Review

The Axiom of Choice: The Last Great Controversy in Mathematics

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Abstract: This paper explores multiple perspectives and proofs regarding the validity of what is considered to be one of the – if not THE - most controversial proofs in the field of mathematics, historical and contemporary applications alike. We consider the various explanations and equivalents of the axiom along with the more widely receptive alternatives. Furthermore, we review the resistance the axiom has encountered in various fields and its potential use in the research and expansion of said fields.

Keywords: Axiom of Choice; Zermello; Set Theory

1. Introduction

Mathematics is not the field that typically comes to mind when one is instructed to imagine a controversial discipline, however, the field of mathematics has been riddled with controversy at every turn. What started as a clean-cut system to keep track of items and measure distance in the beginnings of society has since evolved into an endless cycle of the presentation and subsequent debates of indefinite and undeterminable concepts of the intangible. With arguably unresolved topics such as the nature of the number zero, irrational and imaginary numbers, as well as the race to construct and disprove equations and patterns for finding new prime numbers, mathematics remains every bit as controversial as it was in the days Pythagoras was rumored to have murdered the mathematician who discovered irrational numbers. One axiom that contributes to the endless cycle of deliberation is the Axiom of Choice.

2. Origins and Explanations

2.1. Origins

The Axiom of Choice, commonly shortened to AC, was one conceived out of convenience and, in the words of Ilya Egorychev, a doctor of philosophical science, "seems to be both extremely obvious and, at the same time, a completely mystical statement."[1] Formulated by German mathematician Dr. Ernest Zermelo in 1904, the Axiom of Choice was devised as an axiom to prove that every cardinal has an initial ordinal in the form of the "Well-Ordering Theorem", another one of Zermelo's proposals, which posited that *Every set can be given an order relationship, such as less than, under which it is well ordered.* [2]

Proof of Equivalency: Well-ordering theorem ⇒ Axiom of Choice

Let A be a non-empty set comprised of non-empty sets, and let there be $W = \cup A$ where $\leq$ is the well-ordering of W. Thus, the function $f : A \rightarrow W$ that is defined by $f(A) = min(A)$ is a choice function on A.[3]

The Well-ordering theorem, the Axiom of Choice, as well as Zorn's lemma – the proposition that if every ordered subset of a nonempty set has an upper bound, then the aforementioned nonempty set has at least one maximal element – were determined to be equivalent, thus allowing a mathematician to prove the other two if one was assumed to be true. What makes the AC unique regarding other axioms in mathematics, specifically in the field of set theory, is that the axiom presents the idea that a set exists but does not state any elements nor a method with which to derive the elements. While this idea might not have seemed as though AC had any practical use within the constraints of finite sets, its true potential is explored in more unusual scenarios that were not commonly encountered in the early twentieth century. When utilized on finite sets, the axiom follows other set theory axioms. It isn't until AC is applied in the context of infinite, non-empty sets, can it be utilized in a way other axioms in set theory hadn't before. Despite its seemingly useful potential in these new scenarios, the axiom was faced with a significant resistance which will be further explored in section 4.1.

2.2. Explanations and Examples

In Zermelo's initial publication of his ideas, he begins his explanation by establishing some arbitrary set, M, and some arbitrary, nonempty subset of M, M'. Next, he establishes an associate element m' in every M', thus, each of these associate elements creates a "covering" – as he calls it – of the set M. "The number of these coverings is equal to the product [of the cardinalities of all the subsets M'] and is certainly different from 0."[4] This initial presentation has been adapted over time and is now commonly explained with terms of choice functions.

Definition 1:

Let there be some collection of non-empty sets C. If one chooses an element from each set in C, there exists some function f defined on C such that set S in C, there f(S) is in S.

This function f is a choice function.[5] This can be further shown by example.

Example 1.1

Let C be a collection of all non-empty subsets of {1,2,3,…}. f can be defined by letting f(S) be the smallest element of S.

Example 1.2

Let C be a collection of all intervals of real, positive numbers with finite lengths. f can be defined by letting f(S) be the midpoint of interval S.

