

$$A_1 \in A_1A_i\,, i = 2,3,\dots,k \text{ and } A_2 \in A_2A_j\,, j = 3,4,\dots,k$$

meet the corresponding sides of the other complete k-points in the point of line p formed by the corresponding vertices B_1, B_2 of the complete plane k-points $B_1B_2 \dots B_k$; i.e.,

$$B_1 \in B_1B_i\,, i = 2,3,\dots,k;\;\; B_2 \in B_2B_j\,, j = 3,4,\dots,k.$$

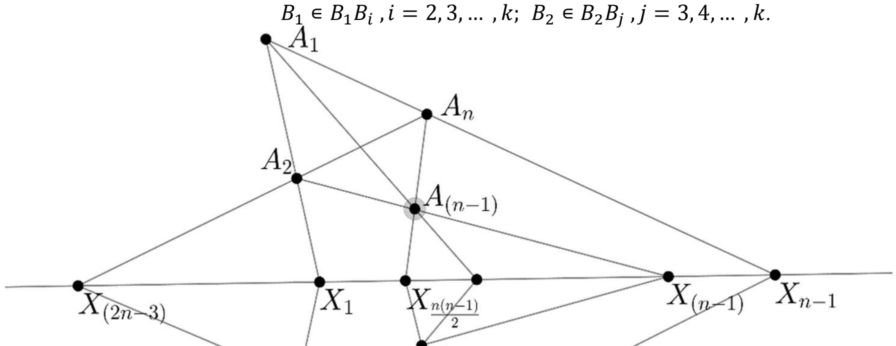

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Figure 3. Perspectivity of two complete n-points from a line.

We consider the complete plane $(k-1)$ points $A_1A_2 \dots A_{k-1}$ and $B_1B_2 \dots B_{k-1}$.

According to hypothesis *ii)*, they are perspective from the axis $p(X_1, X_2, \dots ,)$ and we note that $2(k-1)-3=2k-5$ intersection points are the following collinear points:

$$A_1A_2 \cap B_1B_2 = X_1$$

$$\dots\dots\dots\dots\dots\dots$$

$$A_1A_{k-1} \cap B_1B_{k-1} = X_{k-2}$$

$$A_2A_3 \cap B_2B_3 = X_k$$

$$\dots\dots\dots\dots\dots\dots$$

$$A_2A_{k-1} \cap B_2B_{k-1} = X_{2k-5}$$

$$\dots\dots\dots\dots\dots\dots$$

$$A_{k-2}A_{k-1} \cap B_{k-2}B_{k-1} = X_{\frac{(k-2)(k-1)}{2}}$$

Lines A_1B_1, $i = 1, 2, 3, \dots, (k-1)$ are concurrent with point O.

Therefore, the others intersect points

$$A_3A_4 \cap B_3B_4 = X_{2k-2}$$

$$\dots\dots\dots\dots\dots\dots$$

$$A_3A_{k-1} \cap B_3B_{k-1} = X_{3k-7}$$

$$\dots\dots\dots\dots\dots\dots$$

$$A_{k-2}A_{k-1} \cap B_{k-2}B_{k-1} = X_{\frac{k(k-1)}{2}-2}$$

are collinear with the same line p, and lines A_iB_i, $i = 1, 2, 3, \dots, k-1$, are concurrent with point O:

$$O \in A_iB_i \ , \ i = 1, 2, \ \dots \ , k-1 \qquad (1)$$

The two complete plane $(k-1)$ points, $A_1A_2 \dots A_{k-2}A_k$ and $B_1B_2 \dots B_{k-2}B_k$, are perspective related to the $p(X_1, X_2, \dots)$ axis according to hypothesis ii), because for two complete plane $(k-1)$-points, $A_1A_2 \dots A_{k-2}A_k$ and $B_1B_2 \dots B_{k-2}B_k$, we have $2(k-1)-3=2(k-5)$ collinear points:

$$A_1A_2 \cap B_1B_2 = X_1$$

$$\dots\dots\dots\dots\dots\dots$$

$$A_1A_{k-2} \cap B_1B_{k-2} = X_{k-3}$$

$$A_1A_k \cap B_1B_k = X_{k-1}$$

$$A_2A_3 \cap B_2B_3 = X_k$$

$$\dots\dots\dots\dots\dots\dots$$

$$A_2A_{k-2} \cap B_2B_{k-2} = X_{2k-5}$$

$$A_2A_k \cap B_2B_k = X_{2k-3}$$

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Therefore, the other intersect points

