

PERSPECTIVES ON THE YANG–BAXTER EQUATION IN BCK-ALGEBRAS

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Abstract

We present set-theoretical solutions of the Yang-Baxter equation in *BCK*-algebras. Some solutions in *BCK*-algebras are not solutions in other structures (such as *MV*-algebras). Related to our investigations we also consider some new structures: Boolean coalgebras and a unified braid condition – quantum Yang-Baxter equation. Finally, we will see how poetry has accompanied the development / history of the Yang–Baxter equation.

Keywords: BCK-algebras, Yang-Baxter Equation, Quasi-negation operator, Boolean coalgebras, poetry

MSC: 03G25, 16T25, 06F35

1. INTRODUCTION

The Yang-Baxter equation was first discovered by Nobel laureate C.N. Yang in theoretical physics ([1]) and by R.J. Baxter in statistical mechanics ([2, 3]). It turned out to be one of the main equations in mathematical physics, integrable systems, quantum algebraic systems, the theory of quantum groups, quantum computing, knot theory, braided categories, etc (see [4]). Yang initially considered the matrix equation $F(x)G(x+y)F(y) = G(y)F(x+y)G(x)$, and found an explicit solution where $F(x)$ and $G(x)$ are rational functions. Other versions of this equation were later proved to be very useful, and many scientists have used the axioms of various algebraic structures (Hopf algebras, Yetter-Drinfeld categories, quandles, group actions, Lie (super)algebras, (co)algebra structures, Boolean algebras, relations on sets and so on) in order to obtain solutions for these versions of the Yang-Baxter equation ([5]). F.F. Nichita et al. obtained results on Jordan algebras and Jordan coalgebras, and related them with the Yang-Baxter equations (see, for example, [6, 7] and the references therein). Constructions of quantum gates and link invariants from solutions of the Yang-Baxter equation were described in [8, 9]. Some solutions for the Yang-Baxter equation in *MV*-algebras, Wajsberg-algebras, *MTL*-algebras, weak implication algebras, and lattice effect algebras were investigated in [10, 11, 12, 13].

In this paper we consider BCK-algebras, which were introduced by Y. Imai and K. Iseki ([14]). BCK-algebras are generalizations of the notion of algebraic sets with subtraction and the notion of implication algebra ([14, 15]). These notions are derived in two different ways: one of them is based on set theory and the other is based on non-classical and classical propositional calculus system.

In the next section we will recall some fundamental definitions, lemmas and theorems which are needed for constructing solutions of Yang–Baxter equation in BCK-algebras. We will also define Boolean coalgebras. In Section 3, we will present explicit set-theoretical solutions. Our propositions, lemmas and theorems will hopefully give a new perspective on Yang–Baxter equation (in BCK-algebras). We also propose a braid-quantum Yang-Baxter equation, whose solutions include both solutions of the braid equation and solutions to the quantum Yang-Baxter equation. Finally, a section on poetry related to the Yang-Baxter equation concludes our paper.

2. PRELIMINARIES

Throughout of this section, we give fundamental definitions, lemmas and theorems about the structure of *BCK*–algebras. These notions are taken from [16].

Definition 2.1. *An algebra $\mathcal{A} = (A; \rightarrow, 1)$ of type $(2, 0)$ is said to be a *BCK*–algebra if it verifies the following identities*

- (i) $(x \rightarrow y) \rightarrow ((y \rightarrow z) \rightarrow (x \rightarrow z)) = 1$,
 - (ii) $x \rightarrow ((x \rightarrow y) \rightarrow y) = 1$,
 - (iii) $x \rightarrow x = 1$,
 - (iv) $x \rightarrow 1 = 1$,
 - (v) $x \rightarrow y = 1$ and $y \rightarrow x = 1$ imply $x = y$.
- for each $x, y \in A$.

Lemma 2.2. *The binary relation \leq on A given by*

$$x \leq y \Leftrightarrow x \rightarrow y = 1$$

is a partial order on A with 1 as the biggest element.

As opposed to Lemma 2.2, the poset $(A; \leq)$ has no particular property because any poset $(P; \leq)$ with 1 can be made a *BCK*–algebra by setting $a \rightarrow b := 1$ for $a \leq b$, and $a \rightarrow b := b$ otherwise for any $a, b \in P$.