While it is nothing extraordinary when it comes to finite examples, these choice functions are what make the AC so adaptable when it comes to unknowns and infinites. From Zermelo's original publication came an assertion that is referred to as AC1. This assertion originally stated:

AC1 : *Any collection of nonempty sets has a choice function.*

To account for variability in sets, the collection of sets A = {Ai : i ∈ I} was created as a variable set that varied over the index I where each Ai was a value of A at i. A choice function on the collection A is a variable element of A, thus the original assertion AC1 was redefined in terms of variability and relations into the equivalent assertions AC2 and AC3, respectively.

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AC2 : *Any indexed collection of sets has a choice function*.

AC3 : *For any relation R between sets A and B,* $\forall x \in A\ \exists y \in B\ [R(x,y)] \Rightarrow \exists f\ [f : A \rightarrow B \wedge \forall x \in A[R(x, fx)]]$.

AC3 was proven to be equivalent to and subsequently simplified into written terms in the form of AC4.

AC4 : *Any surjective function has a right inverse.*

These four assertions marked the beginning of the Axiom of Choice's journey through the field of mathematics.[6]

3. Chronology and Historical Applications

3.1. Chronology

Initially published in 1904, the axiom was immediately faced with scrutiny, and though recognized by famed British polymath Bertrand Russell early on, it wasn't until 1922, when Israeli mathematician Abraham Fraenkel presented the "permutation method" as a way to establish the axiom's independence, that it gained any traction amongst mathematicians.

This permutation method was built upon several times as mathematicians continued to establish the validity of the axiom. In 1936, Lindenbaum and Mostowski used a more processed version of this method to prove the independence of multiple ideas in set theory that are considerably weaker than the axiom of choice, and the axiom's independence was again proven in 1963 by Paul Cohen using the standard axioms of set theory. In addition to independence, the consistency of the axiom was found, again using axioms in set theory, by Kurt Gödel in the years of 1935-1938. [6] With the expansion and exploration of the field of set theory, the axiom of choice, while still maintaining some resistance, was slowly becoming incorporated into proofs, if even just implicitly or via its alternatives.

3.2. Historical Applications

Originally, the axiom was derived out of a need to prove the well-ordering theorem, though its appearances following that have been scarce. Until relatively recently, and even to this day, the axiom of choice has been very rarely used. Some mathematicians still consider the axiom to not be a valid method for proving ideas, and even more refuse to explicitly use it due to this stigma. Because of its history of scrutiny, most of its modern and historical applications come in the form of implicit or alternative usage. One of the exceptions to this precedent is the Banach-Tarski Paradox. This paradox asserted that there exists some 3-dimensional ball B, where B = {(x,y,z) ∈ R3 : x2 + y2 + z 2 < 1}, such that the ball could be broken into some finite number of pieces and rearranged with rotations and translations into two identical copies of the original ball. While it may seem to violate physical laws, the axiom of choice side-stepped the Law of Conservation of Mass with its immeasurable sets. As an immeasurable set has no defined volume, there is no violation of physical laws using the axiom of choice, thus, a paradox in the physical world.

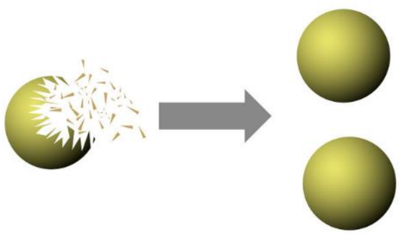

Figure 1: A visualization of the Banach-Tarski Paradox. Image from https://www.thisisdavidjones.com/blog/2018/03/banachtarski/ [14]

Aside from the glamour of being used in paradoxes and proofs, there are several theorems and statements that have come to rely on the axiom as the field of set theory has developed. These include, but are not limited to[7]:

- The Loewenheim-Skolem Theorem
- The Ultrafilter Theorem
- The Baire Category Theorem

These theorems are prime examples of the importance of the axiom of choice. While it may not be explicitly used in many proofs, there are many, more utilized, theorems and statements that are reliant on the axiom.

