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INVARIANT PSEUDOPARALLEL SUBMANIFOLDS OF AN ALMOST α-COSYMPLECTIC (κ, μ, ν)-SPACE

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ABSTRACT. In this article, the geometry of pseudoparallel, Ricci-generalized pseudoparallel and 2-Ricci-generalized pseudoparallel invariant submanifolds of an almost α-cosymplectic (κ, μ, ν) space has been searched under the some conditions. We also give some characterizations for such submanifolds. I think that obtained new results contribute to differential geometry.

1. INTRODUCTION

An almost contact manifold is odd-dimensional manifold $\widetilde{M}^{2n+1}$ which carries a field ϕ of endomorphism of the tangent space, a vector field ξ, called characteristic, and a 1-form η-satisfying

$$\phi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \tag{1}$$

where I denote the identity mapping of tangent space of each point at M. From (1), it follows

$$\phi\xi = 0, \quad \eta \circ \phi = 0 \quad rank(\phi) = 2n. \tag{2}$$

An almost contact manifold $\widetilde{M}^{2n+1}(\phi, \xi, \eta)$ is said to be normal if the tensor field $N = [\phi, \phi] + 2d\eta \otimes \xi = 0$, where $[\phi, \phi]$ denote the Nijenhuis tensor field of ϕ. It is well known that any almost contact manifold $\widetilde{M}^{2n+1}(\phi, \xi, \eta)$ has a Riemannian metric such that

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{3}$$

for any vector fields X, Y on $\widetilde{M}$[5]. Such metric g is called compatible metric and manifold $\widetilde{M}^{2n+1}$ together with the structure (ϕ, η, ξ, g) is called an almost contact metric manifold and denoted by $\widetilde{M}^{2n+1}(\phi, \eta, \xi, g)$. The 2-form Φ of $\widetilde{M}^{2n+1}(\phi, \eta, \xi, g)$ is defined $\Phi(X, Y) = g(\phi X, Y)$ is called the fundamental form of $\widetilde{M}^{2n+1}(\phi, \eta, \xi, g)$. If an almost contact metric

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manifold such that η and Φ are closed, that is, $d\eta = d\Phi = 0$, then it called cosymplectic manifold[6].

An almost α-cosymplectic manifold for any real number α which is defined as

$$d\eta = 0, \quad d\Phi = 2\alpha\eta \wedge \Phi. \tag{4}$$

A normal almost α-cosymplectic manifold is said to be α-cosymplectic manifold[4].

It is well known that on a contact metric manifold $\widetilde{M}^{2n+1}(\phi, \xi, \eta, g)$, the tensor h, defined by $2h = L_\xi \phi$, the following equalities satisfies;

$$\widetilde{\nabla}_X \xi = -\phi X - \phi h X, \quad h\phi + \phi h = 0, \quad trh = tr\phi h = 0, h\xi = 0, \tag{5}$$

where $\widetilde{\nabla}$ is the Levi-Civita connection on $\widetilde{M}^{2n+1}$[3].

In [4], the authors studied the almost α-cosymplectic (κ, μ, ν)-spaces under different conditions and gave an example in dimension 3.

Going beyond generalized (κ, μ)-spaces, in [2], the notation of (κ, μ, ν)-contact metric manifold was introduced as follows;

$$\widetilde{R}(X, Y)\xi = \eta(Y)[\kappa I + \mu h + \nu\phi h]X - \eta(X)[\kappa I + \mu h + \nu\phi h]Y, \tag{6}$$

for some smooth functions κ, μ and ν on $\widetilde{M}^{2n+1}$, where $\widetilde{R}$ denotes the Riemannian curvature tensor of $\widetilde{M}^{2n+1}$ and X, Y are vector fields on $\widetilde{M}^{2n+1}$.

They proved that this type of manifold is intrinsically related to the harmonicity of the Reeb vector on contact metric 3-manifolds. Some authors have studied manifolds satisfying condition (6) but a non-contact metric structure. In this connection, P. Dacko and Z. Olszak defined an almost cosymplectic (κ, μ, ν)-spaces as an almost cosymplectic manifold that satisfies (6), but with κ, μ and ν functions varying exclusively in the direction of ξ in[6]. Later examples have been given for this type manifold[7].

Pseudoparallel submanifolds have been studied in different structures and working on[8, 9, 10]. In the present paper, we generalize the ambient space and research cases of existence or non-existence pseudoparallel submanifold in α-cosymplectic (κ, μ, ν)-space.

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Proposition 1.1. *Given $\widetilde{M}^{2n+1}(\phi,\xi,\eta,g)$ an almost α-cosymplectic (κ,μ,ν)-space, then*

$$
\begin{aligned}
&(7) & h^2 &= (\kappa+\alpha^2)\phi^2,\\
&(8) & \xi(\kappa) &= 2(\kappa+\alpha^2)(\nu-2\alpha)\\
&(9) & \widetilde{R}(\xi,X)Y &= \kappa[g(X,Y)\xi-\eta(Y)X]+\mu[g(hX,Y)\xi-\eta(Y)hX]\\
&(10) & &+ \nu[g(\phi hX,Y)\xi-\eta(Y)\phi hX]\\
&(11) & (\widetilde{\nabla}_X\phi)Y &= g(\alpha\phi X+hX,Y)\xi-\eta(Y)(\alpha\phi X+hX)\\
&(12) & \widetilde{\nabla}_X\xi &= -\alpha\phi^2X-\phi hX,
\end{aligned}
$$

for all vector fields X,Y on $\widetilde{M}^{2n+1}$[5].