$$A_3A_4 \frown B_3B_4 = X_{2k-1}$$

$$\dots\dots\dots\dots\dots\dots\dots$$

$$A_3A_{k-2} \frown B_3B_{k-2} = X_{3k-8}$$

$$A_3A_k \frown B_3B_k = X_{3k-6}$$

$$\dots\dots\dots\dots\dots\dots\dots$$

$$\dots\dots\dots\dots\dots\dots\dots$$

$$A_{k-2}A_k \frown B_{k-2}B_k = X_{\frac{k(k-1)}{2}-1}$$

are collinear with line p, and lines A_iB_i; $i = 1, 2, \dots, k-2, k$ are concurrent with point O:

$$O \in A_iB_i\ ,\ i = 1, 2, \dots, k-2, k \qquad (2)$$

We must prove that the points $A_{k-1}A_k \frown B_{k-1}B_k = X_{\frac{k(k-1)}{2}}$ are collinear with the points

$$X_i\ , i = 1, 2, \dots \dots, k-2, k.$$

In fact, the triangles $A_1A_{k-1}A_k$ and $B_1B_{k-1}B_k$ are perspective-related to point O; therefore, they are perspective from the line p according to Theorem 1 (Desargues' theorem).

Then, the pairs of homologous sides meet at collinear points:

$$A_1A_{k-1} \frown B_1B_{k-1} = X_{k-2}$$

$$A_1A_k \frown B_1B_k = X_{k-1}$$

$$A_{k-1}A_k \frown B_{k-1}B_k = X_{\frac{k(k-1)}{2}}$$

However, the points X_{k-2} and X_{k-1} lie on the line p, and the point $X_{\frac{k(k-1)}{2}}$ must lie on the line p.

Thus, considering relations (1) and (2), the two n-points are perspective from the point O, which proves that the GCD theorem is true for $\forall\, n \in \mathbb{N}$.

The GCD theorem is also valid when the line p passes through diagonal points of the complete n-points.

Now, in an analogical way, we can prove the generalized Desargues' theorem (GD).

For this purpose, in similar way, let there be two given coplanar (or noncoplanar) complete n-points, $A_1A_2 \dots A_n$ and $B_1B_2 \dots B_n$. Additionally, let there be $X_{r(i,j)n} = A_iA_j \cdot B_iA_j$ points of intersections of the correspondent sides, where $r(A_iA_j)_n = j + (i-1)\cdot n - \frac{i\,(i+1)}{2}$; $i, j = 1, 2, 3, \dots, n$, and $i < j$, and let there be a P-intersection point of the lines determined by homologues vertices:

$$0 \ni A_iB_i, i = 1,2,3, \dots, n.$$

GD Theorem (Generalized Desargues' Theorem). *If two complete plane n-points are in perspective from a point P and the $(n-1)$-sides passing through one vertex meet the corresponding sides of the other n-point in the colinear points of a line, then the two complete n-points are perspective from a line.*

Proof. Let there be two given coplanar (or noncoplanar) complete plane n-points, $A_1A_2 \dots A_n$ and $B_1B_2 \dots B_n$. (Figure 3).

i) For $n = 3$, GD theorem is equivalent to Theorem 1 (Desargues' theorem).

ii) We suppose that GD theorem is true for $n = k-1$.

Thus, we can prove that the GD theorem is true for $n = k$.

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The complete plane k-points $A_1A_2 \dots A_k$ and $B_1B_2 \dots B_k$ are $(1-1)$ correspondence; k-*pairs* of homologous vertices are joined by lines concurrent at one point O; and $k-1$ pairs of homologous sides intersect at the collinear points of the line p.