4. Resistance and Counterarguments

4.1. Resistance and Arguments Against

The struggle for recognition and acceptance for the axiom of choice is due to a myriad of different reasons, many of which not originating from the discipline of mathematics. The first argument was sparked early after the initial proposal of the axiom. Scholars in the field of philosophy countered the validity of the axiom with the argument that the concept of proving the existence of intangibles while not being able to explicitly construct these sets conflicted with philosophical principles. [8]

On the opposite end of the spectrum, another argument was presented from the field of computer science. Unlike the field of philosophy, computer science is a rather new discipline in academia. The continuity of scrutiny regarding the axiom, however, remains a constant. With the development of artificial intelligence and computer programs to solve modern problems in mathematics that are physically unable to be performed by the human hand or mind, the implementation of the axiom of choice has presented many computer scientists with a uniquely modern dilemma: the programs cannot replicate the human mind. The axiom of choice's unusual logic and presentation is one that computer scientists like Dr. Nicolas Tabareau of IMT Atlantique Bretagne-Pays De La Loire have reported trouble replicating.[9] While this is not a direct contradiction to the axiom itself, the sheer human aspect of the axiom, a departure for the normalcy of theorems and rules, is one that turns many in the field away.

Returning to the familiar territory of mathematics, many mathematicians have critiqued the axiom's counterintuitive consequences. To accept the axiom of choice as a valid axiom is to accept the Banach-Tarski Paradox, which is something that some academics are opposed to.[2] Additionally, some are against it merely because the mathematician who created it did so to prove his own theorem. The sentiment there lies in the idea that because Zermelo had the incentive to make an axiom to prove a theorem of his, the axiom of choice would be workshopped to fit his needs. While the initial publication was rough, it was refined in a later 1908 publication and has since been proven multiple times using various methods that have been built upon, such as the permutation method.

Another major argument against the validity of the axiom of choice is the existence of statements that are consistent with the negation of the axiom. In Zermelo-Fraenkel set theory, there are several models and statements that are true in which the axiom of choice is false. Because of this division in Zermelo-Fraenkel set theory into Zermelo-Fraenkel set theory with the negation of the axiom of choice, often shortened to ZF¬C, many disregard the branch altogether in favor of general set theory.

The final main argument originating in the field of mathematics is the interpretation of the axiom. In the original axiom, one is told they can "choose" elements from each set in the collection of sets and that there "exists" a function f – the choice function. The issue emerges at the separation of thought into the constructivist and nonconstructivist thought processes. In the constructive branches of mathematics, there is a strong emphasis placed on the ability to find explicitly rather than prove with unknowns. Thus, if one were to interpret the word in a constructivist manner, the axiom would be false as the word "exist" would be interpreted as "find," and one cannot find something that is intangible. Due to this distinction, most mathematicians interpret the axiom in a neutral or

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nonconstructive manner. As a result, the axiom is strongly resisted by those who only practice or research in the constructive branches of mathematics.

4.2. Arguments For

Just as there are many reasons that have been proposed in opposition to the use of the axiom of choice, there are many in its favor. In section 3.2, several theorems were used as an example of the axiom of choice's implicit usage. There are more examples of statements, concepts, and theorems that are reliant on the validity of the axiom that are far less controversial and far more accepted by mathematicians. To deem the axiom to be invalid would be to deem these branching stems as invalid as well.

Furthermore, there are additional, similarly widely accepted, equivalents to the axiom of choice. These statements and concepts include ideas such as Zorn's Lemma, one of the two previously discussed to be an equivalent along with the well-ordering theorem, the Trichotomy of Cardinals, as well as Tychonoff's Theorem. Similar to the first example in this section, the dismissal of the axiom of choice would render the stems of these ideas to also be invalid.

Lastly, many mathematicians who utilize the axiom do so because it is a convenient way to prove statements that could not be proven otherwise, such as the statement that every vector space has a basis.

