Geometry-Induced Rigidity in Elastic Torus from Circular to Oblique Elliptic Cross-Section

Bo-Hua Sun¹

¹*Institute of Mechanics and Technology & School of Civil Engineering,
Xi'an University of Architecture and Technology, Xi'an 710055, China
<http://imt.xauat.edu.cn>
email: sunbohua@xauat.edu.cn*

For a given material, different shapes of a built structure will be corresponding to different rigidity. In this paper, nonlinear displacement formulation is formulated and numerical simulations will be carried out for circular, normal elliptic, and oblique elliptic torus. Investigations show that both deformation and stress response of an elastic torus are sensitive to the radius ratio, and suggest that the analysis of a torus should be done by using the bending theory of a shell, and also reveal that the inner torus is stronger than the outer torus due to the property of their Gaussian curvature. The desired rigidity can be archived by adjusting the ratio of the minor and main radius and oblique angle.

Keywords: elliptic torus, oblique, nonlinear deformation, vibration, Gauss curvature, Maple

I. INTRODUCTION

We have a common sense, it is easier to crack an egg on its side than at its tip as shown in Fig.1. This rigidity is an intrinsic mechanics feature of curved shells, which is called geometry-induced rigidity. Many natural and man-made objects take the shape of shells to gain better rigidity. Both Lazarus et al. [1] Vella et al. [2] studied egg-like shell's deformation and geometry-induced rigidity of non-spherical pressurized elastic shells(ellipsoidal and cylindrical). Human foot evolution also takes advantage of curvature to increase its stiffness, Venkadesan et al. [3] took a theoretical approach to investigate the role of transverse curvature in stiffening the human foot. They considered the human foot as an elastic cantilever cylindrical shell and demonstrated that, if the transverse curvature of the shell increased, this increased the stiffness of the shell in the longitudinal direction.

FIG. 1: (a) Egg under side force (b) Egg under tip force.

All above-mentioned researches are about the shells that has either positive or zero Gaussian curvature, however, there is another kind of shell, who has a mixed

Gaussian curvature, where the torus geometry has both positive, negative Gaussian curvature and turning points (shown in Fig. 2).

FIG. 2: The sign of Gaussian curvature.

Doughnut and tube might be two of the most popular ones (shown in Fig. 3).

FIG. 3: (a) Doughnut (b) Tube.

Among regular shells, such as circular cylindrical shells, conical shells, spherical shells, and tori, the deformation of the torus is one of the most difficulty topics due to its complicated topology. This paper will focus on the geometry-induced rigidity of elastic torus and demonstrate the rigidity change from circular torus to oblique one (shown in Fig. 4).

FIG. 4: Geometry of torus and rigidity transformation.

The torus has been studied for more than 110 years (Chang (Zhang) 1949 [4], Qian and Liang 1979 [5], Xia and Zhang 1986 [6], Zhang, Ren and Sun 1990 [7], Zhang and Zhang 1991 [8], 1994 [9], Audoly and Pomeau 2002 [10], 2010 [11], Sun 2010 [12] 2012 [13], Clark 1950 [14, 15], Dahl 1953 [16], Novozhilov 1959 [17], Timoshenko and Woinowsky-Krieger 1959 [18], Flügge 1973 [19], Gol'denveizer 1961 [20] Sun 2013 [21], Föppl 1907 [22] Weihs 1911 [23], Wissler 1916 [24], Kuznetsov and Levyakov 2001 [25], Zingoni, Enoma and Govender 2015 [26], Jiammeepreecha and Chucheeepsakul 2017 [27], Enoma and Zingoni 2020 [28], H. Reissner 1912 [29], Meissner 1915 [30], E. Reissner 1949 [31], Tao 1959 [32], Steele 1965 [33], Sun 2018 [34], Sun 2020 [35]). The most of publications have been focused on circular torus, and only small percentage on non-circular cross-section torus [25–28], in which, Zingoni, Enoma and Govender 2015 [26] considered a semi-elliptic toroid and tried to get an approximate bending solution that may be valid in regions adjacent to the horizontal equatorial plane.