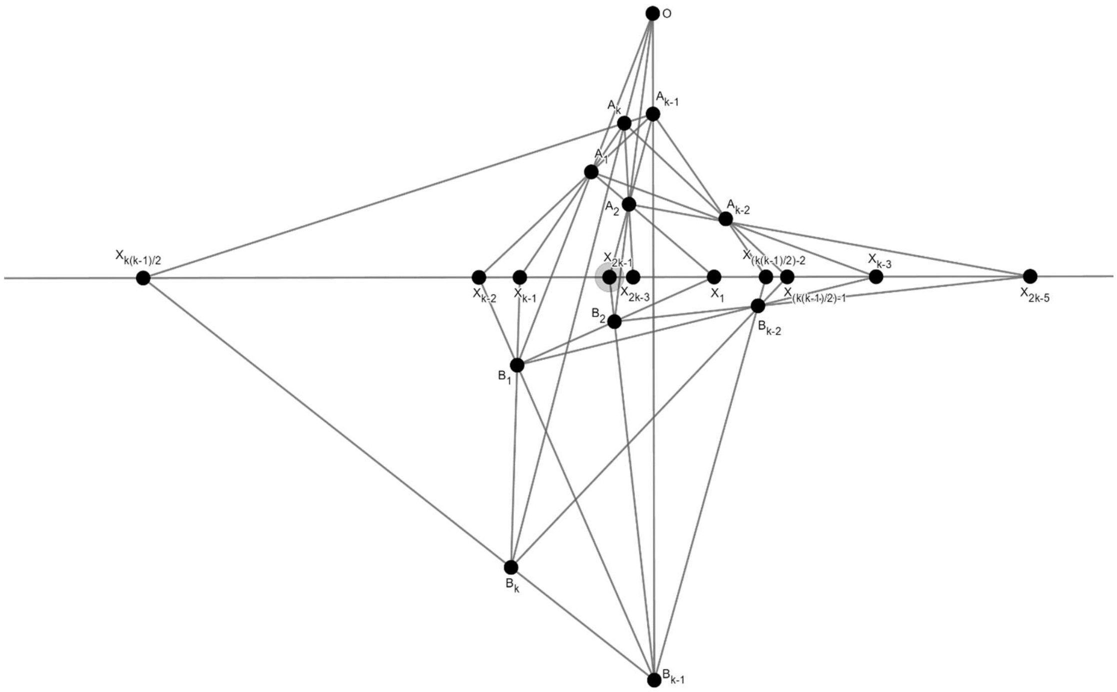

Figure 3. Perspectivity of two n-points from a point.

Based on the hypothesis of the mathematic induction, the two complete plane $(k-1)$-points $A_1A_2 \dots A_{k-1}$ and $B_1B_2 \dots B_{k-1}$ are perspective from the axis (line) $p(X_1, X_2, \dots)$ and from a center O, because for complete plane $(k-1)$-points $A_1A_2 \dots A_{k-1}$ and $B_1B_2 \dots B_{k-1}$, we have $O \in A_iB_i$, $i = 1, 2, \dots, k-1$ and $(k-1)-1 = k-2$.

The collinear points are as follows:

$$A_1A_2 \cap B_1B_2 = X_1$$

$$\dots\dots\dots\dots$$

$$A_1A_{k-1} \cap B_1B_{k-1} = X_{k-2}$$

Therefore, the others intersect points

$$A_2A_3 \cap B_2B_3 = X_k$$

$$\dots\dots\dots\dots$$

$$A_2A_{k-1} \cap B_2B_{k-1} = X_{2k-4}$$

$$A_3A_4 \cap B_3B_4 = X_{2k-2}$$

$$\dots\dots\dots\dots$$

$$\dots\dots\dots\dots$$

$$A_{k-2}A_{k-1} \cap B_{k-2}B_{k-1} = X_{\frac{k(k-1)}{2}-2}$$

are collinear points of the same line p.

Similarly, the two complete plane $(k-1)$-points $A_1A_2 \dots A_{k-2}A_k$ and $B_1B_2 \dots B_{k-2}B_k$ are perspective from the axis $p(X_1, X_2, \dots)$ and from a centre (point) O according to the hypothesis of the theorem

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for complete plane $(k-1)$-points $A_1A_2 \dots A_{k-2}A_k$ and $B_1B_2 \dots B_{k-2}B_k$. Thus, we obtain $O \in A_iB_i$, $i = 1, 2, \dots, k-2, k$ and $(k-1)-1 = k-2. C$ collinear points:

$$A_1A_2 \cap B_1B_2 = X_1$$

$$\dots\dots\dots\dots\dots\dots$$

$$A_1A_{k-2} \cap B_1B_{k-2} = X_{k-3}$$

$$A_1A_k \cap B_1B_k = X_{k-1}$$

Therefore, the other intersect points

$$\mathrm{A_2A_3 \cap B_2B_3 = X_k}$$

$$\dots\dots\dots\dots\dots\dots$$

$$\mathrm{A_2A_{k-2} \cap B_2B_{k-2} = X_{2k-5}}$$

$$\mathrm{A_2A_k \cap B_2B_k = X_{2k-3}}$$

$$\mathrm{A_3A_4 \cap B_3B_4 = X_{2k-2}}$$

$$\dots\dots\dots\dots\dots\dots$$

$$\dots\dots\dots\dots\dots\dots$$

$$\mathrm{A_{k-2}A_k \cap B_{k-2}B_k = X_{\frac{k(k-1)}{2}-1}}$$

are collinear points with the same line p.

We must prove that the point $A_{k-1}A_k \cap B_{k-1}B_k = X_{\frac{k(k-1)}{2}}$ is collinear with the points $X_i, i = 1, 2, \dots$

In fact, the triangles $A_1A_{k-1}A_k$ and $B_1B_{k-1}B_k$ are perspective-related to the point O; therefore, they are perspective from a line (on the basis of Desargues' theorem). Thud, the pairs of homologous sides

$$A_1A_{k-1} \cap B_1B_{k-1} = X_{k-2}$$

$$A_1A_k \cap B_1B_k = X_{k-1}$$

$$A_{k-1}A_k \cap B_{k-1}B_k = X_{\frac{k(k-1)}{2}}$$

are collinear points.

