Competing Conventions with Costly Acquisition of Information

Abstract

We study the evolution of conventions in a "language" game – two groups of agents assign different positive payoffs for coordinating on different actions but zero to miscoordination – under the assumption that agents can pay a cost to learn the type (group) of their opponent. If they pay it, they can play separate actions with different types. We distinguish the analysis in two cases: in the first, the cost is equal to zero and, in the second, it is strictly positive. We look for long-run conventions using stochastic stability analysis. When the cost is zero or sufficiently low, agents always coordinate on their favorite action with their type, but their behavior in mixed interactions depends on their preferences. The favorite action by the type who is more rigid in preferences is the one played by the whole population in equilibrium. When the cost is high enough, two scenarios can happen. When one type is enough more rigid in preferences than the other, every agent plays the action preferred by that type with anyone. When they are both rigid in preferences, a heterogeneous strategy profile that causes miscoordination is the long-run equilibrium: in such a state, the two types play their favorite actions.

1 Introduction

Social scientists usually describe conventions as situations where every person acts in the same way with everybody. How an agent behaves with other people mostly depends on what s/he expects others to do and marginally on her/his preferences[1]. This is why game theorists usually represent social conventions as the outcome of coordination games. Specifically, since the seminal contribution of [6], evolutionary game theorists have used stochastic stability analysis and 2x2 coordination games to study the formation of social conventions. Some of these works focus on coordination games such as the battle of sexes: a class that describes situations in which two groups of people attach value zero to miscoordination but assign higher payoffs on different actions. In these situations, it is not clear which convention will be established in the long-run. Let us think about a person who wants to hang out with a friend: s/he has to choose between proposing her/him going to see a football match or going to the cinema, and s/he does not know what her/his friend prefers. Imagine being in a world without social networks: that person has to pay a high cost in terms of time to learn what her/his friend fancies. If s/he knows that everybody goes to the cinema,

[1] [1] and [2] are classical references from philosophy, while for economics see [3] [4], and [5].

s/he asks her/his friend to go to the cinema, even if s/he favors football matches. However, why does everybody go to the cinema? [7] considers a game similar to the one described above in an evolutionary framework. He considers a population divided into two groups/types. The two types of agents differ in preferences, but each agent decides one single action valid for both types. Hence, it is as if learning the type of an opponent is too costly, and no one ever identifies it. His study finds that if one type is enough more rigid in preferences than the other, then everyone coordinates on the action preferred by that type in the long-run. It also gives conditions on payoffs and group size for the prevalence in the long-run of a heterogeneous strategy profile, which causes miscoordination between the types. Such a state is a situation where everybody plays their favorite action.

Nevertheless, in certain circumstances, it is reasonable to assume that agents can detect the type of person they are playing with at a small cost. In the context of the previous example, a lower cost means, for instance, having social networks, where everyone can learn what their friends prefer. In this case, agents bear this cost, acknowledge who has their same tastes, and coordinate on their favorite actions with them. For instance, those who like cinema ask all the other players what they prefer, and go to the cinema with those who prefer cinema. Similar for those who fancy football. However, expanding the behavior to an entire population: what do people who prefer cinema and people who prefer football do together? That is, which convention does prevail in mixed interactions when the cost is that low? More generally, how does the cost influence the formation of long-run equilibria?

We answer the above questions, formalizing the example previously made and studying the evolution of conventions in a dynamic setting. We model the coordination problem as a repeated language game [7]: we use evolutionary game theory solution concepts and characterize the long-run equilibrium as the one attained when the perturbations are close to zero (see [8], [6] and [9]). What we do differently from [7] is the following. Agents can play a different action with respect to the type they meet if they pay to know this information. If they do not pay, they play the same action with everyone. This change in the strategic set introduces a new possible perturbation. Agents can mistakenly choose to buy the information and also wrongly choose the action. We model two situations: a complete information scenario, where agents are always able to recognize their opponent's type, and an incomplete information one, where agents can notice the opponent's type conditional on paying a cost. In the latter case, we follow a field of the literature regarding costly acquisition of information (see [10] or [11]). Agents decide myopically their best reply based on the current state, which is always observable. However, in the incomplete information case, a player does not know the type of agent s/he is playing with unless s/he pays for that information.

We define rigidity in preferences as the difference between the payoffs when coordinating on the favorite action and on the least favorite action. Given a level of cost, rigidity in preferences is a crucial driver for long-run results. We find that two different scenarios can happen in an entire population, depending on the cost. Firstly, when the cost is zero or sufficiently low, people always recognize who they are playing with, and they always coordinate. In this case, they also coordinate on their favorite action with agents of their type. The convention preferred the most by the more rigid type prevails in mixed interactions. A second outcome occurs when the cost is

high. In this case, agents never apprehend who they are playing with and play the same action with everyone. They risk coordinating on one action that they do not like with people who share their tastes. Indeed, we find that when one group is more rigid than the other, everybody coordinates on that group's favorite action. Even worse, when the cost is high, the two types may play a heterogeneous strategy profile where they play their favorite action and miscoordinate in mixed interactions. We find that this occurs when both types are rigid in preferences.
It is useful to highlight our analysis with respect to the one proposed by [7], from which we started. When the cost is high enough, our results are the same as the previous model. We further show what happens when agents freely see the opponent's type. Comparing these two cases enrich the previous analysis: in this sense, we prove that miscoordination does not occur without incomplete information and a high cost. Under incomplete information, when the cost is high, if both groups care a lot about their favorite outcome, there is miscoordination in mixed interactions. Nevertheless, rigidity alone does not cause miscoordination. Indeed, when the cost is low, the two types always coordinate.

The paper is organized as follows: in Section 2, we explain the basic features of the model, in Section 3 we determine the results for the complete information case where the cost is 0. In Section 4 we derive the results for the case with incomplete information and costly acquisition. We distinguish between 2 cases: c high enough and c low enough. In Section 5 we discuss results and in Section 6 we conclude. We give all proofs in Appendix A.

2 The Model

	a	b
a	Π_A, Π_A	$0, 0$
b	$0, 0$	π_A, π_A

Table 1: Interactions inside group A.

	a	b
a	π_B, π_B	$0, 0$
b	$0, 0$	Π_B, Π_B

Table 2: Interactions inside group B.

	a	b
a	Π_A, π_B	$0, 0$
b	$0, 0$	π_A, Π_B

Table 3: Mixed Interactions.

We consider a population of N agents divided into two groups A and B, $N = N_A + N_B$. We assume $N_A > N_B + 1$ and $N_B > 1$. Each agent can be of two types: A or B. Each group is homogeneous in the type: each player in group $A(B)$ owns type $A(B)$. Agents are randomly matched in pairs to play the 2x2 coordination game represented in Matrix 1 to 3. Types are heterogeneous in preferences towards the two outcomes of coordination: type A prefers to coordinate on action a, while type B on action b. We assume that $\Pi_B \gtreqless \Pi_A > \pi_A \gtreqless \pi_B$ and $\Pi_A + \pi_A = \Pi_B + \pi_B$. Matrix 1 and 2 represent

inside-group interactions, while Matrix 3 represents inter-group interactions (A type row player and B type column player). Name Ω_i the set containing all the possible strategies for player i. With $\omega_i \in \Omega_i$ being a strategy for player i.
We consider a repeated version of the game. Time is discrete: at each step, every agent plays the strategy s/he is pre-programmed to play with whoever s/he meets. We denote by ω_t^i the strategy played by a player i at time t. States of the world express how many agents are playing action a with who[2]. Each state of the world is a vector of four components. $\bar{X} \in X = (X^{\bar{A}A}, X^{\bar{A}B}, X^{\bar{B}A}, X^{\bar{B}B})$. X^{AA} counts how many A types are playing action a with A types, X^{AB}, how many A types are playing action a with B types, and so on. X_t expresses how many players are choosing action a with who at time t. When appropriate, we use the terms states and strategy profiles as synonyms throughout the paper. We suppose that every player has a correct estimate of each component of the state vector (see [6]): at each step t, $i \in A$ knows X_t^{AA} and X_t^{BA}. X_t^{BB} and X_t^{AB} are known to $j \in B$[3].

Definition 1. *The revision protocol is a process such that at each t some players are selected to revise their strategy.*

Assumption 1. *The revision protocol has inertia if the set of players selected to revise their strategy is strictly lower than N.*

At each round, only some players are selected to revise their strategy. The others keep playing what they were pre-programmed to play before. This property is a common assumption in many works in the literature.

Assumption 2. $\forall M \subset N$ *and* $\forall t$, $p(\theta_{t+1} = M) > \sigma$.

The revision protocol reveals who is selected to revise strategy at each time t. We define the revision protocol with *inertia* as $\Theta = \{\overline{\theta}, p\}$, $\overline{\theta} = \{\theta_1, \theta_2, \dots\}$. θ_{t+1} is the realization at time t (the set of players that is effectively selected to revise the strategy), we assume that at each step θ_{t+1} is generated with the same probability p. With an abuse of notation, we also say that p is the probability on all the possible paths of $\overline{\theta}$.

Assumption 3. *A player is myopic best-replier if in choosing her/his strategy s/he best-respond to the current state of the world.*

We assume that an agent who is picked at time t decides her/his best reply strategy for time $t+1$, looking at information s/he has at time t. That is, agents myopically best reply to the current situation. Note that the strategic choice incorporates the purchasing or not of the information: agents decide at time t if they pay a cost to know the type of who they meet at time $t+1$.

To summarize, we assume three relevant properties: *inertia*, *myopia*, and random matching a là [6]. We assume that a player i does not act strategically: s/he chooses the strategy that maximizes the expected payoffs, given what s/he observes at time t. Take $j \neq i \in N$. $\tilde{\omega}_t^j$ represents the strategy of any player that player i can meet at time

[2]Note that this is the same as saying how many players play strategy aa. We believe that for the sake of clarity is better to keep our notation.

[3]Each agent could have also known the entire vector, but just two values matter to each type.

t. Such a choice is unknown and so, a random variable for i. Consider the case when player i is picked at time t to revise her/his strategy. We name $E\left[y^i_{t+1}(\omega^i_{t+1}, \tilde{\omega}^j_{t+1})\right]$ the expected payoff from choosing strategy ω_i for time $t+1$. Given Assumption 3, player i conditions her/his strategy on the information available at time t, namely, s/he assumes that the strategy profile at time t will not change at time $t+1$. Given random matching a là [6], player i correctly conditions her/his decision on the state at time t. That is, player i correctly estimates $E\left[y^i_{t+1}(\omega^i_{t+1}, \tilde{\omega}^j_{t+1})|X_t\right]$.
From now on, we call $E\left[y^i_{t+1}(\omega^i_{t+1}, \tilde{\omega}^j_{t+1})|X_t\right]$ as $E_t(\omega_i)$. Player i chooses strategy ω_i if $E_t(\omega_i) > E_t(\omega'_i)$, $\forall \omega'_i \neq \omega_i \in \Omega_i$. If the previous equation is expressed with the equal sign, player i assigns equal probability to each strategy (*i.e.* indifferent players randomize).
We clarify here the general scheme of our presentation. For every specification (complete and incomplete information), we divide the analysis into unperturbed (agents choose the best reply strategy with probability 1) and perturbed (agents choose a random strategy with a small probability). First, we help the reader understand how each player evaluates her/his best reply strategy and which states are absorbing. Second, we highlight the general structure of the dynamics with perturbation and then give the long-run result. We provide the proofs of all results in the appendix and their intuition in the main text. In the next section, we analyze the case with complete information, hence, when the cost is zero.

3 Complete Information with Free Acquisition

In this section, we assume that each player can freely recognize the type of her/his opponent when randomly matched with her/him. Without loss of generality, we assume that agents always recognize their opponent in this case. We refer to this condition as free acquisition of information. Each player has 4 possible strategies, namely, $\Omega_i = \{aa, ab, ba, bb\}$, $\forall i \in N$. The first letter refers to the own type of the agent, the second to the other type. If $i \in A(B)$ chooses ab, it means that s/he plays a with type $A(B)$ and b with type $B(A)$. Consider $K = \{A, B\}$ and $K' \neq K, = \{A, B\}$.
Define $\pi^K_a = \begin{cases} \Pi_A & \text{if } K = A \\ \pi_B & \text{if } K = B \end{cases}$ and $\pi^K_b = \begin{cases} \pi_A & \text{if } K = A \\ \Pi_B & \text{if } K = B \end{cases}$.
Equation (1) to (4) are the expected payoffs at time t for time $t+1$, for a player $i \in K$ currently playing aa or ab.

$$E_t(aa) = \frac{N_K - 1}{N-1}\frac{X^{KK}_t - 1}{N_K - 1}\pi^K_a + \frac{N_{K'}}{N-1}\frac{X^{K'K}_t}{N_{K'}}\pi^K_a, \tag{1}$$

$$E_t(ab) = \frac{N_K - 1}{N-1}\frac{X^{KK}_t - 1}{N_K - 1}\pi^K_a + \frac{N_{K'}}{N-1}\frac{N_{K'} - X^{K'K}_t}{N_{K'}}\pi^K_b, \tag{2}$$

$$E_t(ba) = \frac{N_K - 1}{N-1}\frac{N_K - X^{KK}_t}{N_K - 1}\pi^K_b + \frac{N_{K'}}{N-1}\frac{X^{K'K}_t}{N_{K'}}\pi^K_a, \tag{3}$$

$$E_t(bb) = \frac{N_K - 1}{N-1}\frac{N_K - X^{KK}_t}{N_K - 1}\pi^K_b + \frac{N_{K'}}{N-1}\frac{N_{K'} - X^{K'K}_t}{N_{K'}}\pi^K_b. \tag{4}$$

The two ratios $\frac{N_K - 1}{N-1}$ and $\frac{N_{K'}}{N-1}$ express how much frequently a player meets type K or K'. From the above equations we obtain the values of X_t s.t. one strategy or another is the best reply for player i.
Note that when an agent calculates her/his expected payoffs, it does not matter what s/he is doing with the other group, as s/he does not count her/himself in that group.