Now, let M be an immersed submanifold of an almost α-cosymplectic (κ,μ,ν)-space $\widetilde{M}^{2n+1}$. By $\Gamma(TM)$ and $\Gamma(T^{\perp}M)$, we denote the tangent and normal subspaces of M in $\widetilde{M}$. Then the Gauss and Weingarten formulae are, respectively, given by

$$\widetilde{\nabla}_XY=\nabla_XY+\sigma(X,Y), \tag{13}$$

and

$$\widetilde{\nabla}_XV=-A_VX+\nabla^{\perp}_XV, \tag{14}$$

for all $X,Y\in\Gamma(TM)$ and $V\in\Gamma(T^{\perp}M)$, where ∇ and $\nabla^{\perp}$ are the induced connections on M and $\Gamma(T^{\perp}M)$ and σ and A are called the second fundamental form and shape operator of M, respectively, $\Gamma(TM)$ denote the set differentiable vector fields on M. They are related by

$$g(A_VX,Y)=g(\sigma(X,Y),V). \tag{15}$$

The covariant derivative of σ is defined by

$$(\widetilde{\nabla}_X\sigma)(Y,Z)=\nabla^{\perp}_X\sigma(Y,Z)-\sigma(\nabla_XY,Z)-\sigma(Y,\nabla_XZ), \tag{16}$$

for all $X,Y,Z\in\Gamma(TM)$. If $\widetilde{\nabla}\sigma=0$, then submanifold is said to be its second fundamental form is parallel.

By R, we denote the Riemannian curvature tensor of the submanifold M, we have the following Gauss equation

$$
\begin{aligned}
\widetilde{R}(X,Y)Z &= R(X,Y)Z+A_{\sigma(X,Z)}Y-A_{\sigma(Y,Z)}X+(\widetilde{\nabla}_X\sigma)(Y,Z)\\
&- (\widetilde{\nabla}_Y\sigma)(X,Z), \qquad (17)
\end{aligned}
$$

for all $X,Y,Z\in\Gamma(TM)$.

For a $(0,k)$-type tensor field T, $k \geq 1$ and a $(0,2)$-type tensor field A on a Riemannian manifold (M,g), $Q(A,T)$-tensor field is defined by

$$\begin{aligned} Q(A,T)(X_1,X_2,...,X_k;X,Y) &= -T((X\wedge_A Y)X_1,X_2,...,X_k)... \\ &- T(X_1,X_2,...X_{k-1},(X\wedge_A Y)X_k),, \end{aligned} \tag{18}$$

for all $X_1,X_2,...,X_k,X,Y \in \Gamma(TM)$[8], where

$$(X\wedge_A Y)Z = A(Y,Z)X - A(X,Z)Y. \tag{19}$$

Definition 1.2. *A submanifold of a Riemannian manifold (M,g) is said to be pseudoparallel, 2-pseudoparallel, Ricci-generalized pseudoparallel and 2-Ricci-generalized pseudoparallel if*

$$\begin{aligned} \widetilde{R}\cdot\sigma \quad &\textit{and} \quad Q(g,\sigma) \\ \widetilde{R}\cdot\widetilde{\nabla}\sigma \quad &\textit{and} \quad Q(g,\widetilde{\nabla}\sigma) \\ \widetilde{R}\cdot\sigma \quad &\textit{and} \quad Q(S,\sigma) \\ \widetilde{R}\cdot\widetilde{\nabla}\sigma \quad &\textit{and} \quad Q(S,\widetilde{\nabla}\sigma) \end{aligned}$$

are linearly dependent, respectively[10].

Equivalently, this cases can be explained by the following way;

$$\widetilde{R}\cdot\sigma = L_1 Q(g,\sigma), \tag{20}$$

$$\widetilde{R}\cdot\widetilde{\nabla}\sigma = L_2 Q(g,\widetilde{\nabla}\sigma), \tag{21}$$

$$\widetilde{R}\cdot\sigma = L_3 Q(S,\sigma), \tag{22}$$

$$\widetilde{R}\cdot\widetilde{\nabla}\sigma = L_4 Q(S,\widetilde{\nabla}\sigma), \tag{23}$$

where the functions L_1, L_2, L_3 and L_4 are, respectively, defined on $M_1 = \{x \in M : \sigma(x) \neq g(x)\}$, $M_2 = \{x \in M : \widetilde{\nabla}\sigma(x) \neq g(x)\}$, $M_3 = \{x \in M : S(x) \neq \sigma(x)\}$ and $M_4 = \{x \in M : S(x) \neq \widetilde{\nabla}\sigma(x)\}$ and S denote the Ricci tensor of M.

Particularly, if $L_1 = 0$(resp. $L_2 = 0$), the submanifold is said to be semiparallel(resp. 2-semiparallel)[9].

2. Invariant Submanifolds of an almost α-cosymplectic (κ,μ,ν) Space

Now, let $\widetilde{M}^{2n+1}(\phi,\xi,\eta,g)$ be an almost α cosymplectic (κ,μ,ν)-space and M an immersed submanifold of $\widetilde{M}^{2n+1}$. If $\phi(T_xM) \subseteq T_xM$, for each point at $x \in M$, then M is said to be an invariant submanifold of $\widetilde{M}^{2n+1}(\phi,\xi,\eta,g)$ with respect to ϕ. After we will easily to see that an

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invariant submanifold with respect to ϕ is also invariant with respect to h.

Proposition 2.1. *Let M be an invariant submanifold of an almost α-cosymplectic (κ,μ,ν)-space $\widetilde{M}^{2n+1}(\phi,\xi,\eta,g)$ such that ξ tangent to M. Then the following equalities hold on M;*

$$\begin{aligned} R(X,Y)\xi &= \kappa[\eta(Y)X-\eta(X)Y]+\mu[\eta(Y)hX-\eta(X)hY] \\ &+ \nu[\eta(Y)\phi hX-\eta(X)\phi hY] \end{aligned} \tag{24}$$

$$(\nabla_X\phi)Y = g(\alpha\phi X+hX,Y)\xi-\eta(Y)(\alpha\phi X+hX) \tag{25}$$

$$\nabla_X\xi = -\alpha\phi^2X-\phi hX \tag{26}$$

$$\phi\sigma(X,Y) = \sigma(\phi X,Y)=\sigma(X,\phi Y), \quad \sigma(X,\xi)=0, \tag{27}$$

where ∇, σ and R denote the induced Levi-Civita connection on M, the shape operator and Riemannian curvature tensor of M, respectively.

Proof. We will not give the proof as it is a result of direct calculations. □

In the rest of this paper, we will assume that M is an invariant submanifold of an α-cosymplectic (κ,μ,ν)-space $\widetilde{M}^{2n+1}(\varphi,\xi,\eta,g)$. In this case, from (5), we have

$$\varphi hX=-h\varphi X, \tag{28}$$

for all $X\in\Gamma(TM)$, that is, M is also invariant with respect to the tensor field h.