Because the points X_{k-2} and X_{k-1} lie on the line p, point $X_{\frac{k(k-1)}{2}}$ also lies on the line p. Thus, we have proven that the GD theorem is true for $\forall n \in \mathbb{N}$.

The duality of the GD theorem is true on the basis of the principle of duality.

Maintaining the synthetical logic of proof of the generalized Desargues' theorems, it is not difficult the analytically prove them.

The proof of the generalized Desargues' theorem is independent of the incidence of the axis of the perspectivity with diagonal points of the complete n-points (n-lines). This is an important fact in the next part of the paper, which discusses the generalization of the harmonic points of the line.

3. The structure of n-harmonic points

Let $A_1A_2 \dots A_n$ be complete n-points and p a coplanar line (Figure 4). Then, the following cases may occur:

- The line p can be incident with at most two vertices of complete n-points.

If, e.g., the line p is incident with vertices A_s and $\mathrm{A_t}$, then we can write $p \equiv A_{r(st)n}$, which means that the line p coincides with the side $A_s\ A_t$. The other sides of the n-points $A_1A_2 \dots A_n$ are intersected by the line p at $\frac{(n-2)(n-3)}{2}$—different points.

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Indeed, every side (in our case the line p) of the complete n-points has $\frac{(n-2)(n-3)}{2}$ opposite sides and $2(n-2)$ adjacent sides.

The line p (which, in this case, is identical to a side of full n-points) is intersected by $2(n-2)$ adjacent sides at points A_s and A_t, while with opposite sides (non-adjacent) at other points P_u, $u = 1,2,\dots,\frac{(n-2)(n-3)}{2}$, which in total are $\frac{(n-2)(n-3)}{2}+2$ points (Figure 4).

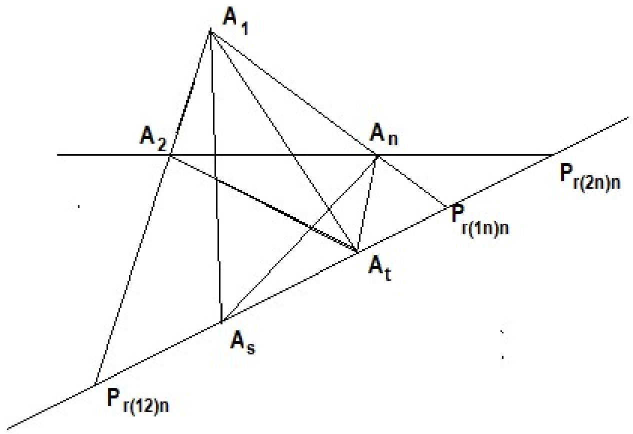

Figure 4. The line is incident with two diagonal points of complete n-points.

Similarly, we can discuss the other four cases:

- The line p can be incident with one vertex of complete n-points.
- The line p can be not incident with any vertices and diagonal points of complete n-points.
- The line p is not incident with any vertices and is incident with one diagonal point of complete n-points.
- The line p is not incident with any vertices but is incident with two diagonal points of complete n-points.

Thus, regarding the intersection points of the straight line and complete plane n-point, we can prove the following proposition:

The intersection points of the complete n-points and a straight line p can be $\frac{n(n-1)}{2}-2$, $\frac{n(n-1)}{2}-(n-1)$, or $\frac{n(n-1)}{2}-k$, where k is the number of diagonal points incidents with the line p, and k=1, 2, 3, …n-2.

Definition 1. A set of points in which the sides of complete plane n-points meet a straight line, where the line is *not incident* with any diagonal points and any vertices of complete n-points, is called a *set of H-points of rank* 0, *which is indicated by* P_n^0.

The number of points of set P_n^0 is $K|P_n^0| = \frac{n(n-1)}{2}$

Definition 2. A set of points in which the sides of a complete plane n-point meet a line, where the line is incident with the k diagonal point and not incident with any vertices of complete n-points, is called *a set of H-points of rank k*, which is indicated by P_n^k.

The number of points of set P_n^k is $K|P_n^k| = \frac{n(n-1)}{2} - k$

Example: A set of the points in which the sides of a complete n–point meet a line p, where the line p is incident with two diagonal points and not incident with any vertices of complete plane n–point, is called a *set of H-points of rank* 2, and it is indicated by P_n^2

The number of points of set P_n^2 is: $K|P_n^2| = \frac{n(n-1)}{2} - 2$

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In the case of $n = 4$, this is the set P_2^2, which presents *four conjugate harmonic points*—a set of points in which the sides of a quadrangle meet a line, where the line is incident with two diagonal points of the quadrangle and not incident with any of vertices of it (see the definition for harmonic points on page 1).