3.1 Unperturbed Dynamics

We begin the analysis for complete information by studying the dynamics of the system when agents play their best reply strategy with probability one. In this case the law of motion can be expressed as $X_{t+1} = F(X_t, \theta_{t+1})$. What happens at time $t+1$ depends on the state at time t (agents myopically best reply to the current state of the world) and on who is selected to revise her/his strategy.
We can separate the dynamics of the system in 3 different dynamics. The two regarding inside-group interactions *i.e.* X_t^{AA} and X_t^{BB} and the one regarding mixed interaction, *i.e.* X_t^{AB} and X_t^{BA}. We call this subset of states $X_t^I = \left(X_t^{AB}, X_t^{BA}\right)$.

Lemma 1. *Under free acquisition of information,* $\forall\bar{\theta}$, $X_{t+1}^{AA} = F_1(X_t^{AA}, \theta_{t+1})$, $X_{t+1}^{BB} = F_4(X_t^{BB}, \theta_{t+1})$ *and* $(X_{t+1}^{AB}, X_{t+1}^{BA}) = F_{2,3}(X_t^{AB}, X_t^{BA}, \theta_{t+1})$.

If the cost is zero, people always acknowledge the type of their opponent. They can separate what they do with their same type from what they do with the other one. For example, take two different ethnicities: they likely differ in preferences, and each ethnicity is freely recognizable by some details (*e.g.*, way of dressing or skin color). Thanks to this result, we can separate Equation (1) to (4) into two parts, the one regarding group K and the one regarding group K'[4]. Note that both X_t^{AA} and X_t^{BB} are one-dimensional dynamics that measure how many players of type $A(B)$ are playing action a with players of their type. X_t^I instead is a two-dimensional dynamics, measuring how many players are playing action a with players of the other type.

Now we search for the absorbing state, that are the unique candidate as long-run equilibria under the unperturbed dynamics.
Consider a subset of 8 states: $X^R = \{(N_A, N_A, N_B, N_B)$, $(0, N_A, N_B, N_B)$, $(N_A, N_A, N_B, 0)$, $(N_A, 0, 0, N_B)$, $(0, N_A, N_B, 0)$, $(N_A, 0, 0, 0)$, $(0, 0, 0, N_B)$ and $(0, 0, 0, 0)\}$.
We call (N_A, N_A, N_B, N_B) and $(0, 0, 0, 0)$ Monomorphic States (MS from now on). Specifically, we refer to the first one as MS_a and to the second as MS_b. We label the remaining six as Polymorphic States (PS from now on). We call $(N_A, N_A, N_B, 0)$ as PS_a and $(N_A, 0, 0, 0)$ as PS_b. In MS, everybody plays the same action with everyone; in PS, at least one type is differentiating the action. In MS_a everybody is playing aa, in MS_b everybody is playing bb. In PS_a A type is playing aa and B type is playing ba. In PS_b, A type is playing ab while B type is playing bb. Both in PS_a and PS_b all players coordinate on their favorite action with their similar. MS, $(N_A, 0, 0, N_B)$ and $(0, N_A, N_B, 0)$ are homogeneous strategy profiles. All the other PS are heterogeneous strategy profiles where agents coordinate in mixed interactions.

Lemma 2. *Under free acquisition of information, if Assumption 1 to 3 hold, the states in* X^R *are the unique absorbing states of the system.*

[4]The first element in the RHS of each equation represents what is optimal to do with group K. The second term what is optimal to do with group K'. If we treat them separately, it does not matter how much frequently one agent meets one type or the other.

We can break these absorbing states into the three dynamics in which we are interested. This simplification helps in understanding why only these states are absorbing. For instance in mixed interactions there are just two possible absorbing states, namely (N_A, N_B) and $(0, 0)$. For what concerns interactions within the same types, N_A and 0 matters for X_t^{AA}, N_B and 0 for X_t^{BB}. Therefore, for each dynamics, the states where everybody plays a or where everybody plays b with one type are absorbing. In this simplification, we can see the importance of Lemma 1. As a matter of fact, in all the dynamics we are studying, there are just two candidates for long-run equilibrium. In the next section, we introduce the perturbations: agents experiment, meaning that they choose a random strategy with a small probability, and then we estimate the long-run conventions.

3.2 Perturbed Dynamics

We now introduce perturbations in the model presented in the previous section. We use tools and concepts developed by [12], [8] and [9]. Agents can experiment while choosing their strategies: there is a small probability that an agent does not choose her/his best response strategy when s/he is picked to do so. We use the uniform error model for mistakes: the probability of experimenting is equal for every agent and every state of the world. We call $\Psi = \left\{\overline{\psi}, \varepsilon\right\}$ the process that that reveals the agents that experiment. $\overline{\psi} = \{\psi_1, \psi_2, \dots\}$ is the realisation of the process at each step. ψ_{t+1} is the set of players who actually experiments at time t, ε is the probability that each player among the picked to revise strategy experiments. Ψ is an *iid* process as a consequence of the uniform mistakes assumptions. What happens at time $t+1$ depends on which players are extracted to revise action and on who among them commits an error. We can think at the dynamics of the system in the following way: $X_{t+1} = F(X_t, \theta_{t+1}, \psi_{t+1})$. Note that Lemma 1 is still valid under this specification.

Concerning long-run equilibria, if we consider a sequence of transition matrices $\{P^\varepsilon\}_{\varepsilon>0}$, with associated stationary distributions $\{\mu^\varepsilon\}_{\varepsilon>0}$, by continuity the accumulation point of $\{\mu^\varepsilon\}_{\varepsilon>0}$ that we call $\mu^\star$, is a stationary distribution of $P := \lim_{\varepsilon\to 0} P^\varepsilon$. Mutations guarantee the ergodicity of the Markov process and the uniqueness of the invariant distribution. We are interested in states which have positive probability in $\mu^\star$.

Definition 2. *A state $\bar{X}$ is stochastically stable if $\mu^\star(\bar{X}) > 0$ and it is uniquely stochastically stable if $\mu^\star(\bar{X}) = 1$.*

Following tree surgery arguments techniques ([12],[9]), we calculate the stochastically stable state as the one with the minimum stochastic potential. The stochastic potential is calculated as the minimum path in terms of errors to reach one absorbing state, starting from all the other states. In our case, thanks to Lemma 1 we can consider dynamics with just two absorbing states. If a state achieves the minimum stochastic potential, it is the uniquely stochastically stable state, our long-run convention. Such a state is the one that is more likely to happen in the long-run since fewer mutations are needed to reach it. If two states have equal stochastic potentials, they are both stochastically stable.

We now define formally the concept of resistances, crucial to understand the stochastic stability analysis. Consider all the states of X as vertexes of a fully connected directed

graph, for every pair of states X_t, X_s call $X_t \to X_s$, the edge from state X_t to state X_s. Call $\upsilon(X_t, \dots, X_s)$ a path from state X_t to state X_s, where a path is the series of edges that link one state to another: $\upsilon(X_t, \dots, X_s) = \{X_{t'} \to X_{t'+1}\}_{t'=0}^{s'-1}$, $X_0 = X_t$ and $X_{s'} = X_s$, $\Upsilon(X_t, \dots, X_s)$ is the set of all paths from state X_t to state X_s.
Define $r(X_t, \dots, X_s)$ as the resistance from state X_t to state X_s, meaning the total number of mistakes needed to reach X_s starting from X_t. A resistance between two states equal to zero corresponds to the transitions that happen with positive probability in the unperturbed dynamics.
We define the basin of attraction of an absorbing state as the set of states from which the system converges to the absorbing state with probability one, under unperturbed dynamics. Consider a situation where there are just two absorbing states $\bar{X}$ and $\bar{X}'$. Thanks to [9] we know that the stochastic potential of $\bar{X}$ is the minimum resistance path that starts from $\bar{X}'$ and exits from its basin of attraction.
We are ready to give the result about the long-run equilibrium under complete information.

Theorem 1. *Under free acquisition of information, if Assumption 1 to 3 hold, for N large enough, if $\frac{\pi_B}{\pi_A} < \frac{N_B}{N_A}$, then PS_b is uniquely stochastically stable. If $\frac{\pi_B}{\pi_A} > \frac{N_B}{N_A}$, then PS_a is uniquely stochastically stable.*

When the cost is null, only Polymorphic States can become long-run conventions. The intuition behind the result is the following. When agents always recognize who they are playing with, they always coordinate on their favorite action with agents of their type. Therefore, X_t^{AA} always converge to N_A and X_t^{BB} always converge to 0. This result rules out 6 of the 8 absorbing states in Lemma 2: only PS_a and PS_b are left. Which of the two is selected depends on the ratio between the payoffs of the two types. We now have the opportunity to explain the concept of rigidity in preferences. If $\Pi_A - \pi_A > \Pi_B - \pi_B$, we say that group A is more rigid than group B: the first value less to coordinate on its not favorite action than the latter. Since $\Pi_A + \pi_A = \Pi_B + \pi_B$, if $\pi_A < \pi_B$, type A is more rigid than type B. If this happens, the cost of exiting from the state where everybody is playing a in mixed interactions is higher than the one of exiting from the state where everybody is playing b. Hence, in mixed interactions, everybody plays a. The opposite happens if type B is enough more rigid than type A. When the cost is zero, we can conclude that whenever a type cares more about coordinating on her/his favorite action than the other, the population coordinates on that action.

It is important to note two things. First, the rigidity of the payoffs must be intended relatively to groups size and vice-versa. Second, the fact that B is the minority group is not crucial, as agents do not look at the entire population when deciding what to do with the other type. If both types are equally rigid in preferences, they coordinate on the outcome preferred by the majority: this is a disadvantage of being the minority group.

We provide two numerical examples to explain how the model works in Figure 1 and 2. We represent just X_t^I, hence, a two-dimensional dynamics[5]. Red states represent the basin of attraction of $(0, 0)$, while green states the one of (N_A, N_B). From grey states the system can reach with paths of resistance 0 both $(0, 0)$ and (N_A, N_B). Any path

[5]For a more exhaustive treatment of lattices, see Appendix A of [7].

that involves more players playing a within red states has a positive resistance. Every path that involves fewer people playing a within green states has a positive resistance. The stochastic potential of (N_A, N_B) is the shortest path which links $(0,0)$ to grey states. Vice-versa for the stochastic potential of $(0,0)$.

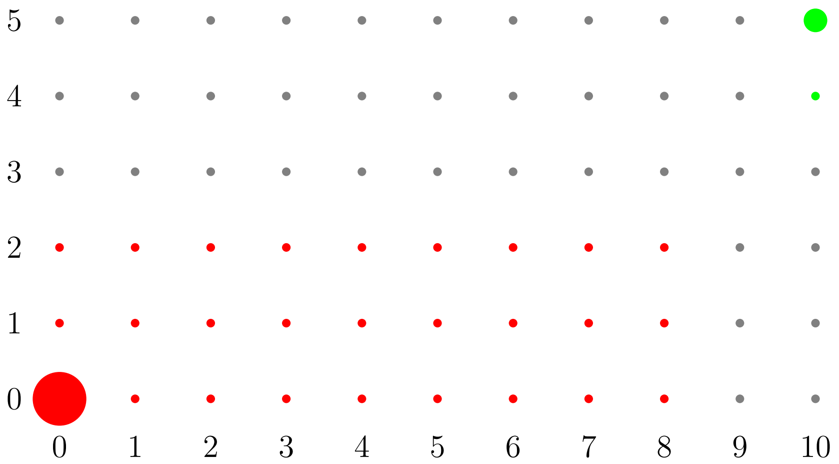

Figure 1: $PS_b = (0,0)$ is uniquely stochastically stable: $\frac{\pi_B}{\pi_A} < \frac{N_B}{N_A}$.

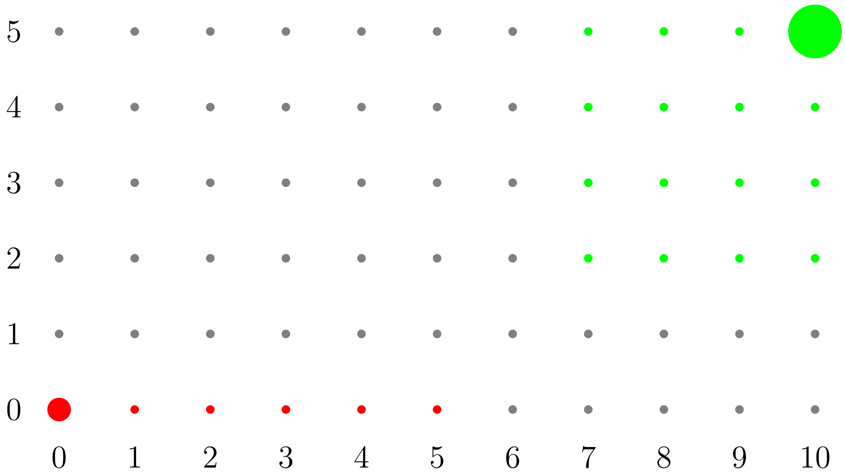

Figure 2: $PS_a = (10,5)$ is uniquely stochastically stable: $\frac{\pi_B}{\pi_A} > \frac{N_B}{N_A}$.