We need the following theorem to quarante for the second fundamental form σ is not always identically zero.

Theorem 2.2. *Let M be an invariant submanifold of an almost α-cosymplectic (κ,μ,ν)-space $M^{2n+1}(\phi,\xi,\eta,g)$. Then the second fundamental form σ of M is parallel M is totally geodesic provided $\kappa\neq 0$.*

Proof. Let us suppose that σ is parallel. From (16), we have

$$(\widetilde{\nabla}_X\sigma)(Y,Z)=\nabla^{\perp}_X\sigma(Y,Z)-\sigma(\nabla_XY,Z)-\sigma(Y,\nabla_XZ)=0, \tag{29}$$

for all vector fields X,Y and Z on M^{2n+1}. Setting $Z=\xi$ in (29) and taking into account (26) and (27), we have

$$\sigma(\nabla_X\xi,Y)=\sigma(\alpha\phi^2X+\phi hX,Y)=0,$$

that is,

$$-\alpha\sigma(X,Y)+\phi\sigma(hX,Y)=0. \tag{30}$$

Writing hX of X in (30) and by using (7) and (27), we obtain

$$\begin{aligned} -\alpha\sigma(hX,Y) &+ \phi\sigma(h^2X,Y)=0,\\ \alpha\sigma(hX,Y) &- (\alpha^2+\kappa)\phi\sigma(X,Y)=0. \end{aligned} \tag{31}$$

From (30) and (31), we conclude that $\kappa\sigma(X,Y)=0$, which proves our assertion. □

Theorem 2.3. *Let M be an invariant pseudoparallel submanifold of an almost α cosymplectic (κ,μ,ν)-space $M^{2n+1}(\phi,\xi,\eta,g)$. Then M is either totally geodesic submanifold or the function L_1 satisfies $L_1 = \kappa \mp \sqrt{(\nu^2-\mu^2)(\kappa+\alpha^2)}$, $\mu\nu(\kappa+\alpha^2)=0$.*

Proof. We suppose that M is an invariant pseudoparallel submanifold of an almost α-cosymplectic $M^{2n+1}(\phi,\xi,\eta,g)$-space. Then there exists a function L_1 on M such that

$$(R(X,Y)\cdot\sigma)(U,V)=L_1Q(g,\sigma)(U,V;X,Y),$$

for all vector fields X,Y,U,V on M. By means of (18) and (20), we have

$$\begin{aligned} R^{\perp}(X,Y)\sigma(U,V) &- \sigma(R(X,Y)U,V)-\sigma(U,R(X,Y)V)\\ &= -L_1\{\sigma((X\wedge_g Y)U,V)+\sigma(U,(X\wedge_g Y)V)\}. \end{aligned} \tag{32}$$

Here taking $Y=U=\xi$ in (32) and taking account Proposition 2.1, we obtain

$$\begin{aligned} R^{\perp}(X,\xi)\sigma(\xi,V) &- \sigma(R(X,\xi)\xi,V)-\sigma(\xi,R(X,\xi)V)\\ &= -L_1\{\sigma((X\wedge_g \xi)\xi,V)+\sigma(\xi,(X\wedge_g \xi)V)\}\\ &= -L_1\{\sigma(X-\eta(X)\xi,V)+\sigma(\xi,\eta(V)X-g(X,V)\xi)\}, \end{aligned}$$

that is,

$$\sigma(R(X,\xi)\xi,V)=L_1\sigma(X,V). \tag{33}$$

By means of Proposition 2.1 and (6), we conclude that

$$(L_1-\kappa)\sigma(X,V)=\mu\sigma(hX,V)+\nu\sigma(\phi hX,V). \tag{34}$$

If hX is substituted for X at (34) and making use of (7) and (27), we obtain

$$(L-\kappa)\sigma(hX,V)=-(\kappa+\alpha^2)[\mu\sigma(X,V)+\nu\phi\sigma(X,V)]. \tag{35}$$

From (34) and (35), we reach at

$$[(L_1-\kappa)^2+(\kappa+\alpha^2)(\mu^2-\nu^2)]\sigma(X,V)=-2\mu\nu(\kappa+\alpha^2)\phi\sigma(X,V).$$

This yields to

$$(L_1-\kappa)^2+(\kappa+\alpha^2)(\mu^2-\nu^2)=0, \mu\nu(\kappa+\alpha^2)=0 \quad or \quad \sigma=0.$$

This completes of the proof. □

From Theorem 2.3, we have following Corollary.

Corollary 2.4. *Let M be an invariant submanifold of an almost α-cosymplectic (κ,μ,ν)-space $M^{2n+1}(\phi,\xi,\eta,g)$. Then M is semiparallel if and only if M is totally geodesic.*

Theorem 2.5. *Let M be an invariant submanifold of an almost α-cosymplectic (κ,μ,ν)-space $M^{2n+1}(\phi,\xi,\eta,g)$. If M is a 2-pseudoparallel submanifold, then M is either totally geodesic or the functions α,κ,μ,ν and L_2 satisfy $L_2=\kappa\mp\sqrt{(\kappa+\alpha^2)(\nu^2-\nu^2)}$ and $\mu\nu(\kappa+\alpha^2)=0$.*

Proof. Let us suppose that M is an 2-pseudoparallel submanifold of (κ,μ,ν)-space $M^{2n+1}(\phi,\xi,\eta,g)$. Then by means of (21), there exists a function L_2 such that

$$(\widetilde{R}(X,Y)\cdot\widetilde{\nabla}\sigma)(U,V,Z)=L_2Q(g,\widetilde{\nabla}\sigma)(U,V,Z;X,Y),$$

for all vector fields X,Y,Z,U,V on M. This implies that

$$\begin{aligned} R^{\perp}(X,Y)(\nabla_U\sigma)(V,Z) &- (\widetilde{\nabla}_{R(X,Y)U}\sigma)(V,Z)-(\widetilde{\nabla}_U\sigma)(R(X,Y)V,Z)\\ &- (\widetilde{\nabla}_U\sigma)(V,R(X,Y)Z)=-L_2\{(\widetilde{\nabla}_{(X\wedge_g Y)U}\sigma)(V,Z)\\ &+ (\widetilde{\nabla}_U\sigma)((X\wedge_g Y)V,Z)+(\widetilde{\nabla}_U\sigma)(V,(X\wedge_g Y)Z)\}. \end{aligned} \tag{36}$$

