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Quantum-like cognition and rationality: biological and artificial intelligence systems

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Abstract

This is a short introductory review on quantum-like modeling of cognition with applications to decision making and rationality. The aim of the review is twofold: a) to present briefly the apparatus of quantum information and probability theory useful for such modeling; b) to motivate applications of this apparatus in cognitive studies and artifical intelligence, psychology, decision making, social and political sciences. We define quantum rationality as decision making that is based on quantum information processing. Quantumly and classically rational agents behaves differently. A quantum-like agent can violate the Savage Sure Thing Principle, the Aumann theorem on impossibility of agreeing to disagree. Such an agent violates the basic laws of classical probability, e.g., the law of total probability and the Bayesian probability inference. In some contexts, "irrational behavior" (from the viewpoint of classical theory of rationality) can be profitable, especially for agents who are overloaded by a variety of information flows. Quantumly rational agents can save a lot of information processing resources. At the same time, this sort of rationality is the basis for quantum-like socio-political engineering, e.g., social laser. This rationality plays the important role in the process of decision making not only by biosystems, but even by AI-systems. The latter equipped with quantum(-like) information processors would behave irrationally, from the classical viewpoint. As for biosystems, quantum rational behavior of AI-systems has its advantages and disadvantages. Finally, we point out that quantum-like information processing in AI-systems can be based on classical physical devices, e.g., classical digital or analog computers.

keywords: mathematical formalism; quantum mechanics; open quantum systems; quantum instruments; quantum Markov dynamics; psychological effects; cognition; decision making, classical rationality, quantum rationality, information overload.

1 Introduction

'Information' is the new paradigm reflecting the main character of modern human society as an information society. The tremendous development of information technologies of the last 20 years has dramatically changed our lifestyle through the world-wide web of the Internet and the mobile-connectivity web that have led to the creation of virtual social networks. Recent information technologies caused the digital transformation of human communications and, consequently, of the whole society. Unavoidably, challenges of the new information culture become the focus of intensive studies and a meeting point for researchers not only from physics, information engineering, and artificial intelligence, but also from biology, psychology, decision making, cognitive, social and political sciences, economics and finance. The aim of this note to review briefly the recent applications of the quantum information and probability theories outside of physics. This line of research is known as quantum-like modeling [1]–[44]. This is not consideration of genuine quantum physical processes in cognitive systems, but modeling behavior of generally macroscopic cognitive or AI agents processing information on the basis of quantum laws.

We want to present in more detail consequences of such information processing for rationality. In classical decision making, rational agents are mathematically modeled as probabilistic information processors using Bayesian update of probabilities: rational = Bayesian. Quantum state update is generally non-Bayesian [44]. We define quantum rationality as decision making that is based on quantum state update. Quantum and classical rational agents behave differently. For instance, a quantum(-like) agent can violate the Savage Sure Thing Principle [45] (see [6, 7, 9]) and the Aumann theorem [46] on impossibility of agreeing to disagree (see [23, 31, 9]).

In quantum-like modeling, the brain is treated as a black box which information processing cannot be described by classical probability [47] (CP) (cf. with geuine quantum physical models of brain's functioning [48]-[55]). And there is a plenty of such nonclassical statistical data - in cognitive psychology, game theory, decision making, social science, economics, finances, and politics. In decision theory, such data was coupled to probability fallacies and irrational behavior of agents. We propose to apply the most well developed non-classical theory of probability, namely, based on the mathematical formalism of quantum theory.

One may think that the appeal to quantum probability [56] (QP) and information to model decision making by humans is too exotic. However, we recall that as early as the 1970s, Tversky (one of the most cited psychologists of all time) and Kahneman (Nobel prize in economics in 2002, for prospect theory, which he co-developed with Tversky) have been demonstrating cases where CP-prescription and human behavior persistently diverge [57]–[61].

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Today, we are at the theoretical cross-roads, with huge divisions across conflicting, entrenched theoretical positions.

Should scientists continue use CP as the basis for descriptive and normative predictions in decision making and accept continuous generation of paradoxes?

Should we abandon probability theory completely and instead pursue explanations based on heuristics, as Tversky and Kahneman proposed?

The use of the probabilistic and statistical methods is really the cornerstone of the modern scientific methodology. Thus, although the heuristic approach to decision making cannot be discarded completely, it seems more natural to search novel probabilistic models for decision making. Our suggestion is to use QP and more generally quantum information, instead of heuristics of Tversky and Kahneman.

We stress that quantum-like modeling does not appeal to genuine quantum physical processes in biosystems. Quantum-like information processing can be performed by macroscopic systems, as cells, animals, or humans. Even artificial intelligence need not be based on quantum physical processors, as, e.g., quantum computers. Quantum-like modeling opens the door to creation of AI-systems processing information by respecting the laws of quantum theory, but equipped with classical physical processors. Some step in this direction was done within the recent studies on quantum(-like) information retrieval (see, e.g., [62]-[64]): algorithms based on the complex Hilbert state representation of information, but driven on classical computers, demonstrate superiority comparing with the traditional ("classical") algorithms. Of course, creation of successfully working genuine quantum computers and simulators would gives the possibility for creation of the genuine quantum AI-systems. However, since behavior of quantum-like and genuine quantum AI-systems is based on the same formalism and methodology, the theory of quantum-like cognition, rationality, and artificial intelligence would be useful even for quantum physical AI-systems.

For readers convenience, this paper contains basics of CP and QP. We present them in two steps. Section 2 is the informal comparative introduction to CP vs.QP; then sections 7, 9 present briefly, but mathematically rigorously the CP and QP formalisms (the later is presented as a part of the quantum formalism: quantum states and observables, the Born rule for probability of outcome of a measurement, the projection postulate and quantum state update). However, these formal presentations of CP and QP are used only in the last part of this work devoted to comparison of classical and quantum versions of the Aumann theorem. The preceding part devoted to classical vs. quantum rationality is written at the heuristic level.