Firstly, consider the case of Figure 1. $N_A = 10$, $N_B = 5$, $\pi_A = 8$, $\Pi_A = 10$, $\pi_B = 3$, $\Pi_B = 15$. Clearly, $\frac{\pi_B}{\pi_A} = \frac{3}{8} < \frac{5}{10} = \frac{N_B}{N_A}$. In this case the stochastic potential of $(10,5)$ is 3, while the stochastic potential of $(0,0)$ is 1. Hence, $(0,0)$ is the uniquely stochastically stable state. We give here a short intuition. Starting from $(0,0)$, it is clear that the shortest way to reach grey states is moving to $(0,3)$. The shortest way to reach grey states from $(10,5)$ is moving to $(9,5)$. Hence, it is easy to exit from the green states, while it is hard to exit the red states. This is why $PS_b = (10,0,0,0)$ is uniquely stochastically stable.

Secondly, consider the case of Figure 2. $N_A = 10$, $N_B = 5$, $\pi_A = 3$, $\Pi_A = 15$, $\pi_B = 8$, $\Pi_B = 10$. Note that $\frac{\pi_B}{\pi_A} = \frac{8}{3} > \frac{5}{10} = \frac{N_B}{N_A}$. In this case the stochastic potential of $(10,5)$ is 1, while the stochastic potential of $(0,0)$ is 4. Hence, $(10,5)$ is the uniquely stochastically stable state. In this case, the easiest way to exit green states is to reach $(6,5)$ or $(10,1)$, while the one to exit the red states is to reach $(0,1)$. Clearly, $PS_a = (10,10,5,0)$ is uniquely stochastically stable in this case.

4 Incomplete Information with Costly Acquisition

In this section, we assume that each player can not freely recognize the type of her/his opponent when randomly matched with her/him. Each agent can buy this information at a strictly positive cost. We refer to this condition as costly acquisition of information[6]. This time $\Omega_i = \{a, b, ab, ba, aa, bb\}$, $\forall i \in N$. If player i chooses ab, s/he pays the cost with every player s/he meets: s/he chooses action a against a player of its type and b against the other types. If s/he chooses strategy a, s/he does not pay the cost and play action action a with everyone. It is trivial to show that there are 4 optimizing behavior out of the 6 strategies, indeed, $E_t(aa) = E_t(a) - c$ and $E_t(bb) = E_t(b) - c$. Hence, $E_t(aa) < E_t(a)$ and $E_t(bb) < E_t(b)$, $\forall c > 0$. We define optimizing behaviors $\Omega_i^o = \{a, b, ab, ba\}$, $\forall i \in N$, with ω_i^o being an optimizing behavior profile of player i.
Consider $K = \{A, B\}$ and $K' \neq K, = \{A, B\}$. Equation (5) to (8) are the expected payoffs at time t for time $t+1$, for a player $i \in K$ currently playing a or ab.

$$E_t(a) = \frac{X_t^{KK} + X_t^{K'K} - 1}{N-1}\pi_a^K, \tag{5}$$

$$E_t(b) = \frac{N - X_t^{KK} - X_t^{K'K}}{N-1}\pi_b^K, \tag{6}$$

$$E_t(ab) = \frac{N_K - 1}{N-1}\frac{X_t^{KK} - 1}{N_K - 1}\pi_a^K + \frac{N_{K'}}{N-1}\frac{N_{K'} - X_t^{K'K}}{N_{K'}}\pi_b^K - c, \tag{7}$$

$$E_t(ba) = \frac{N_K - 1}{N-1}\frac{N_K - X_t^{KK}}{N_K - 1}\pi_b^K + \frac{N_{K'}}{N-1}\frac{X_t^{K'K}}{N_{K'}}\pi_a^K - c. \tag{8}$$

To help the reader visualize the differences between this section and Section 3, we did not explicit in Equation (5) and (6) the frequencies of meetings. Note that if $c = 0$, then $aa = a$ and $bb = b$. From these expressions, we obtain values of X_t and c such that each strategy is the best reply for player i at time t.
We begin the analysis again with the unperturbed dynamics, where agents choose their best reply strategy with probability one.

4.1 Unperturbed Dynamics

So far, there are no more random elements with respect to Section 3. What happens at time $t+1$ is described by the function F_c, denoted in this way to indicate that the cost is now strictly positive. $X_{t+1} = F_c(X_t, \theta_{t+1})$.
We identify absorbing states. We show that just nine states can be absorbing under this specification.

[6]It is trivial to notice that Lemma 1 is not valid anymore. Indeed, since agents learn the type of their opponent conditional on paying a cost, not every player pays it, and the dynamics are no longer separable

State	Condition on $\frac{N_B}{N_A}$	Conditions on c
MS_a	none	none
MS_b	none	none
TS	$\frac{\pi_B}{\Pi_B} < \frac{N_B-1}{N_A}$	$c > \max\left\{\frac{N_B}{N-1}\pi_A, \frac{N_A}{N-1}\pi_B\right\}$
PS_b	none	$c < \frac{N_B}{N-1}\pi_A$
PS_a	1) $\frac{\pi_B}{\Pi_B} > \frac{N_B-1}{N_A}$ 2) $\frac{\pi_B}{\Pi_B} < \frac{N_B-1}{N_A}$	1) $c < \frac{N_B-1}{N-1}\Pi_B$ 2) $c < \frac{N_A}{N-1}\pi_B$
$(0, N_A, N_B, N_B)$	1) $\frac{\pi_A}{\Pi_A} < \frac{N_B}{N_A-1}$ 2) $\frac{\pi_A}{\Pi_A} > \frac{N_B}{N_A-1}$	1) $c < \frac{N_A-1}{N-1}\pi_A$ 2) $c < \frac{N_B}{N-1}\Pi_A$
$(N_A, 0, 0, N_B)$	none	$c < \min\left\{\frac{N_B}{N-1}\pi_A, \frac{N_B-1}{N-1}\pi_B\right\}$
$(0, N_A, N_B, 0)$	1) $\frac{\pi_A}{\Pi_A} < \frac{N_A-1}{N_B}$ and $\frac{\pi_B}{\Pi_B} > \frac{N_B-1}{N_A}$ 2) $\frac{\pi_A}{\Pi_A} > \frac{N_A-1}{N_B}$ and $\frac{\pi_B}{\Pi_B} > \frac{N_B-1}{N_A}$ 3) $\frac{\pi_A}{\Pi_A} < \frac{N_A-1}{N_B}$ and $\frac{\pi_B}{\Pi_B} < \frac{N_B-1}{N_A}$ 4) $\frac{\pi_A}{\Pi_A} > \frac{N_A-1}{N_B}$ and $\frac{\pi_B}{\Pi_B} < \frac{N_B-1}{N_A}$	1) $c < \min\left\{\frac{N_A-1}{N-1}\pi_A, \frac{N_B-1}{N-1}\Pi_B\right\}$ 2) $c < \min\left\{\frac{N_B}{N-1}\Pi_A, \frac{N_B-1}{N-1}\Pi_B\right\}$ 3) $c < \min\left\{\frac{N_A-1}{N-1}\pi_A, \frac{N_A}{N-1}\pi_B\right\}$ 4) $c < \min\left\{\frac{N_B}{N-1}\Pi_A, \frac{N_A}{N-1}\pi_B\right\}$
$(0, 0, 0, N_B)$	none	$c < \frac{N_B-1}{N-1}\pi_B$

Table 4: Necessary and Sufficient Conditions for strict equilibria and absorbing states.

Lemma 3. *Under costly acquisition of information, if Assumption 1 to 3 hold, there are nine possible strict equilibria (conventions):* $X^R \cup (N_A, N_A, 0, 0)$. *The necessary and sufficient condition to be an absorbing state is to be a strict equilibrium.*

All the states that were absorbing under complete information can still be absorbing now. The only difference is that, in this section, some of them are strict equilibria and absorbing under certain conditions: this is why we split the conditions for being strict equilibrium to the one for being absorbing state. There is one more state: $(N_A, N_A, 0, 0)$, a state where there is a heterogeneous strategy profile, and all players are playing their favorite action, miscoordinating in mixed interactions. We call such a state as Type Monomorphic State (TS from now on).
We summarize all the relevant information in Table 4. As we can see from that table, Monomorphic States are always absorbing states for whatever value of the cost and the payoffs. Indeed, when everyone is playing the same action with everyone, it is convenient for everyone to play that action. It is different for the Type Monomorphic State. If the cost is not sufficiently high, every player differentiates the action, and if type B players are not sufficiently rigid, then they play strategy a in TS. For the Polymorphic States, the intuition is the opposite: if the cost is not sufficiently low, agents do not pay the cost and do not differentiate the action.

4.2 Perturbed Dynamics

We now introduce the perturbed dynamics as we did for Section 3. In this case, agents can make two types of errors: they can make a mistake in buying the information and playing the action. We use the word action because the strategy combines the decision on the information acquisition and the action. Therefore, choosing the wrong strategy, in this case, can mean both. Define $\Psi^c = \left\{\overline{\psi^c}, \varepsilon, \eta\right\}$. We assume again uniform mistakes, hence, the two probabilities ε and η are independent each other

and they have the same weight. The first represents the probability of choosing the wrong action, the second the probability of acquiring wrongly the information. $\overline{\psi^c} = \{\psi_1^c, \psi_2^c, \dots\}$ is the realisation of the process at each step. $\overline{\psi^c}$ is a couple of independent variables $\overline{\psi^\eta}$ and $\overline{\psi^\varepsilon}$. Therefore, $\psi_{t+1}^c = \{\psi_{t+1}^\varepsilon, \psi_{t+1}^\eta\}$. ψ_{t+1}^ε is the set of players that wrongly choose the action at time t, and ψ_{t+1}^η is the set of players that wrongly choose to buy the information at time t. The system can be described as following: $X_{t+1} = F_c(X_t, \theta_{t+1}, \psi_{t+1}^c)$.
This new dynamical system points out another important difference with respect to the previous part. In Section 3.2, we introduced $r(X_t, \dots, X_s)$ as the resistance of the path from state X_t to state X_s (how many errors to pass from one state to another). Since there are two types of errors, the concept of resistance changes, we then need to consider three types of resistances.
We call $r_\varepsilon(X_t, \dots, X_s)$ the path from state X_t to state X_s that considers just ε errors (players wrongly choose the action). We call $r_\eta(X_t, \dots, X_s)$ the path that considers only η errors (players wrongly choose to buy the information). Finally, we call $r_{\varepsilon\eta}(X_t, \dots, X_s)$ the path that considers players mistakenly choosing both the action and the purchase of the information. Since we do not make further assumptions on ε and η (probability of making errors uniformly distributed), we can assume $\eta \propto \varepsilon$, meaning

$$\lim_{\varepsilon \to 0} \frac{\eta}{\varepsilon} \in [\underline{a}, \bar{a}], \quad \bar{a} > \underline{a}, \quad \bar{a}, \underline{a} \in \mathbb{R}.$$

We count each error in the path of both ε and η errors as 1, however, $r_{\varepsilon\eta}(X_t, \dots, X_s)$ is always double since it implies a double error. Indeed, we can see this kind of error as the sum of two components, one in η and the other in ε, namely $r_{\varepsilon\eta}(X_t, \dots, X_s) = r_{\varepsilon\eta|_\varepsilon}(X_t, \dots, X_s) + r_{\varepsilon\eta|_\eta}(X_t, \dots, X_s)$.
We explain this difference with an example. Firstly, think about being in TS: if one player from A wrongly choose to buy the information, this player will correctly choose ab, and the system reaches the state $(N_A, N_A - 1, 0, 0)$. Hence, $r_\eta(TS, \dots, (N_A, N_A - 1, 0, 0)) = 1$. Secondly, think about being in MS_a: if one player from B wrongly choose to buy the information and wrongly plays ab, the system reaches the state $(N_A, N_A, N_B - 1, N_B)$. Hence, $r_{\varepsilon\eta}(MS_a, \dots, (N_A, N_A, N_B - 1, N_B)) = 2$. With a similar reasoning, $r_\varepsilon(MS_a, \dots, (N_A - 1, N_A - 1, N_B, N_B)) = 1$, an A player wrongly choose b from MS_a.

In this section we use techniques developed by [13]. Specifically, we use three concepts: Radius, Coradius, and Modified Coradius. The Radius is the minimum number of mutations needed to leave the basin of attraction of one absorbing state. The Coradius can be defined as the maximum path, overall the initial states of minimum resistance paths, starting from any states and ending in the basin of attraction of one absorbing state. The Modified Coradius is defined similarly to the Coradius, except for the fact that we subtract from the Coradius the cost of going from intermediate absorbing state to the starting absorbing state[7]. A state is uniquely stochastically stable if its Radius is strictly bigger than its Coradius (Modified Coradius). We divide this part of the analysis into two cases, the first one where the cost is high and the second one when the cost is low.

[7]Formal definitions are given in the original paper.