Taking $X=Z=\xi$ in (36), we can infer

$$\begin{aligned} R^{\perp}(\xi,Y)(\widetilde{\nabla}_U\sigma)(V,\xi) &- (\widetilde{\nabla}_{R(\xi,Y)U}\sigma)(V,\xi)-(\nabla_U\sigma)(R(\xi,Y)V,\xi)\\ &- (\widetilde{\nabla}_U\sigma)(V,R(\xi,Y)\xi)=-L_2\{(\widetilde{\nabla}_{(\xi\wedge_g Y)U}\sigma)(V,\xi)\\ &+ (\widetilde{\nabla}_U\sigma)((\xi\wedge_g Y)V,\xi)+(\widetilde{\nabla}_U\sigma)(V,(\xi\wedge_g Y)\xi)\}. \end{aligned} \tag{37}$$

Next, we will calculation each of this statements, respectively. Taking account of (16), (26) and (27), we obtain

$$\begin{aligned} R^{\perp}(\xi,Y)(\widetilde{\nabla}_U\sigma)(V,\xi) &= R^{\perp}(\xi,Y)\{\nabla^{\perp}_U\sigma(V,\xi)-\sigma(\nabla_UV,\xi)-\sigma(\nabla_U\xi,V)\}\\ &= -R^{\perp}(\xi,Y)\sigma(\nabla_U\xi,V)\\ &= -R^{\perp}(\xi,Y)\sigma(-\alpha\phi^2U-\phi hU,V)\\ &= -\alpha R^{\perp}(\xi,Y)\sigma(U,V)+R^{\perp}(\xi,Y)\phi\sigma(hU,V. \end{aligned} \tag{38}$$

On the other hand, from (6), (17) and (27), by a direct calculation, we can infer

$$\begin{aligned} R(\xi,X)Y &= \kappa[g(Y,X)\xi-\eta(Y)X]+\mu[g(hY,X)\xi-\eta(Y)hX]\\ &+ \nu[g(X,\phi hY)\xi-\eta(Y)\phi hX]. \end{aligned} \tag{39}$$

Thus

$$
\begin{aligned}
(\widetilde{\nabla}_{R(\xi,Y)U}\sigma)(V,\xi) &= \nabla^{\perp}_{R(\xi,Y)U}\sigma(V,\xi)-\sigma(\nabla_{R(\xi,Y)U}V,\xi)-\sigma(\nabla_{R(\xi,Y)U}\xi,V)\\
&= -\sigma(\nabla_{R(\xi,Y)U}\xi,V)\\
&= \sigma(\alpha\phi^2R(\xi,Y)U+\phi hR(\xi,Y)U,V)\\
&= -\alpha\sigma(R(\xi,Y)U,V)+\sigma(\phi hR(\xi,Y)U,V)\\
&= -\alpha\sigma(-\kappa\eta(U)Y-\mu\eta(U)hY-\nu\eta(U)\phi hY,V)\\
&+ \sigma(-\kappa\eta(U)\phi hY-\mu\eta(U)\phi h^2Y-\nu\eta(U)\phi h\phi hY,V)\\
&= \alpha\kappa\eta(U)\sigma(V,Y)+\alpha\mu\eta(U)\sigma(hY,V)\\
&+ \alpha\nu\eta(U)\sigma(\phi hY,V)-\kappa\eta(U)\sigma(\phi hY,V)\\
&+ \mu(\kappa+\alpha^2)\eta(U)\sigma(\phi Y,V)+\nu(\kappa+\alpha^2)\sigma(V,Y). \qquad (40)
\end{aligned}
$$

Furthermore, by using (26) and ()39, we have

$$
\begin{aligned}
(\widetilde{\nabla}_{U}\sigma)(R(\xi,Y)V,\xi) &= \nabla^{\perp}_{U}\sigma(R(\xi,Y)V,\xi)-\sigma(\nabla_{U}R(\xi,Y)V,\xi)\\
&- \sigma(\nabla_U\xi,R(\xi,Y)V)\\
&= -\sigma(\nabla_U\xi,R(\xi,Y)V)=\sigma(\alpha\phi^2U+\phi hU,R(\xi,Y)V)\\
&= \alpha\sigma(\phi^2U,R(\xi,Y)V)+\sigma(\phi hU,R(\xi,Y)V)\\
&= -\alpha\sigma(U,-\kappa\eta(V)Y-\mu\eta(V)hY-\nu\eta(V)\phi hY)\\
&+ \sigma(\phi hU,-\kappa\eta(V)Y-\mu\eta(V)hY-\nu\eta(V)\phi hY)\\
&= \kappa\alpha\eta(V)\sigma(U,Y)+\mu\alpha\eta(V)\sigma(hY,U)\\
&+ \alpha\nu\eta(V)\sigma(U,\phi hY)-\kappa\eta(V)\sigma(\phi hU,Y)\\
&- \mu\eta(V)\sigma(\phi hU,hY)+\nu\eta(V)\sigma(hU,hY)\\
&= \kappa\alpha\eta(V)\sigma(U,Y)+\mu\alpha\eta(V)\sigma(hY,U)\\
&+ \alpha\nu\eta(V)\sigma(U,\phi hY)-\kappa\eta(V)\sigma(\phi hU,Y)\\
&+ \mu(\kappa+\alpha^2)\eta(V)\sigma(\phi U,Y)-\nu(\kappa+\alpha^2)\eta(V)\sigma(U,Y). \qquad (41)
\end{aligned}
$$

The fourth term gives us

$$
\begin{aligned}
&(\nabla_U\sigma)(V,R(\xi,Y)\xi)\\
&= (\nabla_U\sigma)(V,\kappa[\eta(Y)\xi-Y]-\mu hY-\nu\phi hY). \qquad (42)
\end{aligned}
$$