4.3. Consistency

One of the causes of the shift towards the acceptance of the axiom is the consistent results that have been proven using the axiom. As noted in section 3.1, in the years of 1935-1938, logician Kurt Gödel reported the relative consistency of the axiom using axioms in the field of set theory. In his proofs, he introduced what is known as the "constructability" hierarchy of sets. Based on the cumulative type hierarchy, his proofs differed from the independence proofs of Fraenkel, Lindenbaum, and Mostowski of years prior in that he focused on the definability of the axiom. With P(x) being a power set of x, α being an ordinal, and λ being a limit ordinal, Gödel derived the following[6]:

$$V_0 = \emptyset$$
$$V_{\alpha+1} = P(V_\alpha)$$
$$V_\lambda = \cup_{\alpha<\lambda} V_\alpha$$

Assuming that the Zermelo-Fraenkel set theory – often shortened to ZF - axioms are true, Gödel showed that this derivative of the cumulative type hierarchy was a model of ZF, thus proving the consistency of AC in ZF – sometimes shortened to ZFC for Zermelo-Fraenkel set theory with the axiom of choice. It was later shown by various scholars in addition to Gödel, such as Myhill, Scott, Takeuti, and Post, that one could use ordinal definability to formulate a more concise proof of relative consistency. [6]

Zermelo-Fraenkel axioms	
(1)	*Axiom of extension.* If A and B are sets and if, for all x, $x \in A$ if and only if $x \in B$, then $A = B$.
(2)	*Axiom of the empty set.* There exists a set A such that, for all x, it is false that $x \in A$.
(3)	*Axiom schema of separation.* If A is a set, there exists a set B such that, for all x, $x \in B$ if and only if $x \in A$ and $S(x)$. Here, $S(x)$ is any condition on x in which B is not free (it must be bound by a quantifier such as "all" or "some").
(4)	*Axiom of pairing.* If A and B are sets, there exists a set (symbolized $\{A, B\}$ and called the unordered pair of A and B) having A and B as its sole members.
(5)	*Axiom of union.* If C is a set, there exists a set A such that $x \in A$ if and only if $x \in B$ for some member B of C.
(6)	*Axiom of power set.* If A is a set, there exists a set B, called its power set, such that $x \in B$ if and only if $x \subseteq A$.
(7)	*Axiom of infinity.* There exists a set A such that $\emptyset \in A$ and, if $x \in A$, then $(x \cup \{x\}) \in A$, in which $x \cup \{x\}$ is the set x with x adjoined as a further member.
(8)	*Axiom of choice.* If A is a set the elements of which are nonempty sets, then there exists a function f with domain A such that, for member B of A, $f(B) \in B$.
(9)	*Axiom schema of replacement.* If A is a set and $f(x, y)$ a formula (in which x and y are free) such that for $x \in A$ there is exactly one y such that $f(x, y)$, then there exists a set B the members of which are the y's determined by $f(x, y)$ as x ranges over A.
(10)	*Axiom of restriction (foundation axiom).* Every nonempty set A contains an element B such that $A \cap B = \emptyset$; i.e., A and B have no elements in common.

Figure 2: A chart of Zermelo-Fraenkel Axioms. Image by Britannica. https://www.britannica.com/science/axiom-of-choice [2]

4.4. Independence

As stated in section 3.1, Fraenkel's 1922 presentation of the independence of the axiom of choice was one of the first major events in the history of the axiom and arguably the idea that initiated the movement towards its acceptance. In the early to mid-twentieth century, multiple mathematicians and logicians published proofs demonstrating the axiom's independence in support of normalizing its use in proofs. In Fraenkel's proof, he used the set theory concept of "atoms",

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which are described as individuals with no elements within them that are distinct from empty sets. Using this, Fraenkel showed how one could make a universe V of all infinite set A of atoms by adding all of the subsets of A and adjoining them, thus producing V(A), a model of set theory using atoms.[6] Fraenkel expanded upon this establishment with the idea of permutations. He showed that any permutation of this intangible, infinite set A would produce a permutation of the universe V(A) while still maintaining its structure. From this came the permutation model, which was refined by Lindenbaum and Mostowski in 1936.

5. The Marriage of the Axiom and Related Fields

5.1. Set and Category Theory

As seen by the arguments used both in favor of and opposition of the application of the axiom of choice in the field of set theory, set theory and the axiom are closely intertwined. Set theory, as one can tell by the name, is defined as a branch of mathematics that studies sets and their properties. The axiom of choice allows one to operate intangible sets, which grants the opportunity for those conducting research in this branch to probe further into the unknown than the time prior. However, in section 4.1, it was shown that there is a subtype of set theory – Zermelo-Fraenkel set theory – that is at odds with the axiom of choice. While some statements and theorems in ZF rely on the axiom of choice, some are only true when the axiom is false, leading to the divide of ZF into ZFC and ZF¬C. This division deters many, so while the axiom of choice could lead to the advancement of the field, it could also prove to be a hindrance in the long run. [10] Category Theory, similar to set theory, is a branch of mathematics that has exponentially grown since the start of the twentieth century.