4.2.1 High Cost

In this part of the analysis, we focus on a case when only MS and TS are absorbing states. This happens because we arbitrarily choose a level of c high enough, that it is never optimal to buy the information for agents[8]. Therefore, PS can not be long-run conventions under this specification.

Define the following set of values:

$$\Xi_{PS} = \left\{ \frac{N_B}{N-1}\pi_A, \frac{N_A}{N-1}\pi_B, \frac{N_B-1}{N-1}\Pi_B, \frac{N_A-1}{N-1}\pi_A, \frac{N_B}{N-1}\Pi_A, \frac{N_B-1}{N-1}\pi_B \right\}.$$

Corollary 1. *Under costly acquisition of information, if Assumption 1 to 3 hold, if $c > argmax\{\Xi_{PS}\}$ and $\frac{\pi_B}{\Pi_B} < \frac{N_B-1}{N_A}$, then no PS is a strict equilibrium, and both MS and TS are absorbing states. If $\frac{\pi_B}{\Pi_B} \geq \frac{N_B-1}{N_A}$, then only MS are strict equilibria and absorbing states.*

The proof is straightforward from Table 4 and therefore, we omit it. We previously give the intuition behind this result: if the cost is that high, nobody ever pays to know the opponent's type, the only candidates to be stochastically stable are the two MS and TS. Let us firstly consider the case in which TS is not an absorbing state, hence, the case when $\frac{\pi_B}{\Pi_B} \geq \frac{N_B-1}{N_A}$.

Theorem 2. *Under costly acquisition of information, if Assumption 1 to 3 hold, for N large enough, take $\frac{\pi_B}{\Pi_B} \geq \frac{N_B-1}{N_A}$ and $c > argmax\{\Xi_{PS}\}$. If $N_A > \frac{2N\pi_A+\Pi_A-\pi_A}{\Pi_A+\pi_A}$, then MS_a is uniquely stochastically stable. If $N_A < \frac{2N\pi_A+\Pi_A-\pi_A}{\Pi_A+\pi_A}$, then MS_b is uniquely stochastically stable.*

We give the intuition behind it and the explanation of the result. If $\frac{\pi_B}{\Pi_B}$ is high enough, B type is not rigid, and TS is nor a strict equilibrium or an absorbing state. Therefore, the result depends on type A. When group A is neither large enough nor rigid enough, the system can end up in a situation where everybody plays strategy b. Nobody pays the cost and play action b with everyone. If group A is large enough or rigid enough, everybody plays strategy a in the long-run.
Now we analyze the case when TS is a strict equilibrium.

Theorem 3. *Under costly acquisition of information, if Assumption 1 to 3 hold, for N large enough, take $\frac{\pi_B}{\Pi_B} < \frac{N_B-1}{N_A}$ and $c > argmax\{\Xi_{PS}\}$.*

- *If $N(\pi_B - \pi_A) > N_B\Pi_B - N_A\pi_B - \Pi_B + \pi_B + \Pi_A$, then MS_a is uniquely stochastically stable.*
- *If $N(\pi_A - \pi_B) > N_A\Pi_A - N_B\pi_A - \Pi_A + \Pi_B + \pi_A$, then MS_b is uniquely stochastically stable.*
- *If $\min\{N_A\Pi_A - N_B\pi_A + \pi_A, N_B\Pi_B - N_A\pi_B + \pi_B\} - \Pi_A - \Pi_B > N(\pi_A + \pi_B)$, then TS is uniquely stochastically stable.*

[8]It is possible to prove this result formally. Since we do not need it to prove theorems, we do not show it here.

Moreover when the conditions of the following system hold

$$\begin{cases} N\left(\pi_B - \pi_A\right) \leq N_B \Pi_B - N_A \pi_B - \Pi_B + \pi_B + \Pi_A \\ N\left(\pi_A - \pi_B\right) \leq N_A \Pi_A - N_B \pi_A - \Pi_A + \Pi_B + \pi_A \\ \min\left\{N_A \Pi_A - N_B \pi_A + \pi_A, N_B \Pi_B - N_A \pi_B + \pi_B\right\} - \Pi_A - \Pi_B \leq N\left(\pi_A + \pi_B\right) \end{cases}$$

- *If* $N\left(\pi_B - \pi_A\right) > N_B(\Pi_B + \pi_A) - N_A(\Pi_A + \pi_B) + \Pi_A - \pi_A + \pi_B - \Pi_B$*, then* MS_a *is uniquely stochastically stable.*
- *If* $N\left(\pi_A - \pi_B\right) > N_A(\Pi_A + \pi_B) - N_B(\Pi_B + \pi_A) - \Pi_A + \pi_A - \pi_B + \Pi_B$*, then* MS_b *is uniquely stochastically stable.*
- *If* $N\left(\pi_A - \pi_B\right) = N_A(\Pi_A + \pi_B) - N_B(\Pi_B + \pi_A) - \Pi_A + \pi_A - \pi_B + \Pi_B$*, then both* MS_a *and* MS_b *are stochastically stable.*

We divide the statement of the theorem into two parts for technical reasons. Since we can not rule out every possibility with the first three cases, we need to add the case where the first three conditions fail simultaneously. However, the reader can understand the result just by looking at the first three conditions. If $(\pi_B - \pi_A)$ is high enough, it means that type A is by far more rigid than type B. Hence, there is an asymmetry in the cost of exiting the two basins of attraction of MS_a and MS_b. Exit from the first is harder than exit from the second. Moreover, reaching MS_a from TS is easier than reaching MS_b from TS. This happens because type B is more willing to play strategy a than how much A type is willing to play strategy b. This is why $R(MS_a) > CR(MS_a)$ and MS_a becomes the long-run equilibrium in this case. The opposite happens if $(\pi_A - \pi_B)$ is high enough. The third bullet point is a case where both types are rigid in preferences. When this happens, A type is willing to play b as much as type B is willing to play a. It is relatively easy to reach TS starting from both MS_a and MS_b, but relatively hard to exit from TS. In this case $R(TS) > CR(TS)$. Indeed, TS is the state where both types are playing their favorite action, and, given their rigidity, they do not want to give up on playing that action. We conclude from this section that two conditions must hold for miscoordination to happen in the long-run. First, the cost to pay to know the opponent's type should be so high to make agents desist from differentiating the action. Second, both types should be so rigid that they bear the risk of miscoordinating to play their favorite action.

4.2.2 Low Cost

In this section, we discuss the case when c is as low as possible but greater than 0.

Corollary 2. *Under costly acquisition of information, if Assumption 1 to 3 hold, if* $0 < c < \min\{\frac{\pi_A}{N-1}, \frac{\pi_B}{N-1}\}$*,* MS *and* PS *are absorbing states, while* TS *is not an absorbing state.*

The proof is straightforward as a consequence of Table 4. In this case, we are left with 8 absorbing states, hence, 8 possible stochastically stable equilibria.

Theorem 4. *Under costly acquisition of information, if Assumption 1 to 3 hold, for N large enough, take $0 < c < \min\{\frac{\pi_A}{N-1}, \frac{\pi_B}{N-1}\}$. If $\frac{\pi_B}{\pi_A} < \frac{N_B}{N_A}$, then PS_b is uniquely stochastically stable. If $\frac{\pi_B}{\pi_A} > \frac{N_B}{N_A}$, then PS_a is uniquely stochastically stable.*

The intuition is quite similar to the one of Theorem 1: the conditions are the same. To understand why, it is sufficient to explain the role of the cost. When the cost is low enough, whenever a player can buy the information, s/he does it. This fact enlarges the basins of attraction of all the PS until they reach the dimension they had in Section 3. Nevertheless, the basins of attraction of both MS are scaled-down as the cost decreases. Indeed, there are more states in which it is optimal to buy the information. Among the six PS, there are just two candidates likely to be selected in the long-run: these are the two states where agents coordinate on their favorite action with their similar. These two states are preferred in this case also to MS. When the cost is low, agents always prefer to pay the cost and differentiate the action. In this way, they can play their favorite action with their type.

5 Discussion

We must intend the results of our model at the intersection between three fields of the literature. Firstly, we contribute to the literature on social conventions. Secondly, we contribute to the literature regarding stochastic stability analysis, and lastly, we contribute to the literature on costly acquisition of information.
For what concerns social conventions, we contribute to the literature on language games or, more generally, to the one on coordination games. Many works in this field are concerned about the existence in the long-run of heterogeneous strategy profiles. We started from the original model of [7], which considers agents heterogeneous in preferences, but with a smaller strategic set. His work gives conditions for the stochastic stability of heterogeneous strategy profiles that causes miscoordination in mixed interactions in a random matching case[9]. [19] expands the previous idea to investigate the role of different classes of graphs on the long-run result. They find conditions on graphs such that a heterogeneous strategy profile is stochastically stable. They also consider the choice of a social planner that wants to induce heterogeneous or homogeneous behavior in a population. [20] considers a similar model, where agents choose their actions from a set of culturally constrained possibilities. The author gives a negative connotation to the heterogeneous strategy profile since he labels it as miscoordination. He finds that cultural constraints are a crucial driver for miscoordination. [21] studies a game between agents with heterogeneous preferences and who feel pressure from behaving differently. They characterizes the circumstances under which a biased norm can prevail on a non-biased norm. [22] studies how local dialects survive in a society with an official language. [23] studies the evolution of egalitarian and inegalitarian conventions. To do so, they consider a framework with asymmetry similar to the language game. Similarly, [24] examines the evolution and the persistence of inferior cultural conventions.
We give the conditions for the stability of two types of heterogeneous strategy profiles: those where agents miscoordinate in mixed interactions and those where they coor-

[9] Heterogeneity has been discussed in previous works such as [14], [15], [16], [17] or [18].

dinate. We show that incomplete information, high cost, and rigidity in preferences are key drivers for miscoordination.
Many works propose a version of the language game in a network context. [25] experiments the language game testing whether agents segregate or conform to the majority. [26] suggests a variant of the language game similar to our version but in a model with networks to study the influence of partner-specific behavior on coordination outcomes. Concerning auctions theory, [27] studies a framework where each individual of a population divided into two types has to choose between two skills: a "majority" and a "minority" one. She finds that minorities are advantaged in competition context rather than in coordination one. [28] tests the role of compromise in the battle of sexes with an experiment. A parallel field is the one of bilingual games such as the one proposed by [29] or [30]: these models consider situations in which agents are homogeneous in preferences but can become bilingual at a given cost.

[31] has recently suggested the applicability of tipping points theories to policy and interventions. In my opinion, this could be a field from which drawing explanations and further research questions using language games (see [19] again). Indeed, in our model, there are situations in which the majority conforms to the action preferred by the minority. This fact happens even in inside-group interactions.

Concerning the technical literature on stochastic stability, we contribute by applying standard techniques of stochastic stability to an atypical context such as the costly acquisition of information. Since the seminal works by [32], and [33], many studies have focused on testing the role of different error models in the selection of the equilibrium. We used a uniform error model, and we think that introducing different models could be an interesting exercise for further studies. Among the many models that can be used, there are four relevant variants: payoff-dependent mistakes ([34], [35] and [36]), cost-dependent mistakes ([37] and [38]), intentional mistakes ([39] and [40]) and condition dependent mistakes ([41]). Important experimental works in this literature have been done by [42], [43], [44], and [45].

Other works contribute to the literature on stochastic stability from the theoretical point of view (see [46] for an exhaustive review of the field). Recently, [47] has expanded the domain of behavioral rules regarding the result of stochastic stability. [48] shows that with loss aversion individuals, the stochastic stability of risk dominant equilibria is no longer guaranteed. [49] analyzes the stochastic stability of best response dynamics introducing reference-dependent preferences. [50] examines stochastic stability in an asymmetric binary choice coordination game.
As a hint for further research, we would like to break down a strong assumption in many evolutionary game theory models, such as the random matching a là [6]. In our model, every agent is correctly able to estimate the current state of the world. [51] is one of the most famous works breaking down this property. It would be interesting to go deep into this assumption and modify the way agents estimate the state of the world. Such an analysis would exploit the role of bounded rationality in the formation of social conventions. Specifically, we are interested in applying models such as the one of [52].

[53] introduces the possibility that agents can establish costly links with other players. The main finding is that if a small number of players play the Payoff-Dominant action, other players connect with them and play the Payoff-Dominant action. The work by

[11] extends [53]. They firstly twine evolutionary game theory and costly interactions. This model introduces the fact that interacting with a different type might be costly for an agent. They find that when the cost is low, the Payoff-Dominant strategy is also the stochastically stable one. When the cost is high, the two types in the population coordinate on two different strategies. One on the Risk-Dominant and the other on the Payoff-Dominant. Similarly, [54] studies the role of cultural intolerance and assortativity in a coordination context. They divide the population into two cultural groups who sustain a cost from interacting with the other group. They find interesting conditions under which cooperation can emerge even with cultural intolerance.
Concerning evolutionary game theory literature, we contribute by showing the effect of incomplete information in a context where agents are heterogeneous in preferences. We contribute to the literature regarding costly acquisition of information by applying that model to an evolutionary game theory setting. Many works have contributed to this field of literature recently: few of them using evolutionary game theory. Many works place this framework in a sender-receiver game. This is the case of [10], who builds a model of costly communication in a sender-receiver setup. More recently in this literature are [55], [56], [57] and [58]. [59] applies this model to persuasion games with labeling, [60] studies these kinds of models to dual-process theories in psychology. Finally, [61], is the first to apply costly acquisition of information to the analogy-based reasoning theory developed by [52].