When linear problem of the torus was first studied, high-order and complicated governing equations of a torus under symmetric loads were reduced to lower-order, ordinary differential equation (ODE) by Hans Reissner (1912)[29] when he was a professor at ETH in Switzerland. His colleague at ETH, Meissner (1915) [30] derived a complex-form equations for the shell of revolution. Hence, the first complex-form equation of the shells of revolution including torus is called the Reissner-Meissner equation, which is a ODE system about shear force Q and rotation χ (Flügge 1973 [19]). In 1959, Novozhilov published his celebrated monograph on the complex-form theory of shells, and formulated symmetrical deformation of a torus in his complex-form (Novozhilov 1959 [17]).

Regarding formulation of elastic torus, governing equations that all publications adopted are either Reissner-Meissner's mixed formulation or Novozhilov's complex-form one. Although those formulations have some advantages, however they can not be used for vibration

analysis and nonlinear problem. Therefore, it is desired to have nonlinear displacement formulations for elastic torus. Sun 2010 [12] who derived linear displacement-type equations for torus and obtained a closed-form solution for special case as radius ratio $\frac{a}{R} \rightarrow 0$. This special solution can not capture full mechanics nature of torus and therefore are needed to be improved further. This study wishes to carry on my previous works (Sun 2010 [12]), and investigates mechanical response for arbitrary ratio $\frac{a}{R}$.

The content is organised as follows: Section 2 derives nonlinear displacement formulation of torus. Section 3 carries out numerical investigations on circular, normal and oblique elliptic torus. Section 4 demonstrates geometry-induced rigidity of different torus. Section 5 provides conclusions.

II. NONLINEAR DISPLACEMENT FORMULATION OF ELASTIC ELLIPTIC TORUS

For the torus shown in Fig. 11, the positions of points on the middle surface will be determined by the angles θ and φ . Further, let r_1 be the radius of curvature of the meridian and r_2 the radius of curvature of the normal section, tangential to the parallel circle. This second radius is equal to the segment of the perpendicular to the middle surface between this surface and the axis of the torus.

FIG. 5: Geometry of torus.

The principal radii of curvature of the oblique elliptic torus in Fig.5 are given by

$$r_1 = \frac{1}{\cos \beta} \frac{a^2 b^2}{(a^2 \sin^2 \phi + b^2 \cos^2 \phi)^{3/2}}, \quad (1)$$

$$r_2 = \frac{R}{\sin \phi} + \frac{1}{\cos \beta} \frac{a^2}{(a^2 \sin^2 \phi + b^2 \cos^2 \phi)^{1/2}},$$

where β is oblique angle. The Lame coefficients are $A_1 = r_1$, $A_2 = r_2 \sin \phi$.

From the above relations, we can see that the oblique angle will strongly influence on the mechanics of elliptic torus via affecting the main principal radii r_2 . The above radii will be reduced to normal elliptic torus when $\beta = 0$.

FIG. 6: Loading, forces and moments of torus

Regarding the forces shown in Fig.6, the balance equations [39] are

$$\begin{aligned} \frac{d}{d\phi}(rN_\phi) - N_\theta r_1 \cos \phi + rQ_\phi - rN_\phi \vartheta + r_1 r f_\phi &= 0, \\ \frac{d}{d\phi}(rQ_\phi) - r_1 r \left(\frac{N_\phi}{r_1} + \frac{N_\theta}{r_2} \right) - \frac{d}{d\phi}(rN_\phi \vartheta) + r_1 r f_\zeta &= 0, \end{aligned} \quad (2)$$

where $r = r_2 \sin \phi$, distributed loads f_ϕ , f_ζ along ϕ , ζ direction, and shear force

$$Q_\phi = \frac{1}{r_1 r} \frac{d}{d\phi}(rM_\phi) - \frac{1}{r} \cos \phi M_\theta, \quad (3)$$

where the resultant membranae forces are $N_\phi = K(\varepsilon_\phi + \nu \varepsilon_\theta)$, $N_\theta = K(\nu \varepsilon_\phi + \varepsilon_\theta)$, and resultant bending moments are $M_\phi = B(\kappa_\phi + \nu \kappa_\theta)$, $M_\theta = B(\nu \kappa_\phi + \kappa_\theta)$.