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2 Brief comparison: Classical vs. quantum probability

CP was mathematically formalized by Kolmogorov (1933) [47] (see section 7 for details). This is the calculus of probability measures, where a non-negative weight $p(A)$ is assigned to any event A. The main property of CP is its additivity: if two events O_1, O_2 are disjoint, then the probability of disjunction of these events equals to the sum of probabilities:

$$P(O_1 \vee O_2) = P(O_1) + P(O_2).$$

In fact, powerful integration theory that is needed for calcualtion of averages demands σ-additivity:

$$P(\cup_j O_j) = \sum_j P(O_j), \tag{1}$$

where $O_j \cap O_i = \emptyset, i \neq j$.

QP is the calculus of complex amplitudes or in the abstract formalism complex vectors. Thus, instead of operations on probability measures one operates with vectors. We can say that QP is a *vector model of probabilistic reasoning*. Each complex amplitude ψ gives the probability by the Born's rule: *Probability is obtained as the square of the absolute value of the complex amplitude.*

$$p = |\psi|^2 \tag{2}$$

(for the Hilbert space formalization, see section 9, formula (17). By operating with complex probability amplitudes, instead of the direct operation with probabilities, one can violate the basic laws of CP.

In CP, the *law of total probability* (LTP) is derived by using additivity of probability and *the Bayes formula*, the definition of conditional probability,

$$P(O_2|O_1) = \frac{P(O_2 \cap O_1)}{P(O_1)}, \; P(O_1) > 0. \tag{3}$$

Consider the pair, A and B, of discrete classical random variables. Then

$$P(B = \beta) = \sum_\alpha P(A = \alpha) P(B = \beta | A = \alpha).$$

Thus, in CP the B-probability distribution can be calculated from the A-probability and the conditional probabilities $P(B = \beta | A = \alpha)$.

In QP [56], classical LTP is perturbed by the interference term [5,]; for dichotomous quantum observables A and B of the von Neumann-type, i.e., given by Hermitian operators $\hat{A}$ and $\hat{B}$, the quantum version of LTP has the form:

$$P(B = \beta) = \sum_\alpha P(A = \alpha) P(B = \beta | a = \alpha) \tag{4}$$

$$+2\sum_{\alpha_1<\alpha_2}\cos\theta_{\alpha_1\alpha_2}\sqrt{P(A=\alpha_1)P(B=\beta|A=\alpha_1)P(A=\alpha_2)P(B=\beta|a=\alpha_2)} \tag{5}$$

If *the interference term* is positive, then the QP-calculus would generate a probability that is larger than its CP-counterpart given by the classical LTP (2). In particular, this probability amplification is the basis of the quantum computing supremacy.

3 Classical (Bayesian) vs. quantum (generally non-Bayesian) rationality

In classical theory of decision making, rational behavior of agents is formalized with the *Savage Sure Thing Principle* (STP) [45]:

If you prefer prospect b_+ to prospect b_- if a possible future event A happens $(a=+1)$*; and you prefer prospect b_+ still if future event A does not happen* $(a=-1)$; *then you should prefer prospect b_+, despite having no knowledge of whether or not event A will happen.*

Savage's illustration refers to a person deciding whether or not to buy a certain property shortly before a presidential election, the outcome of which could radically affect the property market:

"Seeing that he would buy in either event, he decides that he should buy, even though he does not know which event will obtain".

STP is considered as the axiom of rationality of decision makers [45]. It plays the important role in decision making and economics in the framework of Savage's subjective utility theory. In the latter, probability is formalized in the classical probabilistic framework [47] and it is endowed with the subjective interpretation.

We remark that STP is a simple consequence of the law of total probability - LTP (see (5)). Violation of LTP implies violation of STP. Thus, the degree of satisfaction of LTP can be used as a statistical test of classical (STP-type) rationality.

In cognitive psychology, violation of STP is known as the *disjunction effect.* A plenty of statistical data was collected in cognitive psychology in experiments demonstrating disjunction effect. For example, in experiments of the Prisoners' Dilemma type [57]–[61]. Such data violate LTP. The latter implies irrationality (from classical viewpoint) of agents participating in experiments (mainly students).

We recall that LTP is derived from two assumptions that are firmly incorporated into the Kolmogorov axiomatics:

1. Additive law for probability.

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2. Baeys formula for conditional probability.

Therefore, violation of LTP and, hence, of STP (and classical rationality) is generated either by violation of additivity of probability or the Bayes law for conditional probability or by the combination of these factors. Generally, this leads to the impossibility to use in decision making Bayesian inference. Quantum(-like) agents proceed with more general inference machinery based on the quantum state update.

Hence, classical rationality is Bayesian inference rationality and quantum rationality is non-Bayesian inference rationality.[1] In the light of above considerations, one can ask:

Are quantum agents irrational?

As was discussed, by using QP it is possible to violate LTP and hence STP. Therefore, generally quantum-like agents are (classically) irrational. However, we can question the classical probabilistic approach to mathematical formalization of decision making and, consequently, the corresponding notion of rationality. We define quantum(-like) rationality as respecting the quantum calculus of probabilities and the quantum formula for interference of probabilities, LTP with the interference term (5).

4 Advantages, disadvantages, and roots of quantum rationality

4.1 Liberalization of decision making

Quantum rationality means more freedom for decision making, liberalization of this process. Generally liberalization has its advantages and disadvantages. The main advantage of quantum rationality is that such agents can come to essentially larger spectrum of possible decisions than classically rational agents. Some quantum decisions are classically unapproachable. One of the main disadvantage is that decisions - events of decisions - can belong to incompatible decision algebras (σ-algebras of events in Kolmogorov axiomatics of CP, see section 7). In such a case, it is impossible to come to consensus. We recall that in CP all possible decision events are unified in one common event-algebra. They can always be joined consistently with operations of conjunction, disjunction, and negation.

We remark that the CP-QP interplay is closely connected to the interplay of classical and quantum logic. Classical logic operates in a single Boolean

[1] Of course, non-Bayesian probability updates are not reduced to quantum, given by state transformations in the complex Hilbert space. One may expect that human decision making violates not only classical, but even quantum rationality.