On the other hand, by view of (19), (26) and (27), we obtain

$$\begin{aligned}(\widetilde{\nabla}_{(\xi\wedge_g Y)U}\sigma)(V,\xi) &= \nabla^{\perp}_{(\xi\wedge_g Y)U}\sigma(V,\xi)-\sigma(\nabla_{(\xi\wedge_g Y)U}V,\xi)\\ &- \sigma(V,\nabla_{(\xi\wedge_g Y)U}\xi)\\ &= \sigma(V,\alpha\phi^2(\xi\wedge_g Y)U+\phi h(\xi\wedge_g Y)U)\\ &= -\alpha\sigma(V,(\xi\wedge_g Y)U)+\sigma(V,(\xi\wedge_g Y)U)\\ &= \alpha\eta(U)\sigma(Y,V)-\eta(U)\sigma(\phi hY,V), \qquad (43)\end{aligned}$$

and

$$\begin{aligned}(\widetilde{\nabla}_U\sigma)((\xi\wedge_g Y)V,\xi) &= \nabla^{\perp}_U\sigma((\xi\wedge_g Y)V,\xi)-\sigma(\nabla_U(\xi\wedge_g Y)V,\xi)\\ &- \sigma((\xi\wedge_g Y)V,\nabla_U\xi)\\ &= \sigma(\alpha\alpha\phi^2U+\phi hU,g(Y,V)\xi-\eta(V)Y)\\ &= \alpha\eta(V)\sigma(Y,U)-\eta(V)\sigma(Y,\phi hU). \qquad (44)\end{aligned}$$

Finally,

$$\begin{aligned}(\widetilde{\nabla}_U\sigma)(V,\eta(Y)\xi-Y) &= -(\widetilde{\nabla}_U\sigma)(V,Y)+(\widetilde{\nabla}_U\sigma)(V,\eta(Y)\xi)\\ &= -(\widetilde{\nabla}_U\sigma)(V,Y)+\nabla^{\perp}_U\sigma(V,\eta(Y)\xi)\\ &- \sigma(\nabla_UV,\eta(Y)\xi)-\sigma(V,\nabla_U\eta(Y)\xi)\\ &= -(\widetilde{\nabla}_U\sigma)(V,Y)-\sigma(V,U[\eta(Y)]\xi+\eta(Y)\nabla_U\xi)\\ &= -(\widetilde{\nabla}_U\sigma)(V,Y)+\eta(V)\sigma(\alpha\phi^2U+\phi hU,V)\\ &= -(\widetilde{\nabla}_U\sigma)(V,Y)-\alpha\eta(Y)\sigma(U,V)\\ &+ \eta(Y)\sigma(\phi hU,V). \qquad (45)\end{aligned}$$

Substituting (38), (40), (41),(42), (43), (44) and (45) into (37), we react at

$$\begin{aligned}&- \alpha R^{\perp}(\xi,Y)\sigma(U,V)+R^{\perp}(\xi,Y)\phi\sigma(U,V)-\kappa\alpha\eta(U)\sigma(V,Y)\\ &- \mu\alpha\eta(U)\sigma(V,hY)-\nu\alpha\eta(U)\sigma(V,\phi hY)+\kappa\eta(U)\sigma(V,\phi hY)\\ &- \mu(\kappa+\alpha^2)\eta(U)\sigma(\phi Y,V)-\nu(\kappa+\alpha^2)\eta(U)\sigma(V,Y)-\kappa\alpha\eta(V)\sigma(U,Y)\\ &- \alpha\mu\eta(V)\sigma(hY,U)-\alpha\nu\eta(V)\sigma(U,\phi hY)+\kappa\eta(V)\sigma(\phi hU,Y)\\ &- \mu(\kappa+\alpha^2)\eta(V)\sigma(\phi U,Y)+\nu(\kappa+\alpha^2)\eta(V)\sigma(U,Y)\\ &- (\nabla_U\sigma)(V,\kappa[\eta(Y)\xi-Y]-\mu hY-\nu\phi hY)=-L_2\{\alpha\eta(U)\sigma(V,Y)\\ &- \eta(U)\sigma(\phi hY,V)+\alpha\eta(V)\sigma(Y,U)-\eta(V)\sigma(Y,\phi hU)\\ &- (\nabla_U\sigma)(V,Y)-\alpha\eta(Y)\sigma(U,V)+\eta(Y)\sigma(\phi hU,V)\}.\end{aligned}$$

Here, taking $V = \xi$ in the last equality and using (27), we conclude that

$$\begin{aligned} L_2\{\alpha\sigma(U,Y) &- \sigma(Y,\phi hU) - (\widetilde{\nabla}_U\sigma)(Y,\xi)\} = \kappa\alpha\sigma(U,Y) + \alpha\mu\sigma(U,hY) \\ &+ \alpha\nu\sigma(U,\phi hY) - \kappa\alpha\sigma(\phi hY,U) + \mu(\kappa+\alpha^2)\sigma(\phi U,Y) \\ &- \nu(\kappa+\alpha^2)\sigma(U,Y) \\ &+ (\widetilde{\nabla}_U\sigma)(\xi,\kappa[\eta(Y)\xi - Y] - \mu hY - \nu\phi hY), \end{aligned} \tag{46}$$

where

$$\begin{aligned} (\widetilde{\nabla}_U\sigma)(Y,\xi) &= -\sigma(\nabla_U\xi,Y) = \sigma(\alpha\phi^2U + \phi hU, Y) \\ &= -\alpha\sigma(U,Y) + \phi\sigma(hU,Y) \end{aligned} \tag{47}$$

and

$$\begin{aligned} &(\widetilde{\nabla}_U\sigma)(\xi,\kappa[\eta(Y)\xi - Y] - \mu hY - \nu\phi hY) \\ = &-\sigma(\nabla_U\xi,\kappa[\eta(Y)\xi - Y] - \mu hY - \nu\phi hY) \\ = &\ \sigma(\alpha\phi^2U + \phi hU,\kappa[\eta(Y)\xi - Y] - \mu hY - \nu\phi hY) \\ = &-\alpha\sigma(U,\kappa[\eta(Y)\xi - Y] - \mu hY - \nu\phi hY) \\ + &\ \sigma(\phi hU,\kappa[\eta(Y)\xi - Y] - \mu hY - \nu\phi hY) \\ = &\ \kappa\alpha\sigma(U,Y) + \alpha\mu\sigma(hY,U) + \alpha\nu\sigma(\phi hY,U) \\ - &\ \kappa\sigma(\phi hU,Y) + \mu(\kappa+\alpha^2)\sigma(\phi U,Y) - \nu(\kappa+\alpha^2)\sigma(U,Y). \end{aligned} \tag{48}$$