On the basis of the principle of duality, we can define a *set of H-lines of rank 0, and it is indicated by* δ_n^0.

The number of lines of set δ_n^0 is $K|\delta_n^0| = \frac{n(n-1)}{2}$

In addition, the *set of H-lines of rank k* is indicated by δ_n^k.

The number of lines of set δ_n^k is $K|\delta_n^k| = \frac{n(n-1)}{2} - k$

Example: A *set of H-lines of rank* 2 is indicated by δ_n^2

The number of lines of set δ_n^2 is $K|\delta_n^2| = \frac{n(n-1)}{2} - 2$

In the case of $n = 4$, this is set δ_n^2 , which present *four conjugate harmonic lines*, that is, four concurrent lines, two of which are diagonal lines, while the other two are lines passing through the two vertices lying on the third diagonal line.

Now, it is important to study the relation between the sets P_n^k for different values of n and k, $k < n + 2$.

If the set P_{n+1}^0 is determined from the complete $(n+1)$-points, and, from these vertices' isolates (exclude), the point n and the rested complete n-points determine the set P_n^0, then it is evident that the set P_n^0 is a subset of the set P_{n+1}^0.

Example: the set P_3^0 is determined by complete three points, $A_1A_2A_3$. If the set P_4^0 is determined by complete 4 points, $A_1A_2A_3A_4$, then $P_3^0 \subset P_4^0$.

In addition, if P_5^0 is determined by complete 5-points, $A_1A_2A_3A_4A_5$, and the derived complete 4-points, $A_1A_2A_3A_4$, then $P_4^0 \subset P_5^0$, etc.

Therefore, the following relation is valid: $P_3^0 \subset P_4^0 \subset P_5^0 \subset \cdots \subset P_n^0 \subset P_{n+1}^0 \ldots$ (*)

On the other hand, for the set P_n^1, the relation of the inclusion of (*) is different.

Let there be $A_1A_2 \ldots A_n$ n-points, which determine the set P_n^1.

The definition of the set P_n^1 implies that there is a diagonal point, which means that two opposite sides of complete n-points $A_1A_2 \ldots A_n$ are incident with the same points of the set P_n^1.

Let there be the sides A_rA_s and A_tA_v, two opposite sides of n-points $A_1A_2 \ldots A_n$.

If we exclude one of the vertices A_r, A_s, A_t or A_v, then we obtain complete $(n-1)$-points; these determine the set P_{n-1}^0, which is $P_{n-1}^0 \subset P_n^0$. In this case, considering relation (*), we have

$$P_3^0 \subset P_4^0 \subset \cdots \subset P_{n-1}^0 \subset P_n^1 \quad (**)$$

We consider the complete $(n+1)$-points $A_1A_2 \ldots A_nA_{n+1}$, which determine the set P_{n+1}^2. The set P_{n+1}^2 includes two diagonal points of complete $(n+1)$-points, $A_1A_2 \ldots A_nA_{n+1}$.

Let there be D_1 and D_2 diagonal points included in the set P_{n+1}^2 and

$$D_1 = A_mA_n \cdot A_pA_q, D_2 = A_rA_s \cdot A_tA_v.$$

If from the complete $(n+1)$ points $A_1A_2A_3 \ldots A_nA_{n+1}$ we exclude one of the vertices $A_m, A_n, A_p, A_q, A_r, A_s$, A_t and A_v, then, we will obtain the complete n-points that determine the set P_n^1 and $P_n^1 \subset P_{n+1}^2$.

Now, considering the relation (**), we have

$$P_3^0 \subset P_4^0 \subset \cdots \subset P_{n-1}^0 \subset P_n^1 \subset P_{n+1}^2 \quad (***)$$

If from the complete (n+1)-points $A_1A_2 \ldots A_nA_{n+1}$ that determine the set P_{n+1}^2, we add the new vertex A_{n+2}, then we will obtain the set P_{n+2}^2 and the relation

$$P_{n+1}^2 \subset P_{n+2}^2$$

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We can continue this process, and considering the relation (***), we obtain the following relation:

$$P_3^0 \subset P_4^0 \subset \cdots \subset P_{n-1}^0 \subset P_n^1 \subset P_{n+1}^2 \subset P_{n+2}^2 \ldots \quad (****)$$

Let us observe the case of complete u-points $A_1A_2 \ldots A_u$ that determine the set $P_u^3,\ u > n+3$.

The set P_u^3 includes three diagonal points of complete u-points $A_1A_2 \ldots A_u$.