Definition 2.3. *An algebra $\mathcal{A} = (A; \rightarrow, 1)$ is said to be a bounded *BCK*–algebra, where $(A; \rightarrow, 1)$ is a *BCK*–algebra with the least element 0 such that $0 \rightarrow x = x$.*

Lemma 2.4. *Let $(A; \rightarrow, 1)$ be a *BCK*–algebra. Then*

- (a) $x \leq y \Rightarrow y \rightarrow z \leq x \rightarrow z$,
- (b) $x \leq y \Rightarrow z \rightarrow x \leq z \rightarrow y$,
- (c) $x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z)$,
- (d) $y \leq x \rightarrow y$,
- (e) $1 \rightarrow x = x$,
- (f) $x \rightarrow y \leq (z \rightarrow x) \rightarrow (z \rightarrow y)$,
- (g) $((x \rightarrow y) \rightarrow y) \rightarrow y = x \rightarrow y$,

are satisfied for all $x, y, z \in A$.

The commutative BCK -algebras can be characterized as join-semilattices by defining the \vee -operation as follows:

$$(1) \quad x \vee y := (x \rightarrow y) \rightarrow y.$$

Definition 2.5. Let $\mathcal{A} = (A; \rightarrow, 0, 1)$ be a bounded BCK -algebra. The mapping N is defined on A as

$$N(x) := x \rightarrow 0$$

for each $x \in A$.

Lemma 2.6. Let $\mathcal{A} = (A; \rightarrow, 0, 1)$ be a bounded BCK -algebra. The mapping N is an antitone involution on A .

Proof. Let $a, b \in A$ and $a \leq b$. By substituting $[c := 0]$ in Lemma 2.4 a , we obtain $b \rightarrow 0 \leq a \rightarrow 0$. By the definition of the mapping N , we get $N(b) \leq N(a)$. Then N is an antitone mapping.

Let $a \in A$. From the Equation 1, we have $N(N(x)) = (x \rightarrow 0) \rightarrow 0 = x \vee 0 = x$. Then $N(N(x)) = x$. So, N is an involution mapping. \square

Definition 2.7. Let $\mathcal{A} = (A; \rightarrow, 0, 1)$ be a bounded BCK -algebra. The binary operations \sqcup and \sqcap are defined as

$$x \sqcup y := (x \rightarrow y) \rightarrow y,$$

$$x \sqcap y := N(N(x) \sqcup N(y)),$$

for each $x, y \in A$.

Definition 2.8. A commutative BCK -algebra is a BCK -algebra that satisfies the identity

$$x \sqcup y = y \sqcup x$$

for each $x, y \in A$.

Theorem 2.9. Let $\mathcal{A} = (A; \rightarrow, 1)$ be a BCK -algebra. Define a unary operation N_k on the section $[k, 1] = \{x \in A : k \leq x\}$ for each $a \in A$ by

$$N_k(x) = x \rightarrow k.$$

Then the structure $\Omega(\mathcal{A}) = (A; \sqcup, N_k, 1)$ is satisfies the following quasi-identities:

- (i) $a \sqcup a = a$,
- (ii) $a \sqcup b = b$ and $b \sqcup a = a$ imply $a = b$,
- (iii) $a \sqcup b = (a \sqcup b) \sqcup b = a \sqcup (a \sqcup b) = b \sqcup (a \sqcup b)$,
- (iv) $(a \sqcup c) \sqcup ((a \sqcup b) \sqcup c) = (a \sqcup b) \sqcup c$,
- (v) $a \sqcup 1 = 1$,
- (vi) $N_a(a) = 1$, $N_a(1) = a$,
- (vii) $a \sqcup b = N_b(N_b(a \sqcup b)) = N_b(N_b(a \sqcup b) \sqcup b)$,
- (viii) $N_b(a \sqcup b) \sqcup N_{b \sqcup c}((a \sqcup c) \sqcup (b \sqcup c)) = N_{b \sqcup c}((a \sqcup c) \sqcup (b \sqcup c))$,
- (ix) $N_{b \sqcup c}(N_c(a \sqcup c) \sqcup (b \sqcup c)) = N_{a \sqcup c}(N_c(b \sqcup c) \sqcup (a \sqcup c))$,
- (ix) $N_a((a \sqcup b) \sqcup a) = N_a(a \sqcup b)$.

for each $a, b, c \in A$.

Lemma 2.10. Let $(A; \sqcup)$ be defined as Theorem 2.9. The binary relation \leq is defined by

$$x \leq y \quad \text{if and only if} \quad x \sqcup y = y.$$

Then, the binary relation \leq is a partial order on A . Moreover, $x \sqcup y$ is the least upper bound of x and y . Dually, $x \sqcap y$ is the greatest lower bound of x and y

Lemma 2.11. Let $\mathcal{A} = (A; \rightarrow, 1)$ be a BCK-algebra. The binary operation \sqcup is defined as Theorem 2.9. Then the following statements

- (i) \mathcal{A} is commutative,
- (ii) $(A; \sqcup)$ is a directoid,
- (iii) $(A; \sqcup)$ is a join-semilattice,

are equivalent to each other.