Substituting (47) and (48) into (46), we get

$$\begin{aligned} &[\alpha L_2 - \kappa\alpha + \nu(\kappa+\alpha^2)]\sigma(U,Y) + [\kappa - L_2 - \alpha\nu]\phi\sigma(hU,Y) \\ - &\ \mu(\kappa+\alpha^2)\phi\sigma(U,Y) - \alpha\mu\sigma(hU,Y) = 0. \end{aligned} \tag{49}$$

If hU is written instead of U in (49) and using (7), (12) and (27), we have

$$\begin{aligned} [\alpha L_2 - \kappa\alpha &+ \nu(\kappa+\alpha^2)]\sigma(hU,Y) - (\kappa+\alpha^2)[\kappa - L_2 - \alpha\nu]\phi\sigma(U,Y) \\ &- \mu(\kappa+\alpha^2)\phi\sigma(hU,Y) + \alpha\mu(\kappa+\alpha^2)\sigma(U,Y) = 0. \end{aligned} \tag{50}$$

From (49) and (50), for $\kappa \neq 0$, we obtain

$$[(L_2 - \kappa)^2 - (\kappa+\alpha^2)(\nu^2 - \mu^2)]\sigma(U,Y) + 2\mu\nu(\kappa+\alpha^2)\phi\sigma(U,Y) = 0.$$

Since the vectors $\phi\sigma(U,Y)$ and $\sigma(U,Y)$ are orthogonal, we conclude that M is a totally geodesic or

$$\mu\nu(\kappa+\alpha^2) = 0,$$

and

$$L_2 = \kappa \mp \sqrt{(\kappa+\alpha^2)(\nu^2 - \nu^2)}.$$

Thus the proof is completed. □

From Theorem 2.5, we have following corollary.

Corollary 2.6. *Let M be an invariant submanifold of an almost α-cosymplectic (κ,μ,ν)-space $M^{2n+1}(\phi,\xi,\eta,g)$. Then M is 2-semiparallel if and only if M is totally geodesic.*

Theorem 2.7. *Let M be an invariant Ricci-generalized pseudoparallel submanifold an almost α-cosymplectic (κ,μ,ν)-space $M^{2n+1}(\phi,\xi,\eta,g)$ Then M is either totally geodesic submanifold or the functions L_3, κ,μ,ν and α satisfy the condition*

$$L_3=\frac{1}{2n}\left(1\mp\frac{1}{\kappa}\sqrt{(\kappa+\alpha^2)(\nu^2-\mu^2}\right),\quad \mu\nu(\kappa+\alpha^2)=0.$$

Proof. We suppose that M is an invariant Ricci-generalized pseudoparallel. Then there exists a function L_3 on M such that

$$(\widetilde{R}(X,Y)\cdot\sigma)(U,V)=L_3Q(S,\sigma)(U,V;X,Y),$$

for all vector fields X,Y,U,V on M. This implies that

$$\begin{aligned}R^{\perp}(X,Y)\sigma(U,V) \;&-\; \sigma(R(X,Y)U,V)-\sigma(U,R(X,Y)V)\\ &= -L_3\{\sigma((X\wedge_S Y)U,V)+\sigma(U,(X\wedge_S Y)V)\}\\ &= -L_3\{\sigma(X,V)S(U,Y)-\sigma(Y,V)S(X,U)\\ &+ \sigma(U,X)S(Y,V)-\sigma(U,Y)S(X,V)\}.\end{aligned}\tag{51}$$

By a direct calculation, we obtain

$$S(X,\xi)=2n\kappa\eta(X).\tag{52}$$

Taking $U=\xi$ in (51)and by view means of (6), (27) and (52), we have

$$\sigma(R(X,Y)\xi,V)=2n\kappa L_2\{\sigma(X,V)-\sigma(Y,V)\},$$

that is,

$$\begin{aligned}2n\kappa L_2\{\sigma(X,V)-\sigma(Y,V)\} &= \sigma(\kappa[\eta(Y)X-\eta(X)Y]+\mu[\eta(Y)hX\\ &- \eta(X)hY]+\nu[\eta(Y)\phi hX-\eta(X)\phi hY],V).\end{aligned}$$

This yields to

$$\kappa(2nL_3-1)\sigma(X,V)=\mu\sigma(hX,V)+\nu\phi\sigma(hX,V).\tag{53}$$

If hX is written instead of X and using (7) and (27), we get

$$\kappa(2nL_3-1)\sigma(hX,V)=-(\kappa+\alpha^2)\{\mu\sigma(X,V)-\nu\phi\sigma(X,V)\}.\tag{54}$$

From (53) and (54), we can derive

$$\begin{aligned}\{\kappa^2(2nL_3-1)^2 \;&+\; (\kappa+\alpha^2)(\mu^2-\nu^2)\}\sigma(X,V)\\ &= -2\mu\nu(\kappa+\alpha^2)\phi\sigma(X,V).\end{aligned}$$

Since σ and $\phi\sigma$ are orthogonal vectors, it follows that

$$\kappa^2(2nL_3-1)^2+(\kappa+\alpha^2)(\mu^2-\nu^2)=0, \quad \mu\nu(\kappa+\alpha^2)=0,$$

which proves our assertions. □

Theorem 2.8. *Let M be an invariant 2-Ricci-generalized pseudoparallel submanifold an almost-α-cosymplectic (κ,μ,ν)-space $M^{2n+1}(\phi,\xi,\eta,g)$ Then M is either totally geodesic submanifold or the function L_4 satisfies*

$$L_4=\frac{1}{2n}\left(1\mp\frac{1}{\kappa}\sqrt{(\kappa+\alpha^2)(\nu^2-\mu^2}\right), \quad \mu\nu(\kappa+\alpha^2)=0.$$