The membranae strains are

$$\begin{aligned} \varepsilon_\phi &= \frac{1}{r_1} \left(\frac{du}{d\phi} + w \right) + \frac{1}{2} \vartheta^2, \\ \varepsilon_\theta &= \frac{1}{r_2} (u \cot \phi + w), \end{aligned} \quad (4)$$

and change of curvature are

$$\begin{aligned} \kappa_\phi &= \frac{1}{r_1} \frac{d\vartheta}{d\phi} + \frac{1}{r_1^2} \vartheta^2, \\ \kappa_\theta &= \frac{\cot \phi}{r_2} \vartheta, \end{aligned} \quad (5)$$

where total rotation $\vartheta = \frac{1}{r_1} (u - \frac{dw}{d\phi})$; and membranae stiffness $K = \frac{Eh}{1-\nu^2}$, bending stiffness $B = \frac{Eh^3}{12(1-\nu^2)}$, thickness h , Young modulus E and Poisson's ratio ν .

With the above strains and curvature change, the resultant membranae force and bending moments can be expressed in terms of displacements u , w as follows:

$$\begin{aligned} N_\phi &= K \left[\frac{1}{r_1} \left(\frac{du}{d\phi} + w \right) + \frac{1}{2} \vartheta^2 + \nu \frac{1}{r_2} (u \cot \phi + w) \right], \\ N_\theta &= K \left[\frac{1}{r_2} (u \cot \phi + w) + \nu \frac{1}{r_1} \left(\frac{du}{d\phi} + w \right) + \frac{1}{2} \vartheta^2 \right], \\ M_\phi &= B \left[\frac{1}{r_1} \frac{d\vartheta}{d\phi} + \frac{1}{r_1^2} \vartheta^2 + \nu \frac{\cot \phi}{r_2} \vartheta \right], \\ M_\theta &= B \left[\nu \left[\frac{1}{r_1} \frac{d\vartheta}{d\phi} + \frac{1}{r_1^2} \vartheta^2 \right] + \frac{\cot \phi}{r_2} \vartheta \right]. \end{aligned} \quad (6)$$

Substituting the shear force Q_ϕ in Eq.3 into Eq.2 and produce

$$\begin{aligned} \frac{d}{d\phi}(rN_\phi) - N_\theta r_1 \cos \phi + \frac{1}{r_1} \left[\frac{d}{d\phi}(rM_\phi) - r \cos \phi M_\theta \right] - rN_\phi \vartheta + r_1 r f_\phi &= 0, \\ \frac{d}{d\phi} \left[\frac{1}{r} \frac{d}{d\phi}(rM_\phi) - \cos \phi M_\theta \right] - r_1 r \left(\frac{N_\phi}{r_1} + \frac{N_\theta}{r_2} \right) - \frac{d}{d\phi}(rN_\phi \vartheta) + r_1 r f_\zeta &= 0. \end{aligned} \quad (7)$$

The Eq. 7 can be further simplified by substituting the constitutive relations into Eq.7, which will generate a final equations that is a six order nonlinear ordinary differential equation system about displacement $u(\phi)$ and $w(\phi)$. The above ODE system will be more complicated if taking into account of the principal radii. We are not going to display the ODE system here due to its complicated format. Obviously the ODE system has no analytical solution but numerical ones.

For simplification of presentation of our results, physical units will not be plotted in all drawings, all physical units are listed in Table I:

TABLE I: Physical units used in this paper

R	a, b	R, h	E	ν	M_ϕ	N_ϕ	Q_ϕ	σ_ϕ	u	w	δ_z, Δ_z
m	m	m	N/m^2	1	Nm	N/m	N/m	N/m^2	m	m	m

III. NUMERICAL CASE STUDIES OF ELASTIC TORUS

To solve the Eq. 7 numerically, a general code for this case is written by Maple [40]. With the help of the code, we have simulate three different torus: circular, normal and oblique elliptic torus.