Theorem 2.12. Let $\Omega = (S; \sqcup, N_k, 1)$ be a structure on S . The binary operation \rightarrow is defined on S as

$$a \rightarrow b := N_b(a \sqcup b).$$

Then, $\mathcal{A}(\Omega) = (S; \rightarrow, 1)$ is a BCK-algebra.

Definition 2.13. Let $\mathcal{A} = (A; \rightarrow, 1)$ be a BCK-algebra.

- If it verifies

$$x \rightarrow (y \rightarrow z) = (x \rightarrow y) \rightarrow (x \rightarrow z)$$

for each $x, y, z \in A$ then, it is called a positive implicative BCK-algebra.

- If it verifies

$$(x \rightarrow y) \rightarrow z = (x \rightarrow z) \rightarrow (y \rightarrow z)$$

for each $x, y, z \in A$ then, it is called a negative implicative BCK-algebra.

At the end of this preliminary section, let us define a new structure which will be used in our search for solutions to the Yang-Baxter equation. Further investigations in the framework of BCK-algebras will continue in the future.

Definition 2.14. A Boolean coalgebra is defined as a 6-tuple $\mathcal{C} = (C, \vee, \Delta, N, 0, 1)$, where \vee , N , 0 and 1 have the usual properties. (So, \vee is an associative and commutative operation, N is an involution, $x \vee 0 = x$, $x \vee 1 = 1$, etc.)

The new structure is $\Delta : C \rightarrow C \times C$, $\Delta(a) = (a_1, a_2)$, and we require:

- (i) Δ is coassociative (i.e., $(\Delta \times I) \circ \Delta = (I \times \Delta) \circ \Delta$),
- (ii) $a_1 \vee a_2 = a \quad \forall a \in C$,
- (iii) $\Delta(0) = (0, 0)$ and
- (iv) $\Delta(a \vee b) = \Delta(a) \vee \Delta(b) \quad \forall a, b \in C$ (Notice that this equality takes place in $C \times C$).

For an arbitrary Boolean algebra, we can associate a Boolean coalgebra with $\Delta(a) = (a \wedge c, a \wedge N(c))$. Also, if we recall that $a \rightarrow b = N(a) \vee b$, we obtain a BCK-algebra with the following property: $\Delta(a \rightarrow b) = \Delta(a \rightarrow 0) \rightarrow \Delta(b) \quad \forall a, b \in C$.

3. A PERSPECTIVE ON YANG-BAXTER EQUATION IN BCK-ALGEBRAS

In this part of this paper, we present some set-theoretical solutions of Yang-Baxter equation in BCK-algebras. Moreover, we define new operators on BCK-algebras, then we obtain new solutions by using these operators.

Let K be a vector space over the field Q . We define the twist map by $\mathcal{U}(p \otimes q) = q \otimes p$, where $\mathcal{U} : K \otimes K \rightarrow K \otimes K$. Besides, the identity map of this vector space is defined $I : K \rightarrow K$. As a Q -linear map, we define $R^{12} = R \otimes I$, $R^{23} = I \otimes R$ and $R^{13} = (I \otimes \mathcal{U})(R \otimes I)(\mathcal{U} \otimes I)$.

Definition 3.1. [17] *A Yang-Baxter operator is an invertible Q -linear map $R : K \otimes K \rightarrow K \otimes K$ and it verifies the braid condition (known as the “Yang-Baxter equation” or the “braid condition”)*

$$(2) \quad R^{12} \circ R^{23} \circ R^{12} = R^{23} \circ R^{12} \circ R^{23}.$$

If R verifies the equation (2), then $\tau \circ R$ and $R \circ \tau$ supply the quantum Yang-Baxter equation (known as QYBE):

$$(3) \quad R^{12} \circ R^{13} \circ R^{23} = R^{23} \circ R^{13} \circ R^{12}.$$

Lemma 3.2. [17] *The equations (2) and (3) are equivalent.*

Lemma 3.3. *The equations (2) and (3) lead each to solutions for the following “braid-quantum Yang-Baxter equation”:*

$$(4) \quad R^{12} \circ X^{13} \circ R^{23} \circ Y^{12} = R^{23} \circ X^{13} \circ R^{12} \circ Y^{23},$$

where $R = XY$.