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algebra. Quantum logic operates in the lattice of orthogonal projectors in complex Hilbert space. This lattice can be represented as union of partially overlapping Boolean algebras representing compatible events given by commuting projectors. We recall that measure-theoretic realization of Boolean algebras are precisely σ-algebras of CP (section 7). However, quantum logic is not just a collection of Boolean algeras. It is based on the consistent transition from one algebra to another - through unitary transformations. The latter can be treated as change of logic coordinates.

By solving a problem, a cognitive system selects one of Boolean subalgebras of quantum logic; typically as simple as possible. Why? We shall discuss this question below.

4.2 Dysfunctional disagreement and information overload

We point out that recently humans started to use widely the quantum decision making technique, especially in social and political decision making. In particular, this can explain the recent trend *in increasing dysfunctional disagreement in e.g. political debates and generally in social life of the modern society.* This disagreement cannot be explained by insufficient information supply. There is a lot of information. The problem is in opposite - in *information overload* [74]. The information flows generated by mass-media and internet are so powerful, that by making a decision on the concrete complex problem humans are not able to construct the joint algebra for all possible events related to the problem under consideration. Different issues involved in a problem are treated in different algebras. Moreover, these algebras can differ essentially depending on agent's social network and his information environment (or it would better to say "a part of the information environment that is selected by this agent").

In the situation when an agent is overloaded with variety of information and he or she should make rapidly the decision on a complex problem, the quantum information processing (without attempting to unify all received information within a single Boolean algebra) demonstrates its superiority over classical one, at least from the viewpoint of minimization of computational resources and speed up of decision making. Thus, in the modern society quantum rationality beats classical rationality, with "just one" casualty - inconsistency of some decisions. However, this is inconsistency w.r.t. classical Boolean logic. If all agents behave quantumly, then such inconsistencies become invisible.

The above discussion is applicable not only to biological systems, as say humans, but also to AI-systems. The latter when operating with powerful information flows would also prefer to use the quantum logic and QP. For each problem, a quantum(-like) AI-system, say robot, selects a proper

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Boolean algebra, but at the same time it keeps the possibility to use other Boolean algebras, corresponding to selection of different orthonormal bases in the Hilbert state space. However, transition from one Boolean algebra to another realized with a unitary operator U in the state space demands a lot of computational resources, because state space has big dimension; for n-qubit state space, it is 2^n. In situation of information overload and temporal constraints, an agent (biological or AI) has no possibility to perform such transition. Moreover, to get more or less complete image of the situation, an agent has to make transitions to a variety of Boolean algebras corresponding to incompatible variables which are represented by noncommuting Hermitian operators.

4.3 Social laser

One of the consequences of information overload is that information loses its content. A human has no possibility analyze deeply the content of communications delivered by mass-media and social networks. People process information without even attempting to construct an extended Boolean algebra of events. They operate with labels such as say covid-19, vaccination, pandemy without trying to go deeper beyond this labels. Contentless information behaves as a bosonic quantum field which is similar to the quantum electromagnetic field. Interaction of humans with such quantum information field can generate a variety of quantum-like behavioral effects. One of them is social lasing, stimulated amplification of social actions (SASA) [75]-[80]. In social laser theory, humans play the role of atoms, social atoms (s-atoms). Interaction of the information field composed of indistinguishable (up to some parameters, as say social energy) excitations with gain medium composed of s-atoms generate the cascade type process of emission of social actions. SASA describes well e.g. color revolutions and other types of mass protests (see [74] for detailed presentation).

5 Classical vs. quantum approach to the problem of agreement on disagree

Aumann's approach [46] to common knowledge and his theorem that rational agents cannot agree on disagree play the crucial role in theory of rationality. This theorem implies that rational agents acting under very natural conditions would never agreeing on disagree. Hence, it couples rationality with consistency of actions of decision makers. The main puzzle raised by the Aumann's theorem is that people often "agree on disagree".

How can we explain this contradiction between the statement which was matehmatically proved in the well-established framework of decision theory and the real behavior of humans?

The typical solution is that one of the two basic assumptions of the Aumann theorem is violated in the real processes of decision making. We recall the Aumann's assumptions:

1. Common knowledge.
2. Common priors.

However, in modern information society these assumptions are very natural. Society is homogeneous and the majority of people have common priors. Information is openly distributed via mass-media and internet. Of course, there are attempts to use insider information. But, such attempts, e.g., at the financial market, are subject for punishment. Violations of the Aumann theorem cannot be reduced to insiders' activity.

In my paper [23] (see also [31, 9]), it was pointed out that the Aumann theorem [46] is also based on the third, so to say hidden, assumption, namely, rationality of agents. As was discussed in section 3, classical notion of rationality is based on Bayesian inference in decision making which is CP-formalized in scientific theory. If real agents are not classically, but quantumly rational, they can violate cthe Aumann theorem, even under the assumptions of common knowledge and common priors.

The basic QP-departure from the classical Aumann's model is the existence of *incompatible information representations of the world* by different agents. To model this situation, instead of the set-theoretical (Boolean) partitions of the space of the states of the world Ω, we have to consider partitions of the unit operator in complex Hilbert space H (space of the quantum states of the world) consisting of the mutually orthogonal projectors. In general these partitions can be incompatible, i.e., the corresponding question-operators of different agents need not commute. Here we proceed with the simplest mathematical model of quantum measurements based on the projection operators, measurements of the von Neumann-Lüders type [65]. Generalization to measurements represented as quantum instruments [66]-[72] seems to be possible, but technically nontrivial.

In short, the main reason for this is that the basic quantum element introducing violations into Aumann's theorem is usage of a more general rule of updating of probabilities, the quantum analog of the Bayesian updating, see section 3 for the general discussion. A few different sources of incompatibility are combined in this rule. Besides the most evident source, namely, possible incompatibility of information representations of agents (decision makers), two other sources also play important roles. These are possible incompatibilities of information representations with quantum events and common prior states. They both can contribute non-trivially into the interference term perturbing the matching of posterior probabilities; even in

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contexts with common prior states and nontrivial common knowledge (see [31, 9] for details).