6 Conclusions

We can summarize our results as follows. When agents easily recognize the type of other people, they always coordinate. They play their favorite action with their similar, while in mixed interactions, they play the favorite action of the type that is more rigid in preferences. If the cost is high, agents never learn the type of their opponents. In this case, either all the agents play the same action with everyone, or all the agents play their favorite action.
Before we discuss the difference between the case with low cost and high cost, we can briefly deepen into the results of Section 3. Indeed, we can interpret the case with null cost as a situation where agents can always recognize other people's type. Each type is recognizable thanks to a specific characteristic. In this way, we can use our contribution to interpreted conventions between two different ethnicities. Indeed, one type can easily recognize the other, and they likely differ in preferences. We are aware that this kind of works may be better in describing short-medium term behaviors: for example, when two different cultures/ethnicities meet for the first time[10]. What we find is both in line with what we expected and with previous literature. Given for granted that in inside-group interactions, each agent coordinates on her/his favorite action, what happens in mixed interactions depends on the relative size of groups and payoffs. From Theorem 1, we know that if one group is more rigid in preferences, it is hard to convince its members not to play their favorite action. Hence, we can conclude that if an ethnicity is more rigid in preferences than another, the first prevails in the short-run: everybody coordinates on the action preferred by that ethnicity[11].

[10] In this sense, studies like [62], [63] and [64] are more appropriate to describe long-run results. Indeed, the mechanism of updating strategies projects the model in a long-run perspective.

[11] In this direction, a recent work by [65], tells us that the integration of minorities, in the long-run,

When the cost is strictly positive, it is misleading to talk about ethnicities because it is never possible to recognize the other type at first glance. For this reason, we should pass on different interpretations. Let us keep the one given in the introduction regarding football and cinema (more generally, free time). In this case, the welfare analysis could take two directions, which need, perhaps, a further investigation. When the cost is sufficiently low, agents always coordinate on their favorite action at least with one group of people. Indeed, agents always coordinate on their favorite action with their similar. In mixed group interactions, we observe coordination: this is why we could say that when c is low enough, there is an improvement in welfare. However, we can not ignore that in this case, there is a sort of "free-riding" effect. Only one type pays the cost, and the other never pays it. In one case, the majority always plays its favorite action and never pays the cost, while the minority affords it. In the other case, the opposite happens. Nevertheless, given that we characterize the equilibria just for cases when the cost is low, this seems not relevant. When the cost is high, there is not such a "free-riding" phenomenon, but there could be the same loss in welfare. Two situations happen: MS are uniquely stochastically stable, or TS is uniquely stochastically stable. In the first situation, many players are coordinating on their second best option even with their similar. In the second situation, agents miscoordinate in mixed interactions, and hence, they suffer from zero gains.

We conclude with a short comparison of our result with the one of [7]. Our results are not qualitatively different from that work, but it is worthwhile to mention a contrast, which is a direct consequence of introducing the possibility to differentiate the action. In the model of [7], a change in payoffs of one type does not affect the behavior of the other type. We can find this effect even in our model when the cost is high. For example, when MS_a is stochastically stable, and B type becomes enough more rigid, the new long-run equilibrium is TS. This means that A type does not change its behavior. However, when the cost is sufficiently low, the change in payoffs of one type influences the behavior, in mixed interactions, of the other type. For instance, when PS_b is stochastically stable, if type A becomes enough more rigid in preferences, PS_a becomes stochastically stable. Both types change the way they behave in mixed interactions. Nevertheless, this is a quantitative difference because from the qualitative point of view, the effect is similar to what we observe in [7]. If the payoffs of one type change, the other keeps playing its favorite action in inside-group interactions, but it changes the action in mixed ones.

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depends similarly on relative payoffs and group sizes.

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A Proofs

Proof of Lemma 1:

We firstly treat the case of X_t^{AA}. Name strategy al the strategy when a player choose action a with her/his type and action $l = \{a, b\}$ with the other type. For an individual $i \in A$,

$$E\left[y_{t+1}^i\left(al, \tilde{\omega}_{t+1}^j\right)|X_t\right] = E\left[y_{t+1}^i\left(al, \tilde{\omega}_{t+1}^j\right)|X_t^{AA}\right].$$

Since i always buy the information, s/he just looks at what A players do. The argument is similar for the dynamics of X_t^{BB}.

For what concerns the dynamics of X_t^I, for a player $i \in A$,

$$E\left[y_{t+1}^i\left(la, \tilde{\omega}_{t+1}^j\right)|X_t\right] = E\left[y_{t+1}^i\left(la, \tilde{\omega}_{t+1}^j\right)|X_t^{BA}\right].$$

Similarly for j against i, j conditions her/his choice only on X_t^{AB}.

□

With abuse of notation, we call best reply (BR), the action optimally taken by a player in one of the three dynamics. For example, if for a player of type A it is convenient to play a against a player of type B, we say that a is her/his BR. We do this in the context of complete information because of the separability of the dynamics.

Proof of Lemma 2:

Thanks to Lemma 1, we can consider the 3 separated dynamics of X_t^{AA}, X_t^{BB} and the one of X_t^I.
Firstly, we prove the result for X_t^{AA} and then the argument stands for X_t^{BB} thanks to symmetry of payoff matrix. We have to show that all the states in X^R have an absorbing component for X_t^{AA}, that is 0 or N_A. When $\bar{X}^{AA} = N_A$, $\forall i \in A$, a is BR against type A at time t. Hence, $F_1(N_A, \theta_{t+1}) = N_A$. Symmetrically if $\bar{X}^{AA} = 0$, b is always BR and so, $F_1(0, \theta_{t+1}) = 0$. Therefore, N_A and 0 are fixed points for X_t^{AA}. Next, we need to show that they are also absorbing, that all the other states are transient, and that there are no cycles. Consider player $i \in A$ picked to revise her/his action with type A at time t. Looking at Equation (1) to (4), depending on the fact that s/he is playing action a or not, s/he chooses action a if $X_t^{AA} > \frac{N_A\pi_A + \Pi_A}{\Pi_A + \pi_A}$ or $X_t^{AA} > \frac{N_A\pi_A - \pi_A}{\Pi_A + \pi_A}$. We call $\frac{N_A\pi_A + \Pi_A}{\Pi_A + \pi_A}$ as $\bar{X}^A$ and $\frac{N_A\pi_A - \pi_A}{\Pi_A + \pi_A}$ as $\underline{X}^A$. $\forall \bar{X}^{AA} > \bar{X}^A$ then a is BR $\forall i \in A$, given that $\bar{X}^A > \underline{X}^A$. Thanks to Assumption 2, in these states, we know that there is always a positive probability to extract a player not playing a and so, $\forall X_t^{AA} \neq N_A$, $p(F_1(X_t^{AA}, \theta_{t+1}) \neq X_t^{AA}) > 0$. Moreover, thanks to the fact that a is

preferred by every player, we know for sure that $\forall X_t^{AA} \geq \bar{X}^A$, $F_1(X_t^{AA}, \theta_{t+1}) \geq X_t^{AA}$. Symmetrically, we can say that $\forall X_t^{AA} < \underline{X}^A$, $F_1(X_t^{AA}, \theta_{t+1}) \leq X_t^{AA}$. Therefore, once at the left of $\underline{X}^A$, the system reaches 0 as time goes by, and once at the right of $\bar{X}^A$ it reaches N_A. The probability of staying in any other state to the left of $\underline{X}^A$ that is not 0 vanishes as time goes by. The same for the probability to stay in a state to the right of $\bar{X}^A$ that is not N_A. We prove the first case, and the result stands for the second, thanks to symmetry in payoff matrices. Consider to be at step s in a state $X_s^{AA} < \underline{X}^A$. For every player, b is BR. The probability of staying in that state from period s to period $s+1$ is σ. Such probability is the probability to extract players who are already playing b at state X_s^{AA}. The probability of staying there for 2 periods is σ^2, after n periods σ^n. When $n \to \infty$, the probability goes to zero. Therefore, the system reaches 0 as time goes by.
Next, consider to be in a state $\underline{X}^A < \bar{X^{AA}} < \bar{X}^A$ at time t, in this case we know that for every i playing a, playing b is BR while, for every i' playing b, the opposite happens. We should not worry about possible absorbing states between these states: in each of them, there is always a path that brings either to the left of $\underline{X}^A$ or to the right of $\bar{X}^A$. Indeed, if only agents playing a are selected, they all choose b, and if enough of them are selected, the system ends up to the left of $\underline{X}^A$, the opposite happens if only players playing b are selected. Hence, for every state between $\underline{X}^A$ and $\bar{X}^A$, there are always paths that lead to 0 and N_A as $t \to \infty$. N_A and 0 are the only absorbing states of the X_t^{AA} dynamics.

We now pass to the analysis of X_t^I. We define 4 important values for X^{AB} and X^{BA}:
$T_A = min\left\{X^{BA} | \bar{X^{BA}} > \frac{\pi_A N_B}{\Pi_A + \pi_A}\right\}$, $T_B = min\left\{X^{AB} | \bar{X^{AB}} > \frac{\Pi_B N_A}{\Pi_B + \pi_B}\right\}$,
$D_A = max\left\{X^{BA} | \bar{X^{BA}} < \frac{\pi_A N_B}{\Pi_A + \pi_A}\right\}$, and $D_B = max\left\{X^{AB} | \bar{X^{AB}} < \frac{\Pi_B N_A}{\Pi_B + \pi_B}\right\}$.
Given these values we also define two sets of states, X^b and X^a:
$X^a = \left\{\bar{X} \in X | \bar{X^{BA}} \geq T_A \text{ and } \bar{X^{AB}} \geq T_B\right\}$ and $X^b = \left\{\bar{X} \in X | \bar{X^{BA}} \leq D_A \text{ and } \bar{X^{AB}} \leq D_B\right\}$.
With similar computation as for X_t^{AA}, we can say that $(0,0)$ and (N_A, N_B) are two fixed points for $F_{2,3}(X_t^I, \theta_{t+1}) = X_{t+1}^I$. Are they absorbing states?
Consider the choice of a player $i \in A$ against player $j \in B$ and vice-versa. There can be four possible combinations of states: states in which a is BR for everyone, states in which b is BR for everyone, states, in which $\forall i \in A$, a is the best reply and $\forall j \in B$, b is the best reply, and states for which the opposite is true. Let us call the third situation as X^{ab} and the fourth as X^{ba}.
Firstly, we prove that inside X^a every agent that is selected to revise her/his strategy ends up choosing a (*i.e.* X^a corresponds to the region where a is BR for everyone). Secondly, we prove that there is no other absorbing state in X^a than (N_A, N_B). Consider a generic step t, a generic state (X_t^{AB}, X_t^{BA}) and assume that $i \in A$ is selected to revise her/his strategy. From Equation (1) to (4), s/he chooses action a against agent $j \in B$ if $X_t^{BA} > \frac{\pi_A N_B}{\Pi_A + \pi_A}$, since T_A is defined as the minimum value s.t. the latter holds, $\forall \bar{X^{BA}} \geq T_A$ every agent $i \in A$ chooses a.
Now, assume that $j \in B$ is picked to revise the strategy, s/he chooses a against $i \in A$ if $X_t^{AB} > \frac{\Pi_B N_A}{\Pi_B + \pi_B}$: since T_B is defined as the minimum values s.t. this relation is true, $\forall \bar{X^{AB}} \geq T_B$, a is best reply $\forall j \in B$. Therefore, by definition of X^a, $\forall \bar{X} \in X^a$, playing action a against the other type is the BR $\forall i \in N$. This means that if $X_0 \in X^a$, $X_s \in X^a, \forall s \geq 0$.

As time goes by, we eventually reach state (N_A, N_B). Indeed, consider being in a generic state $(T_B + d, T_A + d') \in X^a$ at time t, with $d \in [0, N_A - T_B)$ and $d' \in [0, N_B - T_A)$. It is sufficient to use the fact that in every state in X^a there is always at least a small probability σ to select a player that is not playing a and that changes action. This is true by Assumption 2 $\forall \bar{X}^I \neq (N_A, N_B)$, so we know that $p\left(F_{2,3}\left(X_t^I, \theta_{t+1}\right) \geq X_t^I\right) > \sigma,\ \forall X_t^I \in X^a \setminus (N_A, N_B)$[12].
Similar to what we proved before, the probability of staying forever in a state $\bar{X} \in X^a$ that is not (N_A, N_B) vanishes as time goes by.
We now consider X^{ab} and X^{ba}. Take an $X_0 \in X^{ab}$: at each step, there is a positive probability that only agents of type A are picked to revise strategy, since for them a is the best reply, in the next period, there are more or equal agents in A playing a. Hence, if enough players of A that are currently playing b are picked to revise action, the system can end up in a state where for also every player in B, a is the best reply. Therefore, the system can end up in X^a. By the same reasoning, there is also a positive probability of extracting only agents from B, such that in the following period, the system ends up in a state in X^b. The same can be said for every state in X^{ba}. Hence, starting from every state in $X^{ab} \bigcup X^{ba}$, there is always a path that lead the system either to X^a or to X^b. Therefore, there is always a path either to $(0,0)$ or (N_A, N_B) in the long-run.
This means that outside $X^a \bigcup X^b$, all the states are transient, and therefore, there are no absorbing states. This completes the proof.