Obviously, finding all solutions for the *braid-quantum Yang-Baxter equation* is an open problem. A first step to solve it would be to construct some solutions for it, and to make a small analysis on those solutions which are not solutions neither for the braid condition, nor for the quantum Yang-Baxter equation.

Back to BCK-algebras, we recall the following definition.

Definition 3.4. [17] *Let P be any set. The mapping $S(p, q) = (p', q')$ is defined from $P \times P$ to $P \times P$. The mapping S satisfies the Yang-Baxter equation (or equivalently, “ S is a set-theoretical solution of the Yang-Baxter equation”) if it holds the following equation*

$$(5) \quad S^{12} \circ S^{23} \circ S^{12} = S^{23} \circ S^{12} \circ S^{23},$$

which is also equivalent to

$$(6) \quad S^{12} \circ S^{13} \circ S^{23} = S^{23} \circ S^{13} \circ S^{12},$$

where

$$\begin{aligned} S^{12} : P \times P \times P &\rightarrow P \times P \times P, & S^{12}(p, q, r) &= (p', q', r), \\ S^{23} : P \times P \times P &\rightarrow P \times P \times P, & S^{23}(p, q, r) &= (p, q', r'), \\ S^{13} : P \times P \times P &\rightarrow P \times P \times P, & S^{13}(p, q, r) &= (p', q, r'). \end{aligned}$$

Now, we may handle verifying the Yang-Baxter equation in BCK-algebras. First of all, we give the following lemma which is needed for further processing of this work.

Proposition 3.5. [16] *Let $(A; \rightarrow, 1)$ be a BCK-algebra. Then*

- (1) $(0 \rightarrow 0) \rightarrow x = x$
- (2) $(x \rightarrow 0) \rightarrow 0 = x$
- (3) $(z \rightarrow 0) \rightarrow y = (y \rightarrow 0) \rightarrow z$
- (4) $x \rightarrow (y \rightarrow x) = 1$.

hold for all $x, y, z \in A$.

Lemma 3.6. *Let $(A, \rightarrow, 0, 1)$ be a bounded BCK-algebra. Then, the mapping $S(x, y) = (y \rightarrow 0, x \rightarrow 0)$ verifies the braid condition on this structure.*

Lemma 3.7. *Let $(A, \rightarrow, 1)$ be a BCK-algebra. Then, the mapping $S(x, y) = (1 \rightarrow x, 1 \rightarrow y)$ verifies the braid condition on this structure.*

Lemma 3.8. *Let $(A; \rightarrow, 0, 1)$ be a bounded commutative BCK-algebra. Then, the mapping $S(x, y) = ((x \rightarrow 0) \rightarrow y, 0)$ verifies the braid condition on this structure. As a conclusion, the Yang-Baxter equation has a set-theoretical solution in BCK-algebras.*

Proof. We define S^{12} and S^{23} as follows:

$$\begin{aligned} S^{12}(x, y, z) &= ((x \rightarrow 0) \rightarrow y, 0, z), \\ S^{23}(x, y, z) &= (x, (y \rightarrow 0) \rightarrow z, 0). \end{aligned}$$

We show that the equilibrium $S^{12} \circ S^{23} \circ S^{12} = S^{23} \circ S^{12} \circ S^{23}$ are satisfied for each $(x, y, z) \in A \times A \times A$. By the help of the Definition 2.1, Lemma 2.4 (c) and (e) and Proposition 3.5 (2) and (3), we have

$$\begin{aligned} (S^{12} \circ S^{23} \circ S^{12})(x, y, z) &= S^{12}(S^{23}(S^{12}(x, y, z))) \\ &= S^{12}(S^{23}((x \rightarrow 0) \rightarrow y, 0, z)) \\ &= S^{12}((x \rightarrow 0) \rightarrow y, (0 \rightarrow 0) \rightarrow z, 0) \\ &= S^{12}((x \rightarrow 0) \rightarrow y, z, 0) \\ &= (((x \rightarrow 0) \rightarrow y) \rightarrow 0) \rightarrow z, 0, 0) \\ &= (((x \rightarrow 0) \rightarrow y) \rightarrow 0) \rightarrow ((z \rightarrow 0) \rightarrow 0), 0, 0) \\ &= ((z \rightarrow 0) \rightarrow (((x \rightarrow 0) \rightarrow y) \rightarrow 0) \rightarrow 0), 0, 0) \\ &= ((z \rightarrow 0) \rightarrow ((x \rightarrow 0) \rightarrow y), 0, 0) \\ &= ((x \rightarrow 0) \rightarrow ((z \rightarrow 0) \rightarrow y), 0, 0) \\ &= ((x \rightarrow 0) \rightarrow ((y \rightarrow 0) \rightarrow z), 0, 0) \end{aligned}$$