A. An circular torus

FIG. 7: Torus with a cut along its parallel at $\phi = \frac{\pi}{2}$ or $\phi = -\frac{\pi}{2}$ under load F . The boundary condition is: $\phi = \frac{\pi}{2}$: $N_\phi = -\frac{F}{2\pi(R+a)}$, $Q_\phi = 0$, $M_\phi = 0$ and $\phi = -\frac{3\pi}{2}$: $N_\phi = -\frac{F}{2\pi(R+a)}$, $Q_\phi = 0$, $M_\phi = 0$.

The numerical results are shown in Fig.8.

FIG. 8: Circular torus with data: $a = b = 2k[m]$, $R = 10[m]$, $h = a/20[m]$, $\beta = k\pi/8$, $E = 2.0 \times 10^{11} N/m^2$, $\nu = 0.3$, $F = 1[N]$, where $k = 1, 2$

B. A normal elliptic torus

If there were no oblique angle, namely $\beta = 0$, all formulations will be reduced to normal elliptic torus. For a problem as shown in Fig.9.

FIG. 9: Normal torus with a cut along its parallel at $\phi = \frac{\pi}{2}$ or $\phi = -\frac{\pi}{2}$ under load F . The boundary conditions: $\phi = \frac{\pi}{2}$: $N_\phi = -\frac{F \cos \beta}{2\pi(R+\frac{a}{\cos \beta})}$, $Q_\phi = -\frac{F \sin \beta}{2\pi(R+\frac{a}{\cos \beta})}$, $M_\phi = 0$ and $\phi = -\frac{3\pi}{2}$: $N_\phi = -\frac{F \cos \beta}{2\pi(R+\frac{a}{\cos \beta})}$, $Q_\phi = \frac{F \sin \beta}{2\pi(R+\frac{a}{\cos \beta})}$, $M_\phi = 0$.

With the help of the Maple code, some numerical results have been obtained and are shown in Fig. 10.

C. An oblique elliptic torus

For a complete torus with a penetrate cut along the parallel $\theta = \frac{\pi}{2}$ and loaded with vertical force F , the loading condition is shown in Fig. 11.

With the help of the Maple code, some numerical results have been obtained and are shown in Fig. 12.

IV. GEOMETRY-INDUCED RIGIDITY IN TORUS FROM CIRCULAR TO OBLIQUE ELLIPTIC CROSS-SECTION

To compare the rigidity of different torus, we have plotted the vertical opening or relative displacement for circular torus, normal and oblique elliptic torus. From recasts shown in Fig.13, we can see the rigidity can be achieved by adjusting the ratio of a/b and oblique angle β .

Investigations show that both deformation and stress response of an elastic torus are sensitive to the radius ratio, and suggest that the analysis of a torus should be done by using the bending theory of a shell. The results indicate that all quantities such as bending moments, surface forces, shear force, and displacement are strongly effected by oblique angle.

From vertical displacement of above-mentioned results, we can see that the vertical displacement is mainly contributed by outer region of torus, and displacement flat is contributed by inner region of torus, which reveals that the inner torus is stronger than outer torus due to the property of their Gaussian curvature.

FIG. 10: Deformation caused vertical force. Normal elliptic torus with data: $a = 6k[m]$, $b = 15$, $R = 30[m]$, $h = a/15[m]$, $E = 2.0 \times 10^{11} N/m^2$, $\nu = 0.3$, $F = 1[N]$.

FIG. 11: Torus with a cut along its parallel at $\phi = \frac{\pi}{2}$ or $\phi = -\frac{\pi}{2}$ under load F . The boundary conditions: $\phi = \frac{\pi}{2}$: $N_\phi = -\frac{F \cos \beta}{2\pi(R + \frac{a}{\cos \beta})}$, $Q_\phi = -\frac{F \sin \beta}{2\pi(R + \frac{a}{\cos \beta})}$, $M_\phi = 0$ and $\phi = -\frac{3\pi}{2}$: $N_\phi = -\frac{F \cos \beta}{2\pi(R + \frac{a}{\cos \beta})}$, $Q_\phi = \frac{F \sin \beta}{2\pi(R + \frac{a}{\cos \beta})}$, $M_\phi = 0$.