6 Common knowledge: illustrative examples

The notion of *common knowledge* plays the crucial role in various problems of coordination of actions and approaching conventions - in philosophy, economics (including accounting and capital market research), game theory, statistics, computer science, artificial intelligence. This notion is not simple and we shall devote this section to the informal discussion, irrelevant to the use of CP or QP. For illustration, we shall present the examples of the crucial impact of common knowledge in the problems of social coordination. The first one is based on author's personal experience and the second one is commonly used in literature on common knowledge.

We start with the remark that common knowledge is generalization of a simpler notion *mutual knowledge:* everybody in a group of people is aware about some fact or event. Now we present the first illustrative example for the notion of common knowledge:

Example 1. (Earthquake, Japan, 11 March 2011 [9]) At that time I was participating in Quantum Bio-information workshop at the Noda-city campus of Tokyo University of Science, the fourth floor of the conference building. Around 15.00 the building started to shake, strongly shake. [2] This shaking of the building and that this is a sign of a strong earthquake was the mutual knowledge for the workshop participants, everybody seen this. However, people did not try to escape from the building until somebody loudly said: "this is a very strong earthquake." Immediately we sprung outside the building. This announcement of the fact known to everybody in the conference room made the mutual knowledge common and this changed our behavior crucially. After this announcement each workshop's participant knew: "each participant knows that each participant knows that earthquake is very strong." and so on, ad infinitum.

However, this was not the end of this story. After a half-hour staying at some distance from the conference building and seeing that there were no more signs of earthquake, we decided to return to the conference room and continue workshop. In the middle of the session the building started to shake again and sufficiently strongly. However, nobody said publicly that shaking is strong. Session continued. (May be participants expected such an announcement from session's chairman who may be expected it from conference's organizers.) My collaborator who was the last speaker of this

[2] Comment: earthquakes nearby Tokyo happen often; in some periods things in room shake practically everyday. Amplitudes vary day to day; of course each time I estimated their strength. But this is difficult to do subjectively...

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session told that she was really scared during her talk and very angry when people started to ask questions after she finished.

Thus common knowledge is an essentially stronger assumption than simply mutual knowledge. To have *common knowledge means not only that everybody knows some information* E, *but even that everybody knows that everybody knows* E *and that everybody knows that everybody knows that everybody knows* E *and so on, ad infinitum.*

Remark. (Ad infinitum) The definition of common knowledge is based on the *infinite* hierarchy of levels of knowing. This presence of infinity might make the impression that this notion is not useful for concrete applications in which the infinite level of commonality is inapproachable. However, this is not the case. Of course, common knowledge is an ideal notion, but its role in science is similar to the role of other ideal notions, such as, e.g., a point, straight line, irrational number. We cannot proceed mathematically without such ideal notions. This is the exhibition of the transcendental structure of human reasoning.

Example 2. (Blue eyes paradox [73]). There is an island populated by people with blue and green eyes, say k people have blue eyes, others have green eyes. For island's inhabitants, the number k is unknown. At the beginning nobody knows the color of her/his eyes. There is the very strict rule:

If a person finds that she/he has blue eyes, that person must move from the island before sunrise.

At this island everybody knows eye colors of others. But, it is forbidden to discuss eye colors and there are no mirrors or similar devices. Once a stranger comes to the island and announced to all the people:

"At least one of you has blue eyes".

Thus it became common knowledge. The problem: what is the eventual outcome of this public announcement?

Consider first the simplest case, $k = 1$. Some person will recognize that she/he alone has blue eyes (by seeing only green eyes in the others) and leave at the first sunrise. Let now $k = 2$. At the first sunrise nobody leaves the island. Then two people having blue eyes by seeing only one person with blue eyes, and that no one left on the first sunrise understand that $k > 1$. They leave at the second sunrise. And so on by using the inductive argument. The paradox is that if $k > 1$ then the stranger told to people at this island what they already have known: there are blue-eyed people. However, without stranger's announcement this fact was not common knowledge. Its becoming common knowledge had dramatic consequences for inhabitants of the island.

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For k = 2, it is first-order knowledge. Each person having blue eyes knows that there is someone with blue eyes, but she/he does not know that the other blue-eyed person has this same knowledge. For k = 3, it is second order knowledge. After 2 days, each person having blue eyes knows that a second blue-eyed person knows that a third person has blue eyes, but no one knows that there is a third blue-eyed person with that knowledge, until the third day arrives. And so on...

7 Kolmogorov's model of classical probability theory

Since CP-formalization of common knowledge is performed in the measure-theoretic framework, it is useful to recall the Kolmogorov's model [47] of CP.

The *Kolmogorov probability space* [47] is any triple

$$(\Omega, \mathcal{F}, P),$$

where Ω is a set of any origin and $\mathcal{F}$ is a σ-algebra of its subsets, P is a probability measure on $\mathcal{F}$. The set Ω represents random parameters of the model. Kolmogorov called elements of Ω *elementary events*. This terminology is standard in mathematical literature. Sets of elementary events are regarded as *events*.

The essence of Kolmogorov's approach is that not any subset of Ω can be treated as an event. For any stochastic model, the system of events $\mathcal{F}$ is selected from the very beginning. The main mathematical point is that $\mathcal{F}$ has to be a *σ-algebra.*

We remind that a σ-algebra is a system of sets which contains Ω and empty set, it is closed with respect to the operations of countable union and intersection and to the operation of taking the complement of a set. For example, the collection of all subsets of Ω is a σ-algebra. This σ-algebra is used in the case of finite or countable sets:

$$\Omega = \{\omega_1, ..., \omega_n, ...\}. \tag{6}$$

However, for "continuous sets", e.g., $\Omega = [a, b] \subset \mathbf{R}$, the collection of all possible subsets is too large to have applications. Typically it is impossible to describe a σ-algebra in the direct terms. To define a σ-algebra, one starts with a simple system of subsets of Ω and then consider the σ-algebra which is generated from this simple system with the aid of aforementioned operations. In particular, one of the most important for applications σ-algebras, the so called *Borel σ-algebra*, is constructed in this way by staring with the system consisting of all open and closed intervals of the real

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line. In a metric space (in particular, in a Hilbert space), the Borel σ-algebra is constructed by starting with the system of all open and closed balls.

Finally, we remark that in American literature the term σ-field is typically used, instead of σ-algebra.