and

$$\begin{aligned} (S^{23} \circ S^{12} \circ S^{23})(x, y, z) &= S^{23}(S^{12}(S^{23}(x, y, z))) \\ &= S^{23}(S^{12}(x, (y \rightarrow 0) \rightarrow z, 0)) \\ &= S^{23}((x \rightarrow 0) \rightarrow ((y \rightarrow 0) \rightarrow z), 0, 0) \\ &= ((x \rightarrow 0) \rightarrow ((y \rightarrow 0) \rightarrow z), (0 \rightarrow 0) \rightarrow 0, 0) \\ &= ((x \rightarrow 0) \rightarrow ((y \rightarrow 0) \rightarrow z), 0, 0). \end{aligned}$$

Thus, the Yang-Baxter equation is satisfied in BCK-algebras. The mapping $S(x, y) = ((x \rightarrow 0) \rightarrow y, 0)$ is a set-theoretical solution of it on these structures. \square

Lemma 3.9. *Let $(A, \rightarrow, 1)$ be a commutative BCK-algebra. Then, the mapping $S(x, y) = ((x \rightarrow y) \rightarrow y, y)$ verifies the braid condition on this structure. Therefore, the Yang-Baxter equation has a set-theoretical solution in BCK-algebras.*

Proof. We define S^{12} and S^{23} as follows:

$$\begin{aligned} S^{12}(x, y, z) &= ((x \rightarrow y) \rightarrow y, y, z), \\ S^{23}(x, y, z) &= (x, (y \rightarrow z) \rightarrow z, z). \end{aligned}$$

We show that the equality $S^{12} \circ S^{23} \circ S^{12} = S^{23} \circ S^{12} \circ S^{23}$ is satisfied for each $(x, y, z) \in A \times A \times A$. By the Definition 2.8, Lemma 2.4 (c) and (e) and Proposition 3.5 (4), we obtain

$$\begin{aligned} (S^{12} \circ S^{23} \circ S^{12})(x, y, z) &= S^{12}(S^{23}(S^{12}(x, y, z))) \\ &= S^{12}(S^{23}((x \rightarrow y) \rightarrow y, y, z)) \\ &= S^{12}((x \rightarrow y) \rightarrow y, (y \rightarrow z) \rightarrow z, z) \\ &= (((x \rightarrow y) \rightarrow y) \rightarrow ((y \rightarrow z) \rightarrow z)) \rightarrow ((y \rightarrow z) \rightarrow z), \\ &\quad (y \rightarrow z) \rightarrow z, z) \\ &= (((x \rightarrow y) \rightarrow y) \rightarrow ((z \rightarrow y) \rightarrow y)) \rightarrow ((z \rightarrow y) \rightarrow y), \\ &\quad (y \rightarrow z) \rightarrow z, z) \\ &= (((z \rightarrow y) \rightarrow ((x \rightarrow y) \rightarrow y) \rightarrow y) \rightarrow ((z \rightarrow y) \rightarrow y), \\ &\quad (y \rightarrow z) \rightarrow z, z) \\ &= (((z \rightarrow y) \rightarrow ((y \rightarrow (x \rightarrow y)) \rightarrow (x \rightarrow y))) \rightarrow ((z \rightarrow y) \rightarrow y), \\ &\quad (y \rightarrow z) \rightarrow z, z) \\ &= (((z \rightarrow y) \rightarrow (x \rightarrow y)) \rightarrow ((z \rightarrow y) \rightarrow y), (y \rightarrow z) \rightarrow z, z) \\ &= ((x \rightarrow ((y \rightarrow z) \rightarrow z)) \rightarrow ((y \rightarrow z) \rightarrow z), (y \rightarrow z) \rightarrow z, z) \end{aligned}$$

and

$$\begin{aligned} (S^{23} \circ S^{12} \circ S^{23})(x, y, z) &= S^{23}(S^{12}(S^{23}(x, y, z))) \\ &= S^{23}(S^{12}(x, (y \rightarrow z) \rightarrow z, z)) \\ &= S^{23}((x \rightarrow ((y \rightarrow z) \rightarrow z)) \rightarrow ((y \rightarrow z) \rightarrow z), (y \rightarrow z) \rightarrow z, z) \\ &= ((x \rightarrow ((y \rightarrow z) \rightarrow z)) \rightarrow ((y \rightarrow z) \rightarrow z), \\ &\quad (((y \rightarrow z) \rightarrow z) \rightarrow z) \rightarrow z, z) \\ &= (((x \rightarrow ((y \rightarrow z) \rightarrow z)) \rightarrow ((y \rightarrow z) \rightarrow z), \\ &\quad ((z \rightarrow (y \rightarrow z)) \rightarrow (y \rightarrow z)) \rightarrow z, z) \\ &= ((x \rightarrow ((y \rightarrow z) \rightarrow z)) \rightarrow ((y \rightarrow z) \rightarrow z), (y \rightarrow z) \rightarrow z, z). \end{aligned}$$