Let there be D_1, D_2, and D_3, the diagonal points of u-points $A_1A_2 \ldots A_u$:

$$D_1 = A_{r_1}A_{r_2} \cdot A_{r_3}A_{r_4},$$

$$D_2 = A_{s_1}A_{s_2} \cdot A_{s_3}A_{s_4}, \text{ and}$$

$$D_3 = A_{t_1}A_{t_2} \cdot A_{t_3}A_{t_4}, \quad r_i \neq s_i \neq t_i,\ i = 1,2,3,4.$$

If, from complete u-points $A_1A_2 \ldots A_u$, we exclude one of the vertices, $A_{r_i},\ A_{s_i}$, or A_{t_i}, $i = 1,2,3,4$, then we obtain complete (u-1)-points, which determine the set P_{u-1}^3 and the following relation:

$$P_{u-1}^2 \subset P_u^3$$

Now, considering the relation (****), we obtain the following:

$$P_3^0 \subset P_4^0 \subset \cdots \subset P_{n-1}^0 \subset P_n^1 \subset P_{n+1}^2 \subset P_{n+2}^2 \subset \cdots \subset P_{u-1}^2 \subset P_u^3 \quad (*****)$$

This process, in the same way continues, and we obtain the following relations:

$$P_3^0 \subset P_4^0 \subset \cdots \subset P_{n-1}^0 \subset P_n^1 \subset P_{n+1}^2 \subset \cdots \subset P_{u-1}^2 \subset P_u^3 \subset P_{u+1}^3 \subset \cdots$$

$$P_3^0 \subset P_{k_1}^0 \subset P_{k_{1+1}}^1 \subset P_{k_{1+2}}^1 \subset \cdots \subset P_{k_{2-1}}^1 \subset P_{k_2}^2 \subset P_{k_{2+1}}^2 \subset \cdots \subset P_{k_{3-1}}^2 \subset P_{k_3}^3 \subset \cdots$$

where $k_1 = 3,4,5,6 \ldots \quad k_2 = 5,6,7, \ldots \quad k_3 = 5,6,7,8, \ldots$

We present all of these relations of sets of harmonic points in Figure 5.

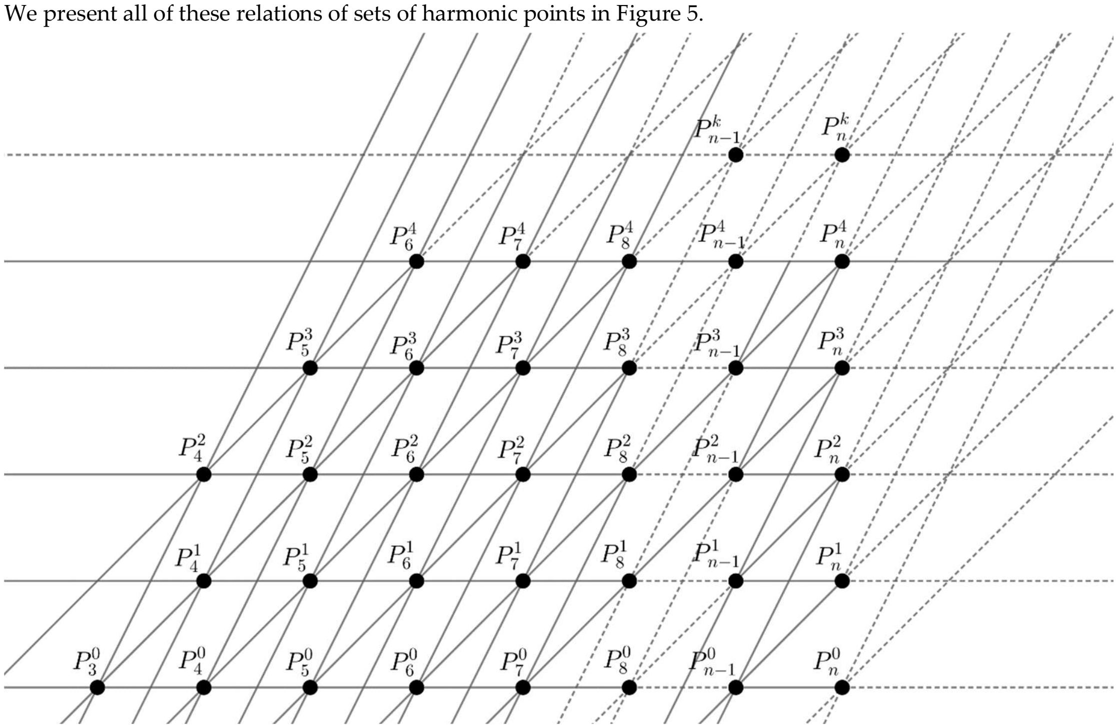

Figure 5. The structure of generalized harmonic points.