The probability is defined as a *measure*, i.e., a map from $\mathcal{F}$ to nonnegative real numbers which is σ-additive:

$$P(\cup_j A_j) = \sum_j P(A_j), \tag{7}$$

where $A_j \in \mathcal{F}$ and $A_i \cap A_j = \emptyset, i \neq j$. The probability measure is always normalized by one:

$$P(\Omega) = 1. \tag{8}$$

In the case of a discrete probability space, see (6), the probability measures have the form

$$P(A) = \sum_{\omega_j \in A} p_j, \ p_j = P(\{\omega_j\}).$$

In fact, any finite measure μ, i.e, $\mu(\Omega) < \infty$, can be transformed into the probability measure by normalization:

$$P(A) = \frac{\mu(A)}{\mu(\Omega)}, A \in \mathcal{F}. \tag{9}$$

A (real) random variable is a map $\xi : \Omega \to \mathbf{R}$ which is measurable with respect to the Borel σ-algebra $\mathcal{B}$ of $\mathbf{R}$ and the σ-algebra $\mathcal{F}$ of Ω. The latter means that, for any set $B \in \mathcal{B}$, its preimage $\xi^{-1}(B) = \{\omega \in \Omega : \xi(\omega) \in B\}$ belongs to $\mathcal{F}$. This condition provides the possibility to assign the probability to the events of the type "values of ξ belong to a (Borel) subset of the real line." The probability distribution of ξ is defined as

$$P_\xi(B) = P(\xi^{-1}(B)). \tag{10}$$

In the same way we define the real (and complex) vector valued random variables, $\xi : \Omega \to \mathbf{R}^n$ and $\xi : \Omega \to \mathbf{C}^n$.

Let $\xi_1, ..., \xi_k$ be real-valued random variables. Their join probability distribution $P_{\xi_1,...,\xi_k}$ is defined as the probability distribution of the vector-valued random variable $\xi = (\xi_1, ..., \xi_k)$. To determine this probability measure, it is sufficient to define probabilities

$$P_{\xi_1,...,\xi_k}(\Gamma_1 \times ... \times \Gamma_k) = P(\omega \in \Omega : \xi_1(\omega) \in \Gamma_1, ..., \xi_k(\omega) \in \Gamma_k)$$

where $\Gamma_j, j = 1, ..., k$, are intervals (open, closed, half-open) of the real line.

We remark once again that LTP (5) is a theorem within Kolmogorov probability theory [47]. We also recall that LTP plays the basic role in Bayesian probability inference.

8 Classical formalization of common knowledge and Aumann theorem

8.1 States of the world

Aumann's considerations are applicable to a finite number of *agents*, call them $i = 1, 2, ..., N$. These individuals are about to learn the answers to various multi-choice *questions*, to make observations.

Mathematically the situation is represented with the aid of classical probability space (based on the Kolmogorov axiomatics, 1933). Typically it is assumed that the state space Ω representing all possible states of the world is finite.

8.2 Agents' information representations

Each agent creates its *information representation* for possible states of the world based on its own possibilities to perform measurements, "to ask questions to the world." Mathematically these representations are given by partitions of $\Omega : \mathcal{P}^{(i)} = (P_j^{(i)})$, where, for each agent i,

$$\cup_j P_j^{(i)} = \Omega \text{ and } P_j^{(i)} \cap P_k^{(i)} = \emptyset, j \neq k.$$

Thus an agent cannot get to know the state of the world ω precisely; she/he can only get to know to which element of its information partition $P_j^{(i)} \equiv P^{(i)}(\omega)$ this ω belongs. In this set-theoretic model of knowledge, by definition the agent i knows an event E in the state of the world ω if the element of his information partition containing this ω is contained in E :

$$P^{(i)}(\omega) \subset E. \tag{11}$$

In logical terms, this can be written as $P^{(i)}(\omega) \Rightarrow E$, the event $P^{(i)}(\omega)$ implies the event E; we also remark that $\{\omega\} \Rightarrow P^{(i)}(\omega)$.

8.3 Common prior

It is assumed that on Ω there is defined a probability p, *the common prior* of all agents. In the accordance with the measure-theoretic model of probability theory (Kolmogorov, 1933) there is given a σ-algebra, say $\mathcal{F}$, of subsets of Ω, its elements represent events ("propositions" in some interpretations), and there is given a probability measure p defined on $\mathcal{F}$. In the knowledge models it is typically assumed that $\mathcal{F}$ is generated by agents' partitions, i.e., this is the minimal σ-algebra containing all systems of set $\mathcal{P}^{(i)}, i = 1, ..., N$. It is important to point out that, in particular, such a σ-algebra contains all

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subsets of the form $P^{(1)}_{j_1} \cap ... \cap P^{(N)}_{j_N}$. Hence, in the classical knowledge model the prior probability is assigned not only to the individual elements of agents' information representations, i.e., $P^{(i)}_j \to p(P^{(i)}_j)$, but even to more complex events

$$P^{(1)}_{j_1} \cap ... \cap P^{(N)}_{j_N} \to p_{j_1...j_N} \equiv p(P^{(1)}_{j_1} \cap ... \cap P^{(N)}_{j_N}). \tag{12}$$

Thus by agreeing on the prior the agents have to agree on numerous conjunctive probabilities.

8.4 CP-formalization of the notion of common knowledge

We consider the systems of sets $\tilde{\mathcal{P}}^{(i)} = \{\cup_m P^{(i)}_{j_m}\}$ consisting of finite unions of the elements of the systems $\mathcal{P}^{(i)}$ and the system $\tilde{\mathcal{P}} = \cap_i \tilde{\mathcal{P}}^{(i)}$. A set O belongs to the system $\tilde{\mathcal{P}}$ if it belongs to any $\tilde{\mathcal{P}}^{(i)}$. Thus, for each i, it can be represented as

$$O = \cup_m P^{(i)}_{j_m}, \tag{13}$$

for some finite set of indexes (depending on i).