Proof. Given M is an invariant 2-Ricci-generalized pseudoparallel submanifold, we have

$$(\widetilde{R}(X,Y)\cdot\widetilde{\nabla}\sigma)(U,V,W)=L_4Q(S,\widetilde{\nabla}\sigma)(U,V,W;X,Y)$$

for all vector fields X,Y,U,V,W on M. This means that

$$\begin{aligned}R^{\perp}(X,Y)(\widetilde{\nabla}_U\sigma)(V,W) &- (\widetilde{\nabla}_{R(X,Y)U}\sigma)(V,W)-(\widetilde{\nabla}_U\sigma)(R(X,Y)V,W)\\ &- (\widetilde{\nabla}_U\sigma)(V,R(X,Y)W)=-L_4\{(\widetilde{\nabla}_{(X\wedge_S Y)U}\sigma)(V,W)\\ &+ (\widetilde{\nabla}_U\sigma)((X\wedge_S Y)V,W)+(\widetilde{\nabla}_U\sigma)(V,(X\wedge_S Y)W)\}. \end{aligned} \tag{55}$$

Taking $X=V=\xi$ in (55), we obtain

$$\begin{aligned}R^{\perp}(\xi,Y)(\widetilde{\nabla}_U\sigma)(\xi,W) &- (\widetilde{\nabla}_{R(\xi,Y)U}\sigma)(\xi,W)-(\widetilde{\nabla}_U\sigma)(R(\xi,Y)\xi,W)\\ &- (\widetilde{\nabla}_U\sigma)(\xi,R(\xi,Y)W)=-L_4\{(\widetilde{\nabla}_{(\xi\wedge_S Y)U}\sigma)(\xi,W)\\ &+ (\widetilde{\nabla}_U\sigma)(\xi\wedge_S Y)\xi,W)+(\widetilde{\nabla}_U\sigma)(\xi,(\xi\wedge_S Y)W)\}. \end{aligned} \tag{56}$$

Now, let's calculate each of these terms separately. Firstly,

$$\begin{aligned}R^{\perp}(\xi,Y)\{-\sigma(\nabla_U\xi,W)\} &= R^{\perp}(\xi,Y)\sigma(\alpha\phi^2U+\phi hU,W)\\ &= -\alpha R^{\perp}(\xi,Y)\sigma(U,W)+R^{\perp}(\xi,Y)\sigma(\phi hU,W). \end{aligned} \tag{57}$$

Making use of (7), (26) and (39), can we calculate second term as

$$
\begin{aligned}
(\widetilde{\nabla}_{R(\xi,Y)U}\sigma)(W,\xi) &= -\sigma(\nabla_{R(\xi,Y)U}\xi,W) = \alpha\sigma(\phi^2R(\xi,Y)U,W) \\
&+ \sigma(\phi h\nabla_{R(\xi,Y)U},W) \\
&= \alpha\kappa\eta(U)\sigma(Y,W) + \alpha\mu\eta(U)\sigma(hY,W) \\
&+ \alpha\nu\eta(U)\sigma(\phi hY,W) \\
&- \kappa\eta(U)\sigma(\phi hY,W) + \mu(\kappa+\alpha^2)\eta(U)\sigma(\phi Y,W) \\
&- \nu\eta(U)\sigma(\phi h\phi hY,W) \\
&= \alpha\kappa\eta(U)\sigma(Y,W) + \alpha\mu\eta(U)\sigma(hY,W) \\
&+ \alpha\nu\eta(U)\sigma(\phi hY,W) \\
&- \kappa\eta(U)\sigma(\phi hY,W) + \mu\eta(U)(\kappa+\alpha^2)\sigma(\phi Y,W) \\
&- \nu(\kappa+\alpha^2)\eta(U)\sigma(Y,W), \qquad (58)
\end{aligned}
$$

$$
\begin{aligned}
&(\widetilde{\nabla}_U\sigma)(R(\xi,Y)\xi,W) \\
&= (\widetilde{\nabla}_U\sigma)(\kappa[\eta(Y)\xi - Y] - \mu hY - \nu\phi hY, W). \qquad (59)
\end{aligned}
$$

In the same way,

$$
\begin{aligned}
(\widetilde{\nabla}_U\sigma)(R(\xi,Y)W,\xi) &= -\sigma(\nabla_U\xi, R(\xi,Y)W) = \sigma(\alpha\phi^2U + \phi hU, R(\xi,Y)W) \\
&= \alpha\kappa\eta(W)\sigma(U,Y) + \alpha\mu\eta(W)\sigma(hY,W) \\
&+ \alpha\nu\eta(W)\sigma(U,\phi hY) - \kappa\eta(W)\sigma(\phi hU,Y) \\
&- \mu\eta(W)\sigma(\phi h^2U,Y) + \nu\eta(W)\sigma(h^2U,Y) \\
&= \alpha\kappa\eta(W)\sigma(U,Y) + \alpha\mu\eta(W)\sigma(hY,W) \\
&+ \alpha\nu\eta(W)\sigma(U,\phi hY) - \kappa\eta(W)\sigma(\phi hU,Y) \\
&+ \mu(\kappa+\alpha^2)\eta(W)\sigma(\phi U,Y)\nu(\kappa+\alpha^2)\eta(W)\sigma(U,Y), \qquad (60)
\end{aligned}
$$

$$
\begin{aligned}
(\widetilde{\nabla}_{(\xi\wedge_S Y)U}\sigma)(\xi,W) &= -\sigma(\nabla_{(\xi\wedge_S Y)U}\xi,W) \\
&= \sigma(\alpha\phi^2(\xi\wedge_S Y)U + \phi h(\xi\wedge_S Y)U,W) \\
&= -\alpha\sigma(S(Y,U)\xi - S(\xi,U)Y,W) \\
&+ \sigma(\phi h[S(Y,U)\xi - S(\xi,U)Y],W) \\
&= 2n\kappa\eta(U)\{\alpha\sigma(Y,W) - \sigma(\phi hY,W)\}, \qquad (61)
\end{aligned}
$$

$$
\begin{aligned}
(\widetilde{\nabla}_U\sigma)((\xi\wedge_S Y)\xi,W) &= -\sigma(\nabla_U(\xi\wedge_S Y)\xi,W)\\
&= (\widetilde{\nabla}_U\sigma)(S(\xi,Y)\xi-S(\xi,\xi)Y,W)\\
&= 2n\{(\nabla_U\sigma)(\kappa\eta(Y)\xi,W)-(\nabla_U\sigma)(\kappa Y,W)\}\\
&= 2n\{-\sigma(U[\kappa\eta(Y)]\xi+\kappa\eta(Y)\nabla_U\xi,W)\\
&- (\nabla_U\sigma)(\kappa Y,W)\}\\
&= 2n\{-\kappa\alpha\eta(Y)\sigma(U,W)+\kappa\eta(Y)\sigma(\phi hU,W)\\
&- (\nabla_U\sigma)(\kappa Y,W)\}. \qquad (62)
\end{aligned}
$$