□

Lemma 4. *Under complete information, if Assumption 1 to 3 hold,*
$p\left(\lim_{t\to\infty} X_t^I = (N_A, N_B)\right) = 1 - p\left(\lim_{t\to\infty} X_t^I = (0,0)\right)$.
$p\left(\lim_{t\to\infty} X_t^{AA} = N_A\right) = 1 - p\left(\lim_{t\to\infty} X_t^{AA} = 0\right)$.
$p\left(\lim_{t\to\infty} X_t^{BB} = N_B\right) = 1 - p\left(\lim_{t\to\infty} X_t^{BB} = 0\right)$.

Proof:

We prove the result for X_t^I, and the argument stands for the two other dynamics thanks to symmetry in the payoff matrix. Firstly, note that whenever the process starts in $X^a \cup X^b$, the lemma is always true thanks to the proof of Lemma 2. We need to show that this is the case, also when the process starts inside $X^{ab} \bigcup X^{ba}$. We prove the result for X^{ab} using the same logic, and the result stands for X^{ba} for symmetry of payoff matrix.
Take $X_0^I \in X^{ab}$, name σ_a the probability of extracting M agents from A that are currently playing b and that would change action a if picked, name σ_b the probability of picking M agents from B currently choosing a that would change action to b if picked. The probability $1 - \sigma_a - \sigma_b$ defines all the other possibilities.
Let us take n steps forward in time:

$$p\left(X_n^I \in X^a\right) \geq (\sigma_a)^n$$

$$p\left(X_n^I \in X^b\right) \geq (\sigma_b)^n$$

[12]Meaning that $X_t^I > X_t^{'I}$ if either $X_t^{AB} > X_t^{'AB}$ and $X_t^{BA} = X_t^{'BA}$ or $X_t^{BA} > X_t^{'BA}$ and $X_t^{AB} = X_t^{'AB}$ or both $X_t^{BA} > X_t^{'BA}$ and $X_t^{AB} > X_t^{'AB}$.

$$p\left(X_n^I \in X^{ab}\bigcup X^{ba}\right) \leq (1-\sigma_a-\sigma_b)^n.$$

This happens because the probability of being in X^a after n steps is the probability of extracting n times, players from A that change action from b to a, plus the probability of reaching X^a from other paths. The same can be said for X^b. Consequently, the probability of not being in one of that two zones is at most $(1-\sigma_a-\sigma_b)^n$, because is the probability of not extracting for n consecutive times neither agents in A playing b nor agents in B playing a minus all the other possibilities that bring the system either to X^a or to X^b.
Consider state $n+t$:

$$p\left(X_{n+t}^I \in X^a\right) \geq (\sigma_a)^n$$

$$p\left(X_{n+t}^I \in X^b\right) \geq (\sigma_b)^n$$

$$p\left(X_{n+t}^I \in X^{ab}\bigcup X^{ba}\right) \leq (1-\sigma_a-\sigma_b)^{n+t}.$$

Clearly, the probability of being in $X^a(X^b)$ is now greater or equal than $(\sigma_a)^n((\sigma_b)^n)$: we know that once in $X^a(X^b)$ the system stays there. The probability of being in $X^{ab}\bigcup X^{ba}$ consequently, is lower than $(1-\sigma_a-\sigma_b)^{n+t}$.
Taking the limit for t that goes to infinity

$$\lim_{t\to\infty}\left(p\left(X_{n+t}^I \in X^{ab}\bigcup X^{ba}\right)\right) \leq 0.$$

This means that if we start in a state in X^{ab} there is no way of ending up in $X^{ab}\bigcup X^{ba}$ in the long-run; hence, the system ends up either in X^a or in X^b, but given this, we know that it ends up either in $(0,0)$ or in (N_A, N_B). This means that the only two possible states where we could be in the long-run are $(0,0)$ and (N_A, N_B) and hence, we can conclude that $p\left(\lim_{t\to\infty} X_t^I = (N_A, N_B)\right) = 1 - p\left(\lim_{t\to\infty} X_t^I = (0,0)\right)$.

□

Corollary 3. *Under complete information, if Assumption 1 to 3 hold,*
$p\left(\lim_{t\to\infty} X_t^I = (N_A, N_B)\right) = 1$ *IFF* $X_0 \in X^a$.
$p\left(\lim_{t\to\infty} X_t^I = (0,0)\right) = 1$ *IFF* $X_0 \in X^b$.
$p\left(\lim_{t\to\infty} X_t^{AA} = N_A\right) = 1$ *IFF* $X_0 \in \left[\bar{X}^A, N_A\right]$, *and*
$p\left(\lim_{t\to\infty} X_t^{AA} = 0\right) = 1$ *IFF* $X_0 \in \left[0, \underline{X}^A\right]$.
$p\left(\lim_{t\to\infty} X_t^{BB} = N_B\right) = 1$ *IFF* $X_0 \in \left[\bar{X}^B, N_B\right]$, *and*
$p\left(\lim_{t\to\infty} X_t^{BB} = 0\right) = 1$ *IFF* $X_0 \in \left[0, \underline{X}^B\right]$.

This result is a consequence of the previous lemmas and therefore, the proof is omitted. Since the only two absorbing states in the dynamics of X_t^I are $(0,0)$ and (N_A, N_B), they are the only two candidates to be stochastically stable states. From now on we call $(0,0)$ as I_X^b and (N_A, N_B) as I_X^a. Thanks to [9], we know that there is at least one path that links I_X^b to I_X^a and vice-versa. Name ρ_a^I the stochastic potential of I_X^a and ρ_b^I, the stochastic potential of I_X^b. Consider X_t^{AA} and X_t^{BB}. For the first, call ρ_a^A the stochastic potential of N_A, ρ_b^A the one of 0. For the second, name ρ_a^B the stochastic potential of N_B and ρ_b^B the one of 0.

Let us call E_A and E_B the two values for which agents in A and in B are indifferent in playing a or b; $E_A = \left\lceil \frac{N_B \pi_A}{\Pi_A + \pi_A} \right\rceil$ and $E_B = \left\lceil \frac{N_A \Pi_B}{\Pi_B + \pi_B} \right\rceil$. From now on we often use values of N large enough to compare the arguments inside ceiling functions.

Lemma 5. *Under free acquisition of information, if Assumption 1 to 3 hold, for N large enough, $\rho_a^I = E_A$ for all values of payoffs and sizes of groups, while*

$$\rho_b^I = \begin{cases} N_A - E_B & \text{if } \frac{\pi_B}{\Pi_A} < \frac{N_B}{N_A} \\ N_B - E_A & \text{if } \frac{\pi_B}{\Pi_A} > \frac{N_B}{N_A} \end{cases}$$

Proof:

We prove that the easiest way to exit the basin of attraction of I_X^a is to reach $(E_B, 0)$ or $(0, E_A)$, and that the one to exit the basin of attraction of I_X^b is to reach either (E_B, N_B) or (N_A, E_A). To prove this statement for I_X^a, firstly, note that once inside X^b every step which involves a passage to a state with more people playing a requires an error. Secondly, note that to exit from X^b we need to be in a state where at least one of the two types is indifferent in playing b or a. In other words, we need to be in a state where either $X^{AB} = E_B$ or $X^{BA} = E_A$ or both. Hence, the minimum resistance path to exit from I_X^b is the one that arrives either to $(E_B, 0)$ or to $(0, E_A)$. It is straightforward to show that all the other paths have greater resistance than the two above. Thanks to Lemma 4 we know that this path is also the minimum resistance path to reach I_X^a. Since we use uniform mistakes, every mutation counts the same value, and without loss of generality, we can count each of them as 1. Since every resistance counts as 1, then $\rho_a^I = \min\{E_B; E_A\}$: $E_A < E_B$ for all values of payoffs and group size, hence, $\rho_a^I = E_A$. Consider ρ_b^I: with a similar resoning it can be $N_A - E_B$ or $N_B - E_A$

$$N_A - E_B < N_B - E_A \iff \frac{\pi_B}{\Pi_A} < \frac{N_B}{N_A}.$$

□

Lemma 6. *Under free acquisition of information, if Assumption 1 to 3 hold, for N large enough, $\rho_a^A = \left\lceil \frac{N_A \pi_A - \pi_A}{\Pi_A + \pi_A} \right\rceil$, $\rho_b^A = \left\lceil \frac{N_A \Pi_A + \Pi_A}{\Pi_A + \pi_A} \right\rceil$, $\rho_a^B = \left\lceil \frac{N_B \Pi_B + \Pi_B}{\Pi_B + \pi_B} \right\rceil$ and $\rho_b^B = \left\lceil \frac{N_B \pi_B - \pi_B}{\Pi_B + \pi_B} \right\rceil$.*

Proof:

The proof is straight forward, indeed, the minimum path in terms of error required to reach one absorbing state starting from the other one is the cost of exit from the basin of attraction of the first. As a matter of fact, let us consider ρ_a^A, once we are out of the basin of attraction of 0, thanks to the proof of Lemma 2 we know that $p(\lim_{t\to\infty} X_t^{AA} \in (\underline{X}^A, \bar{X}^A) = N_A) > 0$. Therefore, $\rho_a^A = \underline{X}^A = \left\lceil \frac{N_A \pi_A - \pi_A}{\Pi_A + \pi_A} \right\rceil$. The same applies for ρ_b^A, ρ_a^B and ρ_b^B.

□

Proof of Theorem 1:

We divide the proof for the three dynamics described so far: for what concerns X_t^{AA}, N_A is uniquely stochastically stable and for what concerns X_t^{BB}, 0 is uniquely stochastically stable, this proof follows directly from Lemma 6, and therefore is omitted. Let us pass to X_t^I. We know from Lemma 5 that $\rho_a^I = E_A$ and that the value of ρ_b^I depends on payoffs and group size. Let us firstly consider the case when $\frac{\pi_B}{\Pi_A} < \frac{N_B}{N_A}$ and $\rho_b^I = N_A - E_B$. If $E_A > N_A - E_B$ then I_X^b is the uniquely stochastically stable, this is the case IFF

$$\frac{\pi_A N_B}{\Pi_A + \pi_A} > \frac{\pi_B N_A}{\Pi_B + \pi_B} \iff \frac{\pi_B}{\pi_A} < \frac{N_B}{N_A}. \tag{9}$$

To complete the proof, we show that whenever $\frac{\pi_B}{\pi_A} > \frac{N_B}{N_A}$, then I_X^a is the uniquely stochastically stable state. Firstly, note that $\frac{\pi_B}{\Pi_A} < \frac{\pi_B}{\pi_A}$, hence, for $\frac{\pi_B}{\pi_A} > \frac{N_B}{N_A} > \frac{\pi_B}{\Pi_A}$, $\rho_b^I = N_A - E_B$ and $E_A = \rho_a^I$. However, Equation (9) is reversed, so, I_X^a is uniquely stochastically stable. For $\frac{\pi_B}{\pi_A} > \frac{\pi_B}{\Pi_A} > \frac{N_B}{N_A}$, $\rho_b^I = N_B - E_A$ and still $\rho_a^I = E_A$, in this case, I_X^a is the uniquely stochastically stable if $E_A < N_B - E_A$, hence, IFF

$$\frac{\pi_A N_B}{\Pi_A + \pi_A} < \frac{\Pi_A N_B}{\Pi_A + \pi_A}.$$

This happens for every value of the payoffs (given that $\Pi_A > \pi_A$) and of the group size. We conclude that whenever $\frac{\pi_B}{\pi_A} < \frac{N_B}{N_A}$, PS_b is uniquely stochastically stable and when $\frac{\pi_B}{\pi_A} > \frac{N_B}{N_A}$, PS_a is uniquely stochastically stable.

□

We now pass to the proof of Section 4. Let us call $\tau_i = \{0, 1\}$, a parameter of choice for each player i in the population. This parameter is equal to 1 if a player decides to acquire the information and to 0 otherwise. For convenience, we call strategy 1 the optimal strategy when a player decides to acquire the information: $1 = argmax(ab, ba, aa, bb)$.

Proof of Lemma 3:

We first show that the nine states are effectively strict equilibria, that there is no other possible equilibrium under Assumption 1 to 3, and that a state is absorbing if and only if it is a strict equilibrium.