We now repeat the definition of common knowledge for two agents (and we continue to proceed with two agents):

ACN *An event E is common knowledge at the state of the world ω if 1 knows E, 2 knows E, 1 knows 2 knows E, 2 knows 1 knows E, and so on.*

In theory of common knowledge the basic role is played by the set of all states of the world for which E is common knowledge; it is denoted by the symbol κE. As was shown by Aumann [46], this set of states of the world belongs to $\tilde{\mathcal{P}}$ and, hence, for each i, it can be represented (in the case $\kappa E \neq \emptyset$) in the form (see (13)):

$$\kappa E = \cup_m P^{(i)}_{j_m}. \tag{14}$$

Let E be an event. For a state of the world ω, each agent i updates the common prior $p(E)$ on the basis of the observation the element $P^{(i)}(\omega)$ of its information partition. (For this agent, it means that the state of the world ω is contained in $P^{(i)}(\omega)$.) This update is given by the conditional probability

$$\mathbf{q}_i(\omega) = p(E \cap P^{(i)}(\omega))/p(P^{(i)}(\omega)).$$

We remark that the conditional probability $\mathbf{q}_i(\omega)$ is defined to be the same for all states of the world ω in a given element of partition. Thus, in fact,

$$\mathbf{q}_i(\omega) \equiv \mathbf{q}_{ik},$$

where $\omega \in P^{(i)}_k = P^{(i)}(\omega)$.

8.5 Aumann theorem

Now, Aumann's theorem states that if both

$$\mathbf{q}_1(\omega) = q_1 \text{ and } \mathbf{q}_2(\omega) = q_2 \tag{15}$$

are common knowledge and prior probabilities are the same, then necessarily $q_1 = q_2$ - simply because

$$q_i = p(E|\kappa C_{q_1 q_2}) = p(E \cap \kappa C_{q_1 q_2})/p(\kappa C_{q_1 q_2}), \tag{16}$$

where $C_{q_1 q_2}$ is the event (15): " the first agent by updating the prior probability of the event E assigns the value q_1 and the second agent the value q_2."

9 Basics of quantum formalism

9.1 States

Denote by H a complex Hilbert space. For simplicity, we assume that it is finite dimensional. Pure states of a system S are given by normalized vectors of H and mixed states by density operators (positive semi-definite operators with unit trace).

9.2 Observables, Born rule for probability, and projection postulate

In the original quantum formalism [65], physical observable A is represented by a Hermitian operator $\hat{A}$. We consider only operators with discrete spectra: $\hat{A} = \sum_x x\, \hat{E}^A(x)$, where $\hat{E}^A(x)$ is the projector onto the subspace of H corresponding to the eigenvalue x. Suppose that system's state is mathematically represented by a density operator ρ. Then the probability to get the answer x is given by *the Born rule*:

$$\Pr\{A = x \| \rho\} = Tr[\hat{E}^A(x)\rho] = Tr[\hat{E}^A(x)\rho\hat{E}^A(x)] \tag{17}$$

and according to the projection postulate the post-measurement state is obtained via the state-transformation:

$$\rho \to \rho_x = \frac{\hat{E}^A(x)\rho\hat{E}^A(x)}{Tr\hat{E}^A(x)\rho\hat{E}^A(x)}. \tag{18}$$

For reader's convenience, we present these formulas for a pure initial state $\psi \in H$. The Born's rule has the form:

$$\Pr\{A = x \| \rho\} = \|\hat{E}^A(x)\psi\|^2 = \langle\psi|\hat{E}^A(x)\psi\rangle. \tag{19}$$

The state transformation is given by the projection postulate:

$$\psi \to \psi_x = \hat{E}^A(x)\psi/\|\hat{E}^A(x)\psi\|. \tag{20}$$

Here the observable-operator $\hat{A}$ (its spectral decomposition) uniquely determines the feedback state transformations $\mathcal{I}_A(x)$ for outcomes x

$$\rho \to \mathcal{I}_A(x)\rho = \hat{E}^A(x)\rho\hat{E}^A(x). \tag{21}$$

The map $x \to \mathcal{I}_A(x)$ given by (21) is the simplest (but very important) example of quantum instrument [66]–[70].

9.3 Quantum logic

Following von Neumann [65] and Birkhoff and von Neumann [81] we represent *events, propositions,* as orthogonal projectors in complex Hilbert space H.

For an orthogonal projector P, we set $H_P = P(H)$, its image, and vice versa, for subspace L of H, the corresponding orthogonal projector is denoted by the symbol P_L.

The set of orthogonal projectors is a *lattice* with the order structure: $P \leq Q$ iff $H_P \subset H_Q$ or equivalently, for any $\Psi \in H$, $\langle\Psi|P\Psi\rangle \leq \langle\Psi|Q\Psi\rangle$. This lattice is known as *quantum logic.* Thus in classical Boolean logic events are represented by sets and in quantum logic events are represented by orthogonal projectors.

We recall that the lattice of projectors is endowed with operations "and" ($\wedge$), conjunction, and "or" ($\vee$), disjunction. For two projectors P_1, P_2, the projector $R = P_1 \wedge P_2$ is defined as the projector onto the subspace $H_R = H_{P_1} \cap H_{P_2}$ and the projector $S = P_1 \vee P_2$ is defined as the projector onto the subspace H_R defined as the minimal linear subspace containing the set-theoretic union $H_{P_1} \cup H_{P_2}$ of subspaces H_{P_1}, H_{P_2} : this is the space of all linear combinations of vectors belonging these subspaces. The operation of negation is defined as the orthogonal complement:

$$P^\perp = \{y \in H : \langle y|x\rangle = 0 : \text{for} \quad \text{all } x \in H_P\}.$$

In the language of subspaces the operation "and", conjunction, coincides with the usual set-theoretic intersection, but the operations "or", disjunction, and "not", negation, are nontrivial deformations of the corresponding set-theoretic operations. It is natural to expect that such deformations can induce deviations from classical Boolean logic.

Consider the following simple example. Let H be two dimensional Hilbert space with the orthonormal basis (e_1, e_2) and let $v = (e_1 + e_2)/\sqrt{2}$. Then $P_v \wedge P_{e_1} = 0$ and $P_v \wedge P_{e_2} = 0$, but $P_v \wedge (P_{e_1} \vee P_{e_2}) = P_v$. Hence, for quantum events, in general *the distributivity law is violated*:

$$P \wedge (P_1 \vee P_2) \neq (P \wedge P_1) \vee (P \wedge P_2) \tag{22}$$

As can be seen from our example, even mutual orthogonality of the events P_1 and P_2 does not help to save the Boolean laws.