Analyzing Figure 5, we note that a set of harmonic points P_k^r is the subset of the set of H-points with a smaller rank or equal to $r+2$; thus, the relation is

$$P_k^r \subset P_{k+1}^r,$$

$$P_k^r \subset P_{k+1}^{r+1} \quad \text{or}$$

$$P_k^r \subset P_{k+1}^{r+2}.$$

In fact, if we add a new vertex to the complete k-points, we obtain a new complete $(k+1)$-points, which determine the following set:

- P_{k+1}^{r} of rank r if the number of the diagonal points is not increased.
- P_{k+1}^{r+1} of rank $r+1$ if the number of the diagonal points is increased by one.
- P_{k+1}^{r+2} of rank $r+2$ if the number of the diagonal points is increased by two.

As the number of diagonal points incidents with the line, it is not possible for it to be more than 2, because each point is determined by two lines.

Thus, if we add the vertex A_{k+1} as an intersection of the two lines to the complete k-points $A_1A_2 \dots A_k$, each of them incidents with a point of the set P_k^r and one of vertices of the complete k-points $A_1A_2 \dots A_k$. This provides us with the set of H-points of rank $r+2$: P_{k+1}^{r+2};

If we add the vertex A_{k+1} as an intersection of the two lines to the complete k-points $A_1A_2 \dots A_k$, only one of them is incident with the points of the set P_k^r and one of the vertices of the complete k-points $A_1A_2 \dots A_k$. This provides us with the set of H-points of rank $r+1$: P_{k+1}^{r+1};

If we add the vertex A_{k+1} as an intersection of the two lines to the complete k-points $A_1A_2 \dots A_k$, not one of them is incident with the points of the set P_k^r, thereby providing us with the set of H-points of rank r: P_{k+1}^{r};

Considering Figure 5 and the relation of inclusion, for the harmonic points P_k^r, we can present the matrices' form:

For example, if we consider the set P_6^0, the series of inclusive sets is P_n^0, $n = 3,4,5$:

$$P_3^0 \subset P_4^0 \subset P_5^0 \subset P_6^0$$

We can denote with the symbol (0000) that the first column presents the set P_3^0, the second column is the set P_4^0, the third column is the set P_5^0, and the fourth column is the P_6^0.

Each of the sets of H-points of rank 0 can be presented by matrices of one row and n-2 columns. The series of the inclusion of H- points of rank 1 is with two rows; for example, the set P_6^1 is presented by the following matrix:

$$P_3^0 \subset P_4^0 \subset P_5^0 \subset P_6^1$$
$$P_3^0 \subset P_4^0 \subset P_5^1 \subset P_6^1$$
$$P_3^0 \subset P_4^1 \subset P_5^1 \subset P_6^1$$

Equivalent matrix:

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

The number of elements in the matrix for the set P_n^1 is $n-3$. Thus, the matrix for the set P_n^1 is $(n-3) \times n-2)$, or, if we exclude the set P_3^0, then the corresponding matrix for P_n^1 is in the form of $(n-3) \times (n-3)$.

The matrix of the inclusion for the set of H- points of rank 2 is reached. The matrix of the set P_6^2 is as follows:

$$P_3^0 \subset P_4^0 \subset P_5^0 \subset P_6^2$$
$$P_3^0 \subset P_4^0 \subset P_5^1 \subset P_6^2$$
$$P_3^0 \subset P_4^1 \subset P_5^1 \subset P_6^2$$
$$P_3^0 \subset P_4^0 \subset P_5^2 \subset P_6^2$$
$$P_3^0 \subset P_4^1 \subset P_5^2 \subset P_6^2$$
$$P_3^0 \subset P_4^2 \subset P_5^2 \subset P_6^2$$

Equivalent matrix:

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$$\begin{pmatrix} 0 & 0 & 0 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 2 & 2 \\ 0 & 1 & 2 & 2 \\ 0 & 2 & 2 & 2 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 & 0 & 2 \\ 0 & 1 & 2 \\ 1 & 1 & 2 \\ 0 & 2 & 2 \\ 1 & 2 & 2 \\ 2 & 2 & 2 \end{pmatrix}$$

The number of rows for the set P_n^2 is $\frac{(n-2)(n-3)}{2}$. Thus, the matrix of inclusion for the h-points P_n^2 is $\left[\frac{(n-2)(n-3)}{2}\right] \times (n-2)$.

If we exclude the set P_3^0, they will have the form $\left[\frac{(n-2)(n-3)}{2}\right] \times n-3)$.

In the same way, we can construct the matrices for each of the sets of the H-points.

The set of the four harmonic points obtained from complete 4-points and a line incident with two diagonal points is the subset of the largest set of H-points constructed before P_n^k, $k \le n-2$.

We can demonstrate the same for the harmonic H-lines.