Then, the Yang-Baxter equation has a one set-theoretical solution $S(x, y) = ((x \rightarrow y) \rightarrow y, y)$ in BCK-algebras. \square

Lemma 3.10. *Let $(A, \rightarrow, 0, 1)$ be a bounded BCK-algebra. Then, the mapping $S(x, y) = ((y \rightarrow (x \rightarrow 0)) \rightarrow 0, 1)$ verifies the braid condition on this structure. Therefore, the Yang-Baxter equation has a set-theoretical solution in BCK-algebras.*

Proof. Let S^{12} and S^{23} be defined as follows:

$$\begin{aligned} S^{12}(x, y, z) &= ((y \rightarrow (x \rightarrow 0)) \rightarrow 0, 1, z), \\ S^{23}(a, b, c) &= (x, (z \rightarrow (y \rightarrow 0)) \rightarrow 0, 1). \end{aligned}$$

By the help of Definition 2.1, Lemma 2.4 (e), Proposition 3.5 (2), we have

$$\begin{aligned} (S^{12} \circ S^{23} \circ S^{12})(x, y, z) &= S^{12}(S^{23}(S^{12}(x, y, z))) \\ &= S^{12}(S^{23}((y \rightarrow (x \rightarrow 0)) \rightarrow 0, 1, z)) \\ &= S^{12}((y \rightarrow (x \rightarrow 0)) \rightarrow 0, (z \rightarrow (1 \rightarrow 0)) \rightarrow 0, 1) \\ &= S^{12}((y \rightarrow (x \rightarrow 0)) \rightarrow 0, z, 1) \\ &= ((z \rightarrow ((y \rightarrow (x \rightarrow 0)) \rightarrow 0) \rightarrow 0) \rightarrow 0, 1, 1) \\ &= ((z \rightarrow (((y \rightarrow 0) \rightarrow 0) \rightarrow (x \rightarrow 0))) \rightarrow 0, 1, 1) \end{aligned}$$

and

$$\begin{aligned} (S^{23} \circ S^{12} \circ S^{23})(x, y, z) &= S^{23}(S^{12}(S^{23}(x, y, z))) \\ &= S^{23}(S^{12}(x, (z \rightarrow (y \rightarrow 0)) \rightarrow 0, 1)) \\ &= S^{23}((((z \rightarrow (y \rightarrow 0)) \rightarrow 0) \rightarrow (x \rightarrow 0)) \rightarrow 0, 1, 1) \\ &= (((z \rightarrow (y \rightarrow 0)) \rightarrow 0) \rightarrow (x \rightarrow 0)) \rightarrow 0, \\ &\quad (1 \rightarrow (1 \rightarrow 0)) \rightarrow 0, 1) \\ &= (((z \rightarrow (y \rightarrow 0)) \rightarrow 0) \rightarrow (x \rightarrow 0)) \rightarrow 0, 1, 1). \end{aligned}$$

Then, the Yang-Baxter equation is satisfied in BCK -algebras. The mapping $S(x, y) = ((x \rightarrow y) \rightarrow y, y)$ is a set-theoretical solution of this equation on BCK -algebras. \square

Example 3.11. Let $X = \{0, x, 1\}$. The operation \rightarrow is defined as the following table:

\rightarrow	0	x	1
0	1	1	1
x	x	1	1
1	0	x	1

Then, $(X, \rightarrow, 0, 1)$ is a bounded BCK -algebra. Moreover, the mapping $S(a, b) = (((b \rightarrow 0) \rightarrow (a \rightarrow 0)) \rightarrow 0, 0)$ verifies the braid condition on this structure. Therefore, the Yang-Baxter equation has a set-theoretical solution in this structure.