Thus quantum logic relaxes some constraints set by classical Boolean logic, in particular, the distributivity between the operations of conjunction and disjunction.

10 Quantum formalization of common knowledge and Aumann theorem with interference term

10.1 Quantum states of the world

In our quantum-like model the "*states of the world*" are given by pure states. Thus the unit sphere $S_1(H)$ in a complex Hilbert space H represents (up to phase factors) all possible states of the world.

10.2 Agents' quantum information representations

Questions posed by agents are mathematically described by Hermitian operators, say $A^{(i)}$. We state again that events (propositions) are identified with orthogonal projectors. For the state of the world Ψ, an event P *occurs* (takes place with probability 1) if Ψ belongs to H_P.

To simplify considerations, we proceed in the case of the finite dimensional state space of the world, $m = \dim \mathrm{H} < \infty$. Each Hermitian operator can be represented as a linear combination of orthogonal projectors to its eigen-subspaces; the questions of agents can be expressed as

$$A^{(i)} = \sum_j a_j^{(i)} P_j^{(i)}, \tag{23}$$

where $(a_j^{(i)})$ are real numbers, all different eigenvalues of $A^{(i)}$, and $(P_j^{(i)})$ are the orthogonal projectors onto the corresponding eigen-subspaces. Here (a_j) encode possible answers to the question of the ith agent.[3] The system of projectors $\mathcal{P}^{(i)} = (P_j^{(i)})$ is the spectral family of $A^{(i)}$. Hence, for any agent i, it is a "disjoint partition of unity":

$$\bigvee_k P_k^{(i)} = I, \; P_k^{(i)} \wedge P_m^{(i)} = 0, k \neq m. \tag{24}$$

[3]Although in quantum physics the magnitudes of these numbers play an important role, in quantum information theory the eigenvalues are merely formal labels encoding information which can be extracted from a state with the aid of an observable. In the case of dichotomous answers, we can simply use zero to encode "no" and one to encode "yes".

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We remark that (24) is simply the lattice-theoretical expression of the following operator equalities:

$$\sum_k P_k^{(i)} = I,\ P_k^{(i)} P_m^{(i)} = 0, k \neq m. \tag{25}$$

This spectral family can be considered as *information representation* of the world by the ith agent. In particular, "getting the answer $a_j^{(i)}$" is the event which is mathematically described by the projector $P_j^{(i)}$.

If the state of the world is represented by Ψ and, for some k_0, $P_\Psi \leq P_{k_0}^{(i)}$, then

$$p_\Psi(P_{k_0}^{(i)}) = \mathrm{Tr} P_\Psi P_{k_0}^{(i)} = 1 \text{ and, for } k \neq k_0,\ p_\Psi(P_k^{(i)}) = \mathrm{Tr} P_\Psi P_k^{(i)} = 0.$$

Thus, in this case, the event $P_{k_0}^{(i)}$ happens with the probability one and other events from information representation of the world by the ith agent have zero probability.

However, opposite to the classical case, in general Ψ need not belong to any concrete subspace[4] $H_{P_k^{(i)}}$. Nevertheless, for any pure state Ψ, there exists the minimal projector $Q_\Psi^{(i)}$ of the form $\sum_m P_{j_m}^{(i)}$ such that $P_\Psi \leq Q_\Psi^{(i)}$. This projector can be constructed in the following way. Each state Ψ determines the set of indexes

$$O_\Psi^{(i)} = \{j : P_j^{(i)} \Psi \neq 0\}. \tag{26}$$

Then the minimal projector majorating the one dimensional projector corresponding to the state Ψ has the form:

$$Q_\Psi^{(i)} = \sum_{j \in O_\Psi^{(i)}} P_j^{(i)}. \tag{27}$$

The projector $Q_\Psi^{(i)}$ represents the ith agent's knowledge about the Ψ-world. We remark that $p_\Psi(Q_\Psi^{(i)}) = 1$.

Consider the system of projectors $\tilde{\mathcal{P}}^{(i)}$ consisting of sums of the projectors from $\mathcal{P}^{(i)}$:

$$\tilde{\mathcal{P}}^{(i)} = \{P = \sum_m P_{j_m}^{(i)}\}. \tag{28}$$

[4]We state again that in the classical probability model the states of the world are encoded by points of Ω. Take one fixed state ω. Since information representation of each agent is a partition of Ω, for each i there exists an element of partition, say $P_j^{(i)}$, containing this ω. For this state of the world, the ith agent should definitely get the answer $a_j^{(i)}$ corresponding the element $P_j^{(i)}$. Thus any agent is able to resolve uncertainty at least for her/his information representation (although she/he is not able to completely resolve uncertainty about the state of the world). In the quantum case an agent is not able to resolve uncertainty even at the level of her/his information representation. And the prior probability is updated in this uncertainty context.

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Then

$$Q_\Psi^{(i)} = \min\{P \in \tilde{\mathcal{P}}^{(i)} : P_\Psi \leq P\}, \tag{29}$$

see (26), (27) for the constructive definition.

Thus in general, for the ith agent, the picture of the world is not only fuzzy (i.e., based on her partition of unity), but also probabilistic corresponding to the set of probabilities, $(p_\Psi(P_k^{(i)}), k = 1, 2, ...)$.

The ith agent picture of the world can be represented by the density operator (mixed quantum state)

$$\rho^{(i)} = \sum_k P_k^{(i)} P_\Psi P_k^{(i)} = \sum_k p_\Psi(P_k^{(i)}) \rho_k^{(i)},$$

where $\rho_k^{(i)} = \frac{P_k^{(i)} P_\Psi P_k^{(i)}}{\mathrm{Tr} \mathrm{P}_\mathrm{k}^{(\mathrm{i})} \mathrm{P}_\Psi \mathrm{P}_\mathrm{k}^{(\mathrm{i})}}$. Since each $\rho_k^{(i)}$ is a pure state, the ith picture of the world is given by the mixture of pure states, corresponding to "cells" $P_k^{(i)}$.