Now, we can prove the following propositions:

Proposition 1. *The set of points* P_n^0 *is uniquely determined when* $(2n-3)$ *collinear points are given.*

Proof. If $\frac{n(n-1)}{2}$ points $T_1, T_2, \dots, T_{\frac{n(n-1)}{2}}$ are on a line p and form a set of P_n^0 defined by a complete plane n-point $A_1A_2 \dots A_n$, then a second complete plane n-point $B_1B_2 \dots B_n$ in the same plane or in any plane passing through p, which has $(2n-3)$ sides passing through $T_i, i = 1, 2, \dots, (2n-3)$ and through any two of the vertices $B_1, B_2, \dots, B_n$, will have its the remaining sides passing through $T_j, j = 2n-2, 2n-1, \dots, \frac{n(n-1)}{2}$.

This was evident from the GCD theorem (the generalized Converse of Desargues' theorem).

Therefore, we draw arbitrarily in any plane through p, $(n-1)$-concurrent lines with a point. Let there be point A_1 and $(n-2)$-concurrent lines with a point; let there be point A_2, $A_2 \in A_1T_1$—one through each of the given $T_1, T_2, \dots, T_{2n-3}$ points. The remaining vertices A_i, $i > 2$ are determined with

$$A_i = A_1 T r_{1i} \cap A_2 T r_{2i}$$

where $r_{1i} = r(A_1A_i)_n = i-1$, $r_{2i} = r(A_2A_i)_n = n+i-3$, and $\frac{n(n-1)}{2}$ sides of the constructed n-point $A_1A_2 \dots A_n$ meet the line p in the desired points:

$$T_i, \; i = 1, 2, \dots, \; 2n-3, 2n-2, \dots, \frac{n(n-1)}{2} \; .$$

The GCD theorem ensures that any other complete plane n-point $B_1B_2 \dots B_n$ constructed in the same way as the complete plane n–points $A_1A_2 \dots A_n$ will determine on p the same points, $T_j, j = 2n-2, 2n-1, \dots, \frac{n(n-1)}{2}$.

Proposition 2. The set of points P_n^1 is uniquely determined when $(2n-4)$ collinear points are given.

Proof. As in Proposition 1, in this case, we have one pair of opposite sides intersecting at T_i, i which is an index of $\{1, 2, \dots, 2n-4\}$.

Based on the GCD theorem, the points $T_j, j = 2n-3, 2n-2, \dots, \frac{n(n-1)}{2} - 1$, are uniquely determined.

Proposition 3. *The set of points* P_n^2 *is uniquely determined when* $(2n-5)$ *collinear points are given.*

Proof. As in Proposition 1, ,in this case, we have one pair of opposite sides intersecting at T_i and a second pair of opposite sides intersecting at T_j; i, j, which is an index of $\{1, 2, \dots, 2n-5\}$. Based on GCD theorem, the points $T_j, j = 2n-3, 2n-2, \dots, \frac{n(n-1)}{2} - 1$ are uniquely determined.

Now, we formulate the main proposition:

Theorem 4. *For every set of* n*-collinear points, the set of* H*–points of rank* 0*, rank* 1*, or rank* 2 *are always uniquely determined.*

Proof. Every natural number n has the form $n = 2k$ or $n = 2k-1$, $k \in \mathbb{Z}^+$.

If $n = 2k$, then $n = 2k = 2(k+2) - 4$. For $(2(k+2) - 4)$ collinear points, Proposition 2 implies that the set of points P^1_{k+2} is uniquely determined.

If $n = 2k - 1$, then $n = 2k - 1 = 2(k+1) - 3$ or $n = 2k - 1 = 2(k+2) - 5$.

For $(2(k+1) - 3)$ collinear points, Proposition 1 implies that the set of points P^0_{k+2} is uniquely determined.

For $(2(k+2) - 5)$ collinear points, Proposition 3 implies that the set of points P^2_{k+2} is uniquely determined.

One interesting case is that when $n = 3$. In this case, for three collinear points A_1, A_2, A_3, Theorem 4 implies that we may constructed sets P^0_3 and P^2_4. The set P^2_4 has four points that represent the well-known *set of four harmonic points*.

Thus, for three collinear points A_1, A_2, A_3, we can determinate the fourth harmonic point $A_4 = H(A_1, A_3; A_2)$ that is $H(A_1, A_3; A_2, A_4)$.

3. Conclusions

The ordering of points on a line according to a rule determined by the relation of the line and the sides of complete n-points is presented as the generalization of harmonic points. According to this rule of ordering the points of a line, it was necessary to generalize the well-known Desargues theorems. Both results will hopefully facilitate a debate in the mathematical community.

It is also necessary to study the algebraic structures of the sets of H-points P^k_n (and the sets of H-lines). In this paper, we examined only the relationship between these sets of H-points of different ranks.

Supplementary Materials: The following are available online at www.mdpi.com/xxx/s1, Figure S1: title, Table S1: title, Video S1: title.

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