It is easy to show that (N_A, N_A, N_B, N_B) and $(0, 0, 0, 0)$ are two strict equilibria, indeed, in each of them every player is playing either a or b. We take the first case, and the argument stands also for the second, thanks to the symmetry of the payoff matrix. In this case, we take a generic player $i \in K$, s/he has the following expected payoffs

$$E_t(a) = \frac{N_K + N_{K'} - 1}{N - 1}\pi_a^K = \pi_a^K,$$

$$E_t(b) = \frac{N - N_K - N_{K'}}{N - 1}\pi_b^K = 0.$$

Note that $1 = aa$, since everyone is playing that action, and a player never chooses 1 in this state. (N_A, N_A, N_B, N_B) is a strict equilibrium since $\pi_a^K > 0$. Next, we can analyze the other 6 states in X^R, firstly let us consider the case of $(N_A, 0, 0, N_B)$. Since in this case, every player is playing ab, the state is a strict equilibrium IFF $\max \omega_i^o = ab$, $\forall i \in N$. Player $i \in K$ faces the following expected payoffs:

$$E_t(a) = \frac{N_K - 1}{N - 1}\pi_a^K,$$

$$E_t(b) = \frac{N_{K'}}{N - 1}\pi_b^K,$$

$$E_t(1) = \frac{N_K - 1}{N - 1}\pi_a^K + \frac{N_{K'}}{N - 1}\pi_b^K - c.$$

Under these conditions for a player of type A, b is never preferred to a since $\frac{N_B}{N-1}\pi_A < \frac{N_A-1}{N-1}\Pi_A$ and a is never preferred to b for a player of type B as $\frac{N_B-1}{N-1}\pi_B < \frac{N_A}{N-1}\Pi_B$. $E_t(1)$ is the highest of the three $\forall i \in N$ IFF $c < \min\left\{\frac{N_B}{N-1}\pi_A, \frac{N_B-1}{N-1}\pi_B\right\}$.
Consider the case of $(0, N_A, N_B, 0)$, since everybody is playing ba, it must be that $\omega_i^o = ba$. $i \in K$ faces the following expected payoffs

$$E_t(a) = \frac{N'_K}{N - 1}\pi_a^K,$$

$$E_t(b) = \frac{N_K - 1}{N - 1}\pi_b^K,$$

$$E_t(1) = \frac{N_K - 1}{N - 1}\pi_b^K + \frac{N_{K'}}{N - 1}\pi_a^K - c.$$

Note that $E_t(a) > E_t(b)$ IFF $\frac{\pi_b^K}{\pi_a^K} < \frac{N'_K}{N_K-1}$, and therefore ba is the best reply strategy IFF $c < \frac{N_K-1}{N-1}\pi_b^K$. When the opposite happens and so, $\frac{\pi_b^K}{\pi_a^K} > \frac{N'_K}{N_K-1}$, ba is the best reply strategy IFF $c < \frac{N'_K}{N-1}\pi_a^K$. These conditions take the form of the ones in Table 4.

Consider the remaining 4 PS, they are characterised by the following fact $BR(X^{\bar{K}K}) = BR(X^{\bar{K}'K})$ but $BR(X^{\bar{K}'K'}) \neq BR(X^{\bar{K}K'})$. In this case it must be that $\tau_i = 0$ is optimal for $i \in K$ while $\tau_j = 1$ is optimal for $j \in K'$. Thanks to the symmetry in payoff matrices we can say that the argument to prove the results for these 4 states is similar to the one described in $(N_A, 0, 0, N_B)$ and $(0, N_A, N_B, 0)$. All the conditions are listed in Table 4.
Next, we need to prove that $(N_A, N_A, 0, 0)$ can be a strict equilibrium, in this case it must be that $\tau_i = 0$ is optimal for every player and moreover that a is the BR $\forall i \in A$ and b, $\forall j \in B$. Consider the case of A, each player who is randomly picked to revise strategy faces the following expected payoffs

$$E_t(a) = \frac{N_A - 1}{N - 1}\Pi_A,$$

$$E_t(b) = \frac{N_B - 1}{N - 1}\pi_A,$$

$$E_t(1) = \frac{N_A - 1}{N - 1}\Pi_A + \frac{N_B}{N-1}\pi_A - c.$$

Given that $E_t(a) > E_t(b)$, a is the best reply strategy IFF $c > \frac{N_B}{N-1}\pi_A$. Consider the choice of B players, they choose between these three expected payoffs

$$E_t(a) = \frac{N_A}{N-1}\pi_B,$$

$$E_t(b) = \frac{N_B - 1}{N-1}\Pi_B,$$

$$E_t(1) = \frac{N_A}{N-1}\pi_B + \frac{N_B - 1}{N-1}\Pi_B - c.$$

In this case when $\frac{\pi_B}{\Pi_B} > \frac{N_B - 1}{N_A}$, b is never best reply and a is best reply hence, the state can not be a strict equilibrium. When $\frac{\pi_B}{\Pi_B} < \frac{N_B-1}{N_A}$ $E_t(b) > E_t(a)$ is and b is best reply IFF $c > \frac{N_A}{N-1}\pi_B$.

The last thing we need to prove is that none of the states outside the nine mentioned in the statement is a strict equilibrium. For what concerns states where not all players of a type are playing the same action with the same type, this is easy to prove. Indeed, by definition, in these states, either not all players are playing their best reply action, or players are indifferent between two or more strategies. In the first case, the state is not a strict equilibrium by definition; in the second case, there is no strictness of the equilibrium since there is not one clear best reply, but more strategies can be best reply simultaneously. Hence, such states can not be strict equilibria. We are left with the 7 states where every player of one type is doing the same thing against the same type. Such states are: $(0, 0, N_B, N_B)$, $(0, N_A, 0, N_B)$, $(N_A, 0, N_B, 0)$, $(0, 0, N_B, 0)$, $(N_A, N_A, 0, N_B)$, $(0, N_A, 0, 0)$, and $(N_A, 0, N_B, N_B)$. It is easy to prove that these states enter in the category of states where not every player is playing her/his best reply. Therefore, they can not be strict equilibria.
Next, we need to prove necessary and sufficient conditions for absorbing states. We first prove the sufficient and necessary conditions to be a fixed point. To prove the sufficient part we rely on the definition of strict equilibrium. In a strict equilibrium, every player is playing her/his BR, and no one has the incentive to deviate. Whoever is called to revise the strategy does not change her/his strategy. Therefore, $F_c(X_t, \theta_{t+1}) = X_t$, for every possible realization of $\overline{\theta}$. To prove the necessary condition think about being in a state that is not a strict equilibrium; in this case, by definition, we know that not all the players are playing their BR. Among them, there are states in which there are no indifferent players, in this case, thanks to Assumption 1 to 3 with positive probability one or more agents who are not playing their BR are selected to revise strategy and they change it, therefore, $F_c(X_t, \theta_{t+1}) \neq X_t$ for some realization of $\overline{\theta}$. In states where some players are indifferent between two or more strategies, thanks to the tie rule, there is always a positive probability that the indifferent agent changes her/his action since s/he is randomizing her/his choice. Moreover, there is also a positive probability to select an agent indifferent between two or more strategies. In this case, s/he changes the one that is currently playing with a positive probability too. Knowing this, we are sure that no state outside strict

equilibria can be a fixed point. In our case, a fixed point is also an absorbing state by definition. Indeed, every fixed point absorbs at least one state: the one where all players except one are playing the same strategy. In this case, if that player is selected s/he changes for sure her/his strategy into the one played by everyone.

□

Here we prove the results of the stochastic stability analysis of Section 4. Given a state $\bar{X}$, we call $R(\bar{X})$ its Radius, $CR(\bar{X})$ its Coradius and $CR^*(\bar{X})$ its Modified Coradius.

Proof of Theorem 2:

We just need to calculate the two Radius of MS_a and MS_b. Since there are just two candidates to be stochastically stable, the Radius of one is the stochastic potential of the other: the state with the biggest Radius is the stochastically stable one. Let us consider $R(MS_a)$, it must be the minimum number of errors such that with positive probability the system ends up in the other absorbing state. Since the basin of attraction of MS_a is a region where a is the best reply strategy for both types, the system must arrive in a situation where one of the two types has b as best reply. For b to be BR for B players, the system must reach a state such that $\bar{X^{AB}} + \bar{X^{BB}} \leq \frac{N\Pi_B - \Pi_B}{\Pi_B + \pi_B}$. This state can be reached with ε mutations, at cost $\frac{N\pi_B + \Pi_B}{\Pi_B + \pi_B}$. If the system reaches a state where $\bar{X^{AA}} + \bar{X^{BA}} \leq \frac{N\pi_A - \pi_A}{\Pi_A + \pi_A}$, b would be BR for A, this happens at cost $\frac{N\Pi_A + \pi_A}{\Pi_A + \pi_A}$. In principle $\frac{N\Pi_A + \pi_A}{\Pi_A + \pi_A} > \frac{N\pi_B + \Pi_B}{\Pi_B + \pi_B}$, hence, $R(MS_a)$ should be $\frac{N\pi_B + \Pi_B}{\Pi_B + \pi_B}$. However, it may not be sufficient to reach such a state. Consider to reach a state s.t. $\bar{X^{AB}} + \bar{X^{BB}} = \frac{N\Pi_B - \Pi_B}{\Pi_B + \pi_B}$, since $\frac{N\Pi_B - \Pi_B}{\Pi_B + \pi_B} > \frac{N\pi_A - \pi_A}{\Pi_A + \pi_A}$, it must be that a is still the best reply $\forall i \in A$, and therefore there is a positive probability to return to MS_a at zero cost. Nevertheless, once in that state, it can happen that only B players are selected, and that they all choose strategy b. This makes the system reach a state $(\bar{X^{AA}}, \bar{X^{AB}}, 0, 0)$ at the same cost of the one described above. Once in this state, if $\bar{X^{AA}} < \frac{N\pi_A - \pi_A}{\Pi_A + \pi_A}$, we are in the basin of attraction of MS_b, and hence, $R(MS_a)$ would be $\frac{N\pi_B + \Pi_B}{\Pi_B + \pi_B}$. This happens only if $\frac{N\pi_A - \pi_A}{\Pi_A + \pi_A} + N_B = \frac{N\Pi_B - \Pi_B}{\Pi_B + \pi_B}$, more generally, considering a positive number k, this happens if $\frac{N\pi_A - \pi_A}{\Pi_A + \pi_A} + N_B = \frac{N\Pi_B - \Pi_B}{\Pi_B + \pi_B} - k$. Fixing payoffs and groups size, $k = \frac{N\Pi_B - \Pi_B}{\Pi_B + \pi_B} - \frac{N\pi_A - \pi_A}{\Pi_A + \pi_A} - N_B$, hence, the cost of this path would be

$$\frac{N\pi_B + \Pi_B}{\Pi_B + \pi_B} + \frac{N\Pi_B - \Pi_B}{\Pi_B + \pi_B} - \frac{N\pi_A - \pi_A}{\Pi_A + \pi_A} - N_B = N_A - \frac{N\pi_A - \pi_A}{\Pi_A + \pi_A}.$$

With a similar reasoning $R(MS_b) = \frac{N\pi_A + \Pi_A}{\Pi_A + \pi_A}$.

We prove that all the other paths with η errors are costlier than ones with ε. We know that a is the BR for every state inside the basin of attraction of MS_a, nobody in the basin of attraction of MS_a optimally buys the information, and every player once bought the information (by mistake) plays strategy aa. Every path with η error also involves a ε error, and hence, is double that of the one described above.

MS_a is stochastically stable IFF $N_A - \frac{N\pi_A - \pi_A}{\Pi_A + \pi_A} > \frac{N\pi_A + \Pi_A}{\Pi_A + \pi_A}$, this is verified when $N_A > \frac{2N\pi_A + \Pi_A - \pi_A}{\Pi_A + \pi_A}$. Therefore, we conclude that MS_a is stochastically stable in the above scenario, while if the opposite happens MS_b is stochastically stable.

□

Proof of Theorem 3

We first calculate Radius, Coradius, and Modified Coradius for the three states we are interested in, and then we compare them to draw inference about long-run stability. The Radius of MS_a is the minimum number of errors required to escape from that state: due to the presence of TS, the easiest way to escape from the basin of attraction of MS_a is to convince B players who are not playing their favorite action. This comes at cost $\frac{N\pi_B+\Pi_B}{\Pi_B+\pi_B}$. The alternative is to convince A players to play b: this path leads to state $(0,0,N_B,N_B)$, and then to $(0,0,0,0)$. This path costs $\frac{N\Pi_A+\pi_A}{\Pi_A+\pi_A}$ ε errors, therefore, $R(MS_a)=\frac{N\pi_B+\Pi_B}{\Pi_B+\pi_B}$. With a similar reasoning we can conclude that $R(MS_b)=\frac{N\pi_A+\Pi_A}{\Pi_A+\pi_A}$. Consider TS: the easiest way to escape from it is to reach either MS_a or MS_b, depending on payoffs. In other words, we need to reach a state where both types have either a or b as the best reply strategy. Let us firstly consider to reach MS_a from TS: in this case, we need to convince B to play strategy a. The state in which type B changes their best reply depends on payoffs and group size. Think about being in a state (N_A,N_A,k',k'), in this case a is the best reply strategy for every player in B if $(N_A+k'-1)\pi_B>(N-N_A-k')\Pi_B$. This inequality is obtained declining Equation (5) to (8), comparing B playing a/ab or b/ba: note that 1 is not taken in consideration, this happens because for sure a is preferred to 1 in the basin of attraction of MS_a. Fixing payoffs, we can calculate the exact value of k' which is $\frac{N_B\Pi_B-N_A\pi_B+\pi_B}{\Pi_B+\pi_B}$, this would be the cost of reaching MS_a from TS. With a similar argument, the cost of reaching MS_b is $\frac{N_A\Pi_A-N_B\pi_A+\pi_A}{\Pi_A+\pi_A}$. Next, we need to show that there are no paths involving η errors that are lower than the two proposed above. The intuition is the following. Consider a situation in which n players of A are selected at one step and they all choose to buy the information. In this case, they all optimally choose strategy ab. This means that at the cost of n the system reaches a state in which N_A-n players are playing b against B, in this state B type continues to play b, while A continues to play a. Hence, with positive probability, the system returns to TS with no error needed. The same happens when B players choose by mistake to buy the information. The only way to exit the basin of attraction of TS is via ε errors, therefore, $R(TS)=\min\left\{\frac{N_B\Pi_B-N_A\pi_B+\pi_B}{\Pi_B+\pi_B},\frac{N_A\Pi_A-N_B\pi_A+\pi_A}{\Pi_A+\pi_A}\right\}$.