10.3 Quantum way of common knowledge formalization

Definition 1. *For the Ψ-state of the world and the event E, the ith agent knows E if*

$$Q_\Psi^{(i)} \leq E. \tag{30}$$

It is evident that if, for the state of the world Ψ, the ith agent knows E, then $\Psi \in H_E$. In general the latter does not imply that E is known (for the state Ψ).[5] However, if $\Psi \in E = P_j^{(i)}$, then this event is known for i. The same is valid for any event of the form $E = P_{j_1}^{(i)} \vee ... \vee P_{j_k}^{(i)} (= P_{j_1}^{(i)} + ... + P_{j_k}^{(i)})$; if $\Psi \in H_E$, then such E is known for i.

We remark that the straightforward analog of the classical definition, see (11), would be based on condition $P_j^{(i)} \leq E$ for

$$P_\Psi \leq P_j^{(i)}, \tag{31}$$

instead of more general condition (30). However, it would trivialize the class of possible states of the world, because condition (31) is very restrictive.

[5]For example, the state space H is four dimensional with the orthonormal basis (e_1, e_2, e_3, e_4), the projectors P_1 and P_2 project H onto the subspaces with the bases (e_1, e_2) and (e_3, e_4), respectively. Here (P_1, P_2) is information representation of an agent. Let E be the projector onto the subspace with the basis (e_1, e_4) and let $\Psi = (e_1 + e_4)/\sqrt{2}$. Then $Q_\Psi = I$, the unit operator. Hence, E is not known for this agent, although it belongs to H_E.

We shall use the standard definition of common knowledge, see **ACN**, but based on the quantum representation of knowing an event, see Definition 1. As in the classical case, we have that "Where something is common knowledge, everybody knows it."

We recall that in the classical case, for each event E, there is considered the set of all states of the world for which E is common knowledge. It is denoted by the symbol κE.

This definition is naturally generalized to the quantum case. Here κE is defined as the projector on the subspace consisting of all states of the world for which E is common knowledge.

Similar to the set-theoretic framework, we introduce the system of projectors

$$\tilde{\mathcal{P}} = \cap_i \tilde{\mathcal{P}}^{(i)}.$$

We remark that (by definition) a projector $P \in \tilde{\mathcal{P}}$ if and only if, for each $i = 1, ..., N$, it can be represented in the form

$$P = \sum_m P^{(i)}_{j_m}. \tag{32}$$

Examples illustrating how this common knowledge structure works can be found in [9] .

Now, before to formulate the quantum version of the Aumann's theorem, we recall that while in the classical Aumann scheme the update of the prior probability distribution on the basis of information representations of agents plays the crucial role. The quantum analog of the Aumann scheme is based on the quantum procedure of the state update as the result of measurement, in the simplest case this is the projection postulate based update.

10.4 Quantum version of Aumann's theorem

The quantum common prior assumption is formulated naturally: both agents assign to possible states of the world the same quantum probability distribution given by the density operator ρ, a priori state. The agents do not know exactly the real state of the world which is always a pure state and in general a possible state of the world appears for them as a mixed quantum state ρ.

Theorem 1. [23] *Under assumptions of common prior and common knowledge, the following interference version of Aumann's theorem holds:*

$$q_i - q_s = \frac{1}{\mathrm{Tr}\rho\kappa \mathrm{C}_{\mathrm{q_1\ldots q_N}}} \Big(\sum_{j \neq m} \mathrm{Tr} \mathrm{P}^{(\mathrm{i})}_{\mathrm{k_j}} \rho \mathrm{P}^{(\mathrm{i})}_{\mathrm{k_m}} \mathrm{E} - \sum_{\mathrm{j \neq m}} \mathrm{Tr} \mathrm{P}^{(\mathrm{s})}_{\mathrm{k_j}} \rho \mathrm{P}^{(\mathrm{s})}_{\mathrm{k_m}} \mathrm{E} \Big). \tag{33}$$

If the amplitude of right-hand side of (33), the interference term between updates of the probability of the event E by two different agents, is small, we can say that the agents named i and s practically agree. The interference term can be considered as a measure of "agreement on disagree" between the agents.

11 Concluding discussion

As was confirmed by plenty of experimental statistical data from cognitive psychology, decision making, social and political sciences, generally human agents make their decisions by violating CP-laws (see introduction). It is natural to look for a more general formalism to model cognition and decision making. The QP-formalism serves as the basis for such extension of CP-based approach. In this paper we discussed the consequences of the quantum-like model for rationality. Quantum rationality, i.e., based on quantum probability inference, differs crucially from classical Bayesian rationality. It has its advantages and disadvantages; some of them were discussed in this paper. The main reason for the use of quantum(-like) information processing is information overload, especially in the situation demanding quick decision making under temporal and computation resource restrictions. Since the modern information society is characterized by intensification of information flows generated by mass-media, social networks, and working process, we can expect increasing use of quantum information processing by human agents and, hence, the phenomenon of quantum rationality will become more common. From the classical rationality viewpoint, this kind of rationality can be viewed as irrationality. However, it seems the only choice in the modern society.

The main danger of quantum rational behavior is that such agents become a very good medium for social engineering; in particular, a good active medium for social lasing. The latter can be used to generate instability throughout the world, in the form of mass-protests and color revolutions.

The quantum-like approach to modeling of cognition, decision making, and rationality will definitely find applications in artificial intelligence, for two types of AI-systems:

- Systems equipped with genuine quantum information processing devices, say quantum computers or simulators.
- Systems equipped with classical information processing devices, say classical digital or analog computers, realizing quantum(-like) information processing.

Personally I do not share the generally high expectation for successful realization of genuine quantum physical computing project, especially hopes that such quantum devices can be useful for AI-systems, say robots. I think that quantum information processing based on classical computational devices has better perspectives. But, since in science it is always difficult to

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make prognoses for future development, both types of AI-systems, genuine quantum and quantum-like, have to be studied. In future, the output of this paper may become useful for modeling rationality in collectives composed of quantum and quantum-like robots and other AI-systems.

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