V. CONCLUSIONS

As conclusion, we have formulated elastic oblique elliptic torus in terms of displacement and successfully solved some typical problems, free vibration of the torus has

FIG. 12: Oblique elliptic torus with data: $a = 1.2b[m]$, $b = 2k[m]$, $R = 10[m]$, $h = a/20[m]$, $\beta = k\pi/8$, $E = 2.0 \times 10^{11} N/m^2$, $\nu = 0.3$, $F = 1[N]$, where $k = 1, 2$

been studied. To verify our formulation, we wrote a computational code in Maple and carried out some numerical simulations. The validation of our numerical results was confirmed and supported by both finite element analysis and Hans Reissner's formulation. Our investigations show that both deformation and stress response of an elastic torus are sensitive to the radius ratio and dramatically vary with the meridian angle, therefore the analysis of a torus should be done by the bending theory of a shell rather than membrane theory of shell. Our numerical studies also reveal that the inner torus is stronger than outer torus due to the property of their Gaussian curvature. Concerning the free vibration of torus, our analysis indicates that both initial in u and w direction must be included otherwise will cause big errors if initial in u di-

FIG. 13: The opening displacement at cut. The results are case of $k=1$: circular torus ($a = 2[m]$), normal elliptic torus ($a = 1.2b[m]$, $b = 2[m]$) and oblique elliptic torus ($a = 1.2b[m]$, $b = 2[m]$, $\beta = \pi/8$).

rection were omitted. Last but not least, the oblique angle has a strong influence on the mechanics of elliptic torus, because the oblique angle will affect the main principal radii.

Acknowledgement:

The author is honored to have benefits from my supervisor Prof. Dr.Ing Wei Zhang (Wei Chang), who was the first Chinese scholar studied the deformation of torus, it is my privilege to dedicate this paper to the memories of Prof. Zhang for his great contribution to the analysis of elastic torus. The author appreciates my students: Mr. Jie Wei, for preparation of Fig.5 (a,b) and Fig.6.

Data availability: The data that support the findings of this study are available from the corresponding author upon reasonable request.

- [1] A. Lazarus, H. C. B. Florijn, and P. M. Reis, Geometry-induced rigidity in nonspherical pressurized elastic shells. *Phys Rev Lett.* 109, 144301 (2012).
- [2] D. Vella, A. Ajdari, A. Vaziri, and A. Boudaoud, Indentation of ellipsoidal and cylindrical elastic shells. *Phys Rev Lett.* 109, 144302 (2012).
- [3] M. Venkadesan, et al. Stiffness of the human foot and evolution of the transverse arch. *Nature* 579, 97-100 (2020).
- [4] W. Chang (W. Zhang), *Druckspannungszustand in kreisringschale und ähnlichen Schalen mit Scheitelkreisringen unter drehsymmetrischer Belastung. Arbeit zur Erlangung des Grades eines Doctor-Ingenieurs der Technischen Hochschule, Berlin, 1944.* (Sci. Rep. Nat. Tsinghua Univ., Ser A. 259-349 (1949))
- [5] W.Z. Qian and S.C. Liang, Complex form equation and asymptotic solution. *J. of Tsinghua University*, 19(1979)(1) 27-47,
- [6] Z.H. Xia and W. Zhang, The general solution for thin-walled curved tubes with arbitrary loadings and various boundary conditions. *Int. J. Pressures and Piping* 26(1986) 129-144.
- [7] W. Zhang, W.M. Ren and B. Sun, Toroidal Shells - history, current situation and future. Fifth Conf. of Space Structures, Lanzhou, China 1990.
- [8] R.J. Zhang and W. Zhang, Turning point solution for thin toroidal shell vibrations. *Int. J. Solids Structures* 27(1991)(10) 1311-1326.
- [9] R.J. Zhang and W. Zhang, Toroidal shells under nonsymmetric loading. *Int. J. Solids Structures*, 31(1994) 2735-2750.
- [10] Audoly, B. and Pomeau, Y., 2002. The elastic torus: anomalous stiffness of shells with mixed type, *C. R. Mecanique* 330, 425-432.
- [11] B. Audoly and Y. Pomeau, *Elasticity and Geometry*. University of Cambridge, Cambridge, 2010
- [12] B. Sun, Closed-form solution of axisymmetric slender elastic toroidal shells. *J. of Engineering Mechanics*, 136 (2010) 1281-1288.
- [13] B. Sun, *Toroidal Shells*. (Nova Novinka, New York, 2012)
- [14] R.A. Clark, R.A. and E. Reissner, Bending of curved tubes, *Advances in Applied Mechanics*, vol.II, Academic Press (1950)
- [15] R.A. Clark, On the theory of thin elastic toroidal shells. *J. Mech. Phys. Solids*, 29(1950) 146-178.
- [16] N.C. Dahl, Toroidal-shell expansion joints. *J. of Applied Mechanics*, ASME, 20(1953) 497-503.
- [17] V.V. Novozhilov, *The Theory of Thin Shell*. (Noordhoff, Groningen, 1959).
- [18] S. Timoshenko and S. Woinowsky-Krieger, *Theory of Plates and Shells*. McGraw-Hill, New York (1959).
- [19] W. Flügge, *Stresses in Shells*. Springer-Verlag Berlin (1973).
- [20] A.L. Gol'denveizer, *Theory of Elastic thin Shells*. Pergamon Press, New York (1961).
- [21] B. Sun, Centenary studies of toroidal shells and in memory of Prof. Zhang Wei. *Mechanics in Engineering*, 37(2013). (In Chinese)
- [22] L. Föppl, *Vorlesungen Über Technische Mechanik*. volume 5.B.G. Teubner, Leipzig, Germany (1907).
- [23] G. Weihs. *Über Spannungs- und Formänderungszustände in dünnen Hohlreifen*. Halle a. S. 1911.
- [24] H. Wissler, *Festigkeiberechnung von Ringsflächen*. Promotionarbeit, Zurich (1916). <https://doi.org/10.3929/ethz-a-000099037>
- [25] V.V. Kuznetsov, S.V. Levyakov, Nonlinear pure bending of toroidal shells of arbitrary cross-section. *Int. J. of Solids and Structures*, 38(2001) 7343-7354.
- [26] A. Zingoni, N. Enoma and N. Govender, Equatorial bending of an elliptic toroidal shell. *Thin-Walled Structures*, 96(2015) 286-294.
- [27] W. Jiammeepreecha and S. Chucheeprasakul, Nonlinear static analysis of an underwater elastic semi-toroidal shell.