Proof. Let S^{12} and S^{23} be defined as follows:

$$\begin{aligned} S^{12}(a, b, c) &= (((b \rightarrow 0) \rightarrow (a \rightarrow 0)) \rightarrow 0, 0, c), \\ S^{23}(a, b, c) &= (a, ((c \rightarrow 0) \rightarrow (b \rightarrow 0)) \rightarrow 0, 0). \end{aligned}$$

By using Definition 2.1, Lemma 2.4 (e), Proposition 3.5 (2), we get

$$\begin{aligned}
 (S^{12} \circ S^{23} \circ S^{12})(a, b, c) &= S^{12}(S^{23}(S^{12}(a, b, c))) \\
 &= S^{12}(S^{23}(((b \rightarrow 0) \rightarrow (a \rightarrow 0)) \rightarrow 0, 0, c)) \\
 &= S^{12}(((b \rightarrow 0) \rightarrow (a \rightarrow 0)) \rightarrow 0, ((c \rightarrow 0) \rightarrow (0 \rightarrow 0)) \\
 &\quad \rightarrow 0, 0) \\
 &= S^{12}(((b \rightarrow 0) \rightarrow (a \rightarrow 0)) \rightarrow 0, ((c \rightarrow 0) \rightarrow 1) \rightarrow 0, 0) \\
 &= S^{12}(((b \rightarrow 0) \rightarrow (a \rightarrow 0)) \rightarrow 0, 0, 0) \\
 &= (((((0 \rightarrow 0) \rightarrow (((b \rightarrow 0) \rightarrow (a \rightarrow 0)) \rightarrow 0) \rightarrow 0) \\
 &\quad \rightarrow 0), 0, 0) \\
 &= (((((b \rightarrow 0) \rightarrow (a \rightarrow 0)) \rightarrow 0) \rightarrow 0) \rightarrow 0, 0, 0) \\
 (7) \quad &= (((b \rightarrow 0) \rightarrow (a \rightarrow 0)) \rightarrow 0, 0, 0),
 \end{aligned}$$

and

$$\begin{aligned}
 (S^{23} \circ S^{12} \circ S^{23})(a, b, c) &= S^{23}(S^{12}(S^{23}(a, b, c))) \\
 &= S^{23}(S^{12}(a, ((c \rightarrow 0) \rightarrow (b \rightarrow 0)) \rightarrow 0, 0)) \\
 &= S^{23}((((((c \rightarrow 0) \rightarrow (b \rightarrow 0)) \rightarrow 0) \rightarrow 0) \rightarrow (a \rightarrow 0)) \\
 &\quad \rightarrow 0, 0, 0) \\
 &= (((((((c \rightarrow 0) \rightarrow (b \rightarrow 0)) \rightarrow 0) \rightarrow 0) \rightarrow (a \rightarrow 0)) \rightarrow 0, \\
 &\quad ((0 \rightarrow 0) \rightarrow (0 \rightarrow 0)) \rightarrow 0, 0) \\
 (8) \quad &= (((((((c \rightarrow 0) \rightarrow (b \rightarrow 0)) \rightarrow 0) \rightarrow 0) \rightarrow (a \rightarrow 0)) \\
 &\quad \rightarrow 0, 0, 0).
 \end{aligned}$$

Since the Equation (7) is equal to the Equation (8) for all $a, b, c \in X$, we obtain that the Yang-Baxter equation is satisfied in this structure. The mapping $S(a, b) = (((b \rightarrow 0) \rightarrow (a \rightarrow 0)) \rightarrow 0, 0)$ is a set-theoretical solution of this equation on this structure, whereas it is not a set-theoretical solution of the Yang-Baxter equation in bounded BCK-algebras. \square

Example 3.12. *The mapping $S(x, y) = (x \rightarrow y, x)$ is a set-theoretical solution of the Yang-Baxter equation in Boolean algebras and implicative BCK-algebras (see [18]) while it is not a set-theoretical solution of the Yang-Baxter equation in MV-algebras:*

$$\begin{aligned}
 (S^{12} \circ S^{23} \circ S^{12})(x, y, z) &= S^{12}(S^{23}(S^{12}(x, y, z))) \\
 &= S^{12}(S^{23}(x \rightarrow y, x, z)) \\
 &= S^{12}(x \rightarrow y, x \rightarrow z, x) \\
 &= ((x \rightarrow y) \rightarrow (x \rightarrow z), x \rightarrow y, x)
 \end{aligned}$$

and

$$\begin{aligned}
 (S^{23} \circ S^{12} \circ S^{23})(x, y, z) &= S^{23}(S^{12}(S^{23}(x, y, z))) \\
 &= S^{23}(S^{12}(x, y \rightarrow z, y)) \\
 &= S^{23}(x \rightarrow (y \rightarrow z), x, y) \\
 &= (x \rightarrow (y \rightarrow z), x \rightarrow y, x),
 \end{aligned}$$