Let us calculate the Coradius of each state. We start from the one of TS, in this case we have to consider the two minimum paths to reach it from MS_a and MS_b, therefore, we consider both $\frac{N\pi_A+\Pi_A}{\Pi_A+\pi_A}$ and $\frac{N\pi_B+\Pi_B}{\Pi_B+\pi_B}$. Firstly, the argument to prove that these two are the minimum error paths to reach TS from MS_a and MS_b are given by the previous part of the proof. Secondly, we have to prove that this path is the maximum among the minimum paths that start from any other state and brings to TS. There are three regions from which we can start and end-up in TS: the basin of attraction of MS_b, the one of MS_a, and all the other states which are not in the basins of attraction of the three states considered. We can say that from this region, there is always a positive probability to end up in MS_a, MS_b, or TS. Hence, we can consider as 0 the cost to reach TS from this region. The other two regions are the one considered above and since we are taking the maximum path to reach TS from any other state we have to take the sum of this two. Hence, $CR(TS)=\frac{N\pi_A+\Pi_A}{\Pi_A+\pi_A}+\frac{N\pi_B+\Pi_B}{\Pi_B+\pi_B}$.

Let us think about MS. Similarly to the two previous proofs also in this case we can focus just on ε paths. Note that in this case, TS is always placed between the two MS. Let us start from MS_b: in this case we can consider 3 different path starting from any state and arriving to MS_b. The first one starts in TS, the second starts in every state outside the basin of attractions of the three absorbing states, and the last starts in MS_a. In the second case the cost to reach MS_b is 0, since like we showed before, this set of states converges with positive probability to MS_b with no mutations needed. Next, consider to start in TS: the minimum resistance path is to convince A players playing b, and we know that this cost is $\frac{N_A\Pi_A - N_B\pi_A + \pi_A}{\Pi_A+\pi_A}$. Now, we need to consider the case of starting in MS_a. The minimum path can be two values depending on payoffs. The first alternative is to convince A players of playing b which comes at cost $\frac{N\Pi_A+\pi_A}{\Pi_A+\pi_A}$. The second alternative is to convince firstly B players of playing b and then once reached TS to convince A players playing b. Hence, $\min r(MS_a, MS_b) = \min\left\{\frac{N\Pi_A+\pi_A}{\Pi_A+\pi_A}, \frac{N\pi_B+\Pi_B}{\Pi_B+\pi_B} + \frac{N_A\Pi_A - N_B\pi_A+\pi_A}{\Pi_A+\pi_A}\right\}$. Since the two numbers in the expression are all greater than $\frac{N_A\Pi_A - N_B\pi_A+\pi_A}{\Pi_A+\pi_A}$ we can say that $CR(MS_b) = \min r(MS_a, MS_b)$. We said before that at most $\frac{N\pi_A-\pi_A}{\Pi_A+\pi_A}$ agents are sufficient to make A agents play b. Next, think about the cost of reaching TS from MS_a: the cost of reaching a state with at least $\frac{N\Pi_B-\Pi_B}{\Pi_B+\pi_B}$ agents playing b. Since $\frac{N\pi_A-\pi_A}{\Pi_A+\pi_A} < \frac{N\Pi_B-\Pi_B}{\Pi_B+\pi_B}$ it must be that the cost of convincing A types playing b is lower than the one of convincing B players playing b. Now, think about the case of reaching MS_b from TS: it must be that few players are playing a in order to convince A players to play b. Such a number must be $\frac{N\pi_A-\pi_A}{\Pi_A+\pi_A}$, but reaching this number from TS is for sure less costly than reaching it from MS_a, since in TS, there are less people playing a. Therefore, $\frac{N\Pi_A+\pi_A}{\Pi_A+\pi_A} \geq \frac{N\pi_B+\Pi_B}{\Pi_B+\pi_B} + \frac{N_A\Pi_A-N_B\pi_A+\pi_A}{\Pi_A+\pi_A}$, hence, $CR(MS_b) = \frac{N\pi_B+\Pi_B}{\Pi_B+\pi_B} + \frac{N_A\Pi_A-N_B\pi_A+\pi_A}{\Pi_A+\pi_A}$. With a similar reasoning, $CR(MS_a) = \frac{N\pi_A+\Pi_A}{\Pi_A+\pi_A} + \frac{N_B\Pi_B-N_A\pi_B+\pi_B}{\Pi_B+\pi_B}$.

Lastly, we calculate the Modified Coradius of the three absorbing states. Firstly, note that $CR(TS) = CR^*(TS)$, since between MS and TS there are no intermediate states. To formally prove this claim, think about reaching TS starting from MS_a, going to MS_b, and reaching TS from MS_b (subtracting the cots of going from MS_a to MS_b) and vice-versa. If we sum these two paths, we obtain $CR^*(TS)$, namely,

$$CR^*(TS) = \min r^*(MS_a, \ldots, TS) + \min r^*(MS_b, \ldots, TS) = \frac{N\pi_A + \Pi_A}{\Pi_A + \pi_A} + \frac{N\pi_B + \Pi_B}{\Pi_B + \pi_B}.$$

Note that $r^*(X_s, X_{s'})$ is the resistance that subtracts the costs for intermediate steps. On the other hand, for MS, $CR^* < CR$. The maximum path of minimum resistance from each MS to the other MS passes through TS. Hence, for each MS, we need to subtract from the Coradius the cost of passing from TS to the other MS. Let us consider $CR^*(MS_a)$, in this case, the maximum path among the minimum to reach the state MS_a is the one that starts in MS_b, reaches TS, and then from it goes to MS_a. We need to subtract to the path described above the cost of passing from TS to MS_b: this follows from the definition of Modified Coradius. Hence,

$$CR^*(MS_a) = \frac{N\pi_A + \Pi_A}{\Pi_A + \pi_A} + \frac{N_B\Pi_B - N_A\pi_B + \pi_B}{\Pi_B + \pi_B} - \frac{N_A\Pi_A - N_B\pi_A + \pi_A}{\Pi_A + \pi_A}.$$

Similarly,

$$CR^*(MS_b) = \frac{N\pi_B + \Pi_B}{\Pi_B + \pi_B} + \frac{N_A\Pi_A - N_B\pi_A + \pi_A}{\Pi_A + \pi_A} - \frac{N_B\Pi_B - N_A\pi_B + \pi_B}{\Pi_B + \pi_B}.$$

Note that both $CR^*(MS_a) = CR(MS_a) - e$, and $CR^*(MS_b) = CR(MS_b) - d$, with both e and b being strictly positive. Hence, it must be that $CR^*(MS_a) < CR(MS_a)$ and $CR^*(MS_b) < CR(MS_b)$.

The conditions of the statement are easily checked with simple calculus. Indeed, whenever $R(MS_a) > CR(MS_a)$, both $R(MS_b) < CR(MS_b)$ and $R(TS) < CR(TS)$. Similarly, when $R(MS_b) > CR(MS_b)$ or $R(TS) > CR(TS)$. When $R(MS_a) \leq CR(MS_a)$, $R(MS_b) \leq CR(MS_b)$, and $R(TS) \leq CR(TS)$, we need to use Modified Coradius. Given that $CR(TS) = CR^*(TS)$ it will never be that $R(TS) > CR^*(TS)$. We can show that when $R(MS_a) > CR^*(MS_a)$, then $R(MS_b) < CR^*(MS_b)$ and vice-versa. The less trivial case is when $R(MS_a) = CR^*(MS_a)$, indeed, it is possible that also $R(MS_b) = CR^*(MS_b)$. Referring to Theorem 3 in [13], we know that either both are stochastically stable, or none of the two is. Note that for the ergodicity of our process the second case is impossible to accomplish, hence, it must be that when both $R(MS_a) = CR^*(MS_a)$ and $R(MS_b) = CR^*(MS_b)$, both $\mu^\star(MS_b) > 0$ and $\mu^\star(MS_a) > 0$.

□

Proof of Theorem 4:

We split the absorbing states into 2 sets and then apply Theorem 1 by [13]. The theorem is general, and is valid for sets of absorbing states and not just singleton sets. Define the two following sets of states: $M_1 = \{PS_a, PS_b\}$ and $M_2 = (PS \setminus M_1) \cup MS$. Firstly, we show that $R(M_1) > R(M_2)$, this gives us the right to affirm that the stochastically stable state must be in M_1. Secondly, we move PS_a into M_2 and show sufficient conditions for the two states in M_1 to be stochastically stable. Let us start by calculating $R(M_1)$, which is the minimum cost to escape the union of the basins of attraction of both PS_a and PS_b.
The dimension of the basins of attraction of the two states is determined by the value of c. Whenever a state is inside the basin of attraction of PS_a, ba is optimal for B, and a is the best reply for A. Similarly, ab is optimal for A inside the basin of attraction of PS_b and b is optimal for B. Hence, in order to escape these basins of attraction, we need ε errors. Now, we need to know the dimension of these basins of attraction for the values of c considered. We start from PS_a and the argument stands for the other states in PS for symmetry of payoffs matrix. There are many roads to exit from the basin of attraction of this state: the shortest depends on payoffs and groups size. Firstly, we consider the one that convinces B players to play a. Consider the choice of a B player inside a category of states where $X^{\bar{B}B} \in \left[0, \frac{N_B\Pi_B - \Pi_B}{\Pi_B + \pi_B}\right)$ and $X^{\bar{A}B} \in \left(\frac{N_A\Pi_B}{\Pi_B + \pi_B}, N_A\right]$. Referring to Equation (5) to (8), the optimal level of c s.t. 1 is the best reply for B players is $c < \min\left\{\frac{N_B\Pi_B - X^{\bar{B}B}(\Pi_B + \pi_B) - \Pi_B}{N-1}, \frac{X^{\bar{A}B}(\Pi_B + \pi_B) - N_A\Pi_B}{N-1}\right\}$. If $0 < c < \min\{\frac{\pi_A}{N-1}, \frac{\pi_B}{N-1}\}$, whenever $X^{\bar{B}B} \in \left[0, \frac{N_B\Pi_B - \Pi_B}{\Pi_B + \pi_B}\right)$ and $X^{\bar{A}B} \in \left(\frac{N_A\Pi_B}{\Pi_B + \pi_B}, N_A\right]$, 1 is the BR for B. Therefore, we are out of the basin of attraction of PS_a if the system reaches a state where $X^{\bar{B}B} \geq \frac{N_B\Pi_B + \pi_B}{\Pi_B + \pi_B}$. Starting from $X^{\bar{B}B} = 0$, this comes at

cost $\frac{N_B\Pi_B+\pi_B}{\Pi_B+\pi_B}$. This cost is determined by ε errors, since once in PS_a it is sufficient that a number of B plays by mistake b. Another possible path is to convince A playing ba. This comes at cost $\frac{N_A\Pi_A+\pi_A}{\Pi_A+\pi_A}$. With similar arguments, it is possible to show that the cost of exit from M_1 starting from PS_b is the same. For this reason, $R(M_1) = \min\left\{\frac{N_B\Pi_B+\pi_B}{\Pi_B+\pi_B}, \frac{N_A\Pi_A+\pi_A}{\Pi_A+\pi_A}\right\}$.
Likewise, we can show that the easiest way to escape M_2 is either to reach PS_a from MS_a, or to reach PS_b from MS_b. Convincing players playing their favorite action inside-group is always the cheapest option. Therefore, $R(M_2) = \min\left\{\frac{N_A\pi_A+\Pi_A}{\Pi_A+\pi_A}, \frac{N_B\pi_B+\Pi_B}{\Pi_B+\pi_B}\right\}$.
With simple algebra, it is easy to show that $R(M_1) > R(M_2)$ in every case. This means that the stochastically stable state must be in M_1.
In this second part of the proof we "move" PS_a in M_2 and give conditions such that $R(M_1') > R(M_2')$, with obviously $M_1' = PS_b$ and $M_2' = MS \cup (PS \setminus M_1')$. We evaluate the 2 Radius: differently from before we can include in $R(M_1')$ the possibility to escape reaching PS_a. Symmetrically, we can escape from M_2' reaching M_1' from PS_a. Let us consider the path that goes from M_1' to PS_a. It is sufficient that $\frac{N_B\pi_A}{\Pi_A+\pi_A}$ players from A start playing a to reach the basin of attraction of PS_a. Indeed, once in $(N_A, \frac{N_B\pi_A}{\Pi_A+\pi_A}, 0, 0)$, we can reach PS_a or MS_a without other errors needed, since at least for A type a is now BR.
Since $\frac{N_B\pi_A}{\Pi_A+\pi_A} < \min\left\{\frac{N_B\Pi_B+\pi_B}{\Pi_B+\pi_B}, \frac{N_A\Pi_A+\pi_A}{\Pi_A+\pi_A}\right\}$, we can say that $R(M_1') = \frac{N_B\pi_A}{\Pi_A+\pi_A}$. With a similar argument it can be shown that $R(M_2') = \frac{N_A\pi_B}{\Pi_B+\pi_B}$: now that PS_a is not in M_1', it is not anymore sufficient to reach the union of the two basins of attraction but we need to reach the one of PS_b, hence, the cost is higher than before. In this case, $R(M_2') \lesseqgtr R(M_1')$ when

$$\frac{N_B}{N_A} \lesseqgtr \frac{\pi_B}{\pi_A}.$$

We conclude saying that if $R(M_2') > R(M_1')$ then PS_a is uniquely stochastically stable. The same but opposite reasoning is valid for PS_b when $R(M_1') > R(M_2')$.

□