Finally

$$
\begin{aligned}
(\widetilde{\nabla}_U\sigma)(\xi,(\xi\wedge_S Y)W) &= -\sigma(\nabla_U\xi,(\xi\wedge_S Y)W)\\
&= \sigma(\alpha\phi^2U+\phi hU,S(Y,W)\xi-S(\xi,W)Y)\\
&= 2n\kappa\alpha\eta(W)\sigma(U,Y)-2n\kappa\eta(W)\sigma(\phi hU,Y). \qquad (63)
\end{aligned}
$$

Consequently, substituting (57), (58), (59), (60), (61), (62) and (63) into (56), we reach at

$$
\begin{aligned}
&- \alpha R^{\perp}(\xi,Y)\sigma(U,W)+R^{\perp}(\xi,Y)\sigma(\phi hU,W)-\alpha\kappa\eta(U)\sigma(Y,W)\\
&- \alpha\mu\eta(U)\sigma(hY,W)-\alpha\nu\eta(U)\sigma(\phi hY,W)+\kappa\eta(U)\sigma(\phi hY,W)\\
&- \mu\eta(U)(\kappa+\alpha^2)\sigma(\phi Y,W)+\nu(\kappa+\alpha^2)\eta(U)\sigma(Y,W)\\
&- (\nabla_U\sigma)(\kappa[\eta(Y)\xi-Y]-\mu hY-\nu\phi hY,W)-\alpha\kappa\eta(W)\sigma(U,Y)\\
&- \alpha\mu\eta(W)\sigma(hY,W)-\alpha\nu\eta(W)\sigma(U,\phi hY)+\kappa\eta(W)\sigma(\phi hU,Y)\\
&- \mu(\kappa+\alpha^2)\eta(W)\sigma(\phi U,Y)+\nu(\kappa+\alpha^2)\eta(W)\sigma(U,Y)\\
&= -L_4\{2n\kappa\alpha\eta(U)\sigma(Y,W)-2n\kappa\eta(U)\sigma(\phi hY,W)-2n\kappa\alpha\eta(Y)\sigma(U,W)\\
&+ 2n\kappa\eta(Y)\sigma(\phi hU,W)-2n(\nabla_U\sigma)(\kappa Y,W)+2n\alpha\kappa\eta(W)\sigma(U,Y)\\
&- 2n\kappa\eta(W)\sigma(\phi hU,Y)\}.
\end{aligned}
$$

In the last equality, putting $W=\xi$, we have

$$
\begin{aligned}
2nL_4\{(\nabla_U\sigma)(\kappa Y,\xi) &- \kappa\alpha\sigma(U,Y)+\kappa\sigma(\phi hU,Y)\}=\nu(\kappa+\alpha^2)\sigma(U,Y)\\
&- \alpha\kappa\sigma(U,Y)-\alpha\mu\sigma(hY,U)-\alpha\nu\sigma(\phi hU,Y)\\
&- \mu(\kappa+\alpha^2)\sigma(\phi U,Y)+\kappa\sigma(\phi hU,Y)\\
&- (\nabla_U\sigma)(\kappa[\eta(Y)\xi-Y]-\mu hY-\nu\phi hY,\xi), \qquad (64)
\end{aligned}
$$

where

$$
\begin{aligned}
(\nabla_U\sigma)(\kappa Y,\xi) &= -\sigma(\nabla_U\xi,\kappa Y)=\sigma(\alpha\phi^2U+\phi hU,\kappa Y)\\
&= -\alpha\kappa\sigma(U,Y)+\kappa\sigma(\phi hU,Y), \qquad (65)
\end{aligned}
$$

and

$$\begin{aligned}
& (\nabla_U\sigma)(\kappa[\eta(Y)\xi - Y] - \mu hY - \nu\phi hY, \xi) \\
= & -\sigma(\nabla_U\xi, \kappa[\eta(Y)\xi - Y] - \mu hY - \nu\phi hY) \\
= & \sigma(\alpha\phi^2 U + \phi hU, \kappa[\eta(Y)\xi - Y] - \mu hY - \nu\phi hY) \\
= & \alpha\kappa\sigma(U,Y) + \alpha\mu\sigma(U,hY) + \alpha\nu\sigma(U,\phi hY) \\
- & \kappa\sigma(\phi hU, Y) + \mu(\kappa+\alpha^2)\sigma(\phi U, Y) - \nu(\kappa+\alpha^2)\sigma(U,Y). \qquad (66)
\end{aligned}$$

(65) and (66) are put in (64), we conclude that

$$\begin{aligned}
[\kappa\alpha(2nL_4 - 1) & + (\kappa+\alpha^2)(\nu - \mu\phi)]\sigma(U,Y) \\
& - [\kappa(2nL_4 - 1)\phi + \alpha(\nu\phi + \mu)]\sigma(hU,Y) = 0. \qquad (67)
\end{aligned}$$

Here hU is written instead of U and taking into account of (7) and (27), we have

$$\begin{aligned}
[\kappa\alpha(2nL_4 - 1) & + (\kappa+\alpha^2)(\nu - \mu\phi)]\sigma(hU,Y) \\
& + [\kappa(2nL_4 - 1)\phi + \alpha(\nu\phi + \mu)](\kappa+\alpha^2)\sigma(U,Y) = 0. \qquad (68)
\end{aligned}$$

From (67) and (68), it follows for $\kappa \neq 0$,

$$[\kappa^2(2nL_4 - 1)^2 + (\mu^2 - \nu^2)(\kappa+\alpha^2)]\sigma(U,V) + 2\mu\nu(\kappa+\alpha^2)\phi\sigma(U,V) = 0.$$

This proves our assertion. □

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