Category theory, which has become a central component of many contemporary fields in the sciences such as computer science, studies structures and systems of structures. Several theorems in this branch have previously been proven using the axiom of choice. Following the same theme as set theory, while the axiom of choice has the potential to be helpful, it is a risk that many researchers specializing in category theory are not willing to take.

5.2. Pure Mathematics and Physics

Pure mathematics and physics are two of the fields that are heavily critical with the usage of the axiom of choice despite the potential. A common complaint amongst pure mathematicians is the inconsistency of the axiom. While its applications in the field aid in generalizations and simplifying unquantifiable concepts, it can lead to conclusions and results that contradict observable phenomena.[5] This is in addition to other complaints regarding the departure from the constructivist thought process as presented previously in section 4.1.

The modern fascination with the axiom of choice stems from physics, above all other fields. The axiom of choice aids with the manipulation of intangibles and infinites, the two biggest categories of which are the infinitely small and infinitely large. Quantum and Astrophysics are two modern disciplines of physics that focus on these extremes. In relation to the generalizations and assumptions that are already standard in physical research, the axiom of choice would not be a very radical addition to the ensemble of formulas and theorems used in these fields. While there is very little literature for the use of the axiom, there are several publications utilizing equivalents and alternatives for the axiom, indicating that it will be a more viable option for those looking to publish with as little resistance and controversy as possible.[11]

This trend of using the equivalent axioms has spanned as far back as the creation of the axiom of choice. Many mathematicians and scholars would seldom choose to risk tarnishing their work by relying on a historically criticized axiom when there are other options available.

6. Equivalents and Their Applications

6.1. Stronger Axioms

One of the main hindrances to the use of the axiom of choice is that it has stronger equivalences. Stronger axioms, such as the axiom of constructability, imply the axiom of choice, making it stronger, and thus preferable. There are also sub-branch specific axioms such as the axiom of global choice. It is again preferable as it applies proper classes to sets. This additional use makes it a stronger axiom for proofs in the sub-branches of set theory. As the axiom of global choice is itself implied by a stronger axiom – the axiom of limitation of size – there are several axioms above the axiom of choice that are more efficient and more widely accepted. [12]

6.2. Equivalent Axioms

Equivalent axioms are defined by the assumption of the axioms of ZF, but not necessarily the axiom of choice or the negation of the axiom of choice.[13] Two major equivalencies that have previously been discussed include the well-ordering theorem and Zorn's lemma. These two are particularly useful substitutes for the axiom of choice as they are both commonly utilized in different branches of mathematics, the well-ordering theorem and Zorn's lemma in set theory and order theory respectively. Both of these branches have a multitude of similarly equivalent concepts. Another theorem that has been previously discussed is Tychonoff's theorem in point-set theory. Abstract algebra, functional analysis, and graph theory are amongst the other fields that have minor equivalencies to the axiom of choice. These options provide the opportunity to manipulate sets in the same manner one could with the axiom of choice without the increased likelihood of doubt.

6.3. Weaker Axioms

While the utilization of a weaker axiom in a proof where the axiom of choice could be used may sound counterintuitive, it may be worth it to sacrifice the elegancy and expend the effort to avoid the additional scrutiny. These weaker axioms are true for all models in ZFC but are not necessarily true for all models in ZF. Similar to the equivalent axioms, there are several examples for multiple fields of mathematics, such as functional analysis, but as they are weaker, there are limitations to the practical uses of these axioms.

7. Conclusions

The axiom of choice has had a very long history relative to its age. It gained its infamy as the world's most controversial axiom, pulling arguments from scholars of all disciplines, even those outside of the field of mathematics. While it has been proved multiple ways, there have been several complicating factors that leave the use of such a historically criticized tool with an air of doubt to this day. While it could hold the key to the expansion and exploration in various fields, it seems the only role it will play in research will be vicariously through its alternate axioms and implicit assumptions.

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