- l. Thin-Walled Structures, 116(2017) 12-18.
- [28] N. Enoma and A. Zingoni, Analytical formulation and numerical modelling for multi-shell toroidal pressure vessels. Computers and Structures, 232(2020), Article 105811.
- [29] H. Reissner, Spannungen in Kugelschalen (Kuppeln). Festschrift Heinrich Müller-Breslau (A. Kröner, Leipzig, 1912), 181-193
- [30] E. Meissner, Über und Elastizität Festigkeit dünner Schalen. Viertelschr. D. natur.Ges., Bd.60, Zurich (1915).
- [31] E. Reissner, On bending of curved thin-walled tubes. Proc. National Academy of Sci., (1949) 204-208.
- [32] L.N. Tao, On toroidal shells. J. of Math. and Physics, 38,(1959) 130-134.
- [33] C.R. Steele, Toroidal pressure vessels. J. Spacecr. Rocket., 2(1965) 937-943,
- [34] B. Sun, Exact solution of Qian's equation of slender toroidal shells. Mechanics in Engineering, 38(2018)(5) 567-569.(in Chinese)
- [35] B. Sun. On symmetrical deformation of toroidal shell with circular cross-section. Preprints 2020, 2020030156 (doi: 10.20944/preprints202003.0156.v2).
- [36] A. Ronveaux, Heun's Differential Equations. Oxford University Press (1995).
- [37] https://en.wikipedia.org/wiki/Heun_function
- [38] J.E. Marsden, L. Sirovich and S.S. Antman eds. Hypergeometric Functions and Their Applications. Texts in Applied Mathematics. 56. New York: Springer-Verlag (1991).
- [39] J.N. Reddy, Theory and Analysis of Elastic Plates and Shells. CRC Press, 2007.
- [40] Maple <https://www.maplesoft.com/>
- [41] A.W. Leissa, Vibration of Shells. Acoustical Society of America, 1993.