Then, the Yang-Baxter equation has a set-theoretical solution $S(x, y) = (x \rightarrow y, x)$ in positive implicative BCK-algebras. Since $x \rightarrow y$ corresponds to $\neg x \oplus y$ in MV-algebras. Then, we get

$$\begin{aligned}(S^{12} \circ S^{23} \circ S^{12})(x, y, z) &= S^{12}(S^{23}(S^{12}(x, y, z))) \\ &= S^{12}(S^{23}(\neg x \oplus y, x, z)) \\ &= S^{12}(\neg x \oplus y, \neg x \oplus z, x) \\ &= (\neg(\neg x \oplus y) \oplus (\neg x \oplus z), \neg x \oplus y, x)\end{aligned}$$

and

$$\begin{aligned}(S^{23} \circ S^{12} \circ S^{23})(x, y, z) &= S^{23}(S^{12}(S^{23}(x, y, z))) \\ &= S^{23}(S^{12}(x, \neg y \oplus z, y)) \\ &= S^{23}(\neg x \oplus (\neg y \oplus z), x, y) \\ &= (\neg x \oplus (\neg y \oplus z), \neg x \oplus y, x)\end{aligned}$$

is not a set-theoretical solution of the Yang-Baxter equation as $\neg(\neg x \oplus y) \oplus (\neg x \oplus z) \neq \neg x \oplus (\neg y \oplus z)$.

Lemma 3.13. Let $(A, \rightarrow, 0, 1)$ be a positive implicative BCK-algebra. Then, the mapping $S(x, y) = (x \rightarrow y, 1 \rightarrow x)$ verifies the braid condition on this structure, i.e., the Yang-Baxter equation has a set-theoretical solution in BCK-algebras.

Proof. Let S^{12} and S^{23} be defined as follows:

$$\begin{aligned}S^{12}(x, y, z) &= (x \rightarrow y, 1 \rightarrow x, z), \\ S^{23}(x, y, z) &= (x, y \rightarrow z, 1 \rightarrow z).\end{aligned}$$

From Lemma 2.4 (e), we have

$$\begin{aligned}(S^{12} \circ S^{23} \circ S^{12})(x, y, z) &= S^{12}(S^{23}(S^{12}(x, y, z))) \\ &= S^{12}(S^{23}(x \rightarrow y, 1 \rightarrow x, z)) \\ &= S^{12}(x \rightarrow y, (1 \rightarrow x) \rightarrow z, 1 \rightarrow (1 \rightarrow x)) \\ &= ((x \rightarrow y) \rightarrow ((1 \rightarrow x) \rightarrow z), 1 \rightarrow (x \rightarrow y), 1 \rightarrow (1 \rightarrow x))\end{aligned}$$

and

$$\begin{aligned}(S^{23} \circ S^{12} \circ S^{23})(x, y, z) &= S^{23}(S^{12}(S^{23}(x, y, z))) \\ &= S^{23}(S^{12}(x, y \rightarrow z, 1 \rightarrow y)) \\ &= S^{23}x \rightarrow (y \rightarrow z), 1 \rightarrow x, 1 \rightarrow y) \\ &= (x \rightarrow (y \rightarrow z), (1 \rightarrow x) \rightarrow (1 \rightarrow y), 1 \rightarrow (1 \rightarrow x)).\end{aligned}$$

Therefore, the Yang-Baxter equation has a one set-theoretical solution $S(x, y) = (x \rightarrow y, 1 \rightarrow x)$ in positive implicative BCK-algebras. \square

Lemma 3.14. Let $(A, \rightarrow, 0, 1)$ be a bounded negative implicative BCK-algebra. The mapping $S(x, y) = (((y \rightarrow 0) \rightarrow (x \rightarrow 0)) \rightarrow 0, x)$ verifies the braid condition on this structure. Therefore, the Yang-Baxter equation has a set-theoretical solution in BCK-algebras.

Proof. Let S^{12} and S^{23} be defined as follows:

$$\begin{aligned} S^{12}(x, y, z) &= (((y \rightarrow 0) \rightarrow (x \rightarrow 0)) \rightarrow 0, x, z), \\ S^{23}(x, y, z) &= (x, ((z \rightarrow 0) \rightarrow (y \rightarrow 0)) \rightarrow 0, y). \end{aligned}$$

We have

$$\begin{aligned} (S^{12} \circ S^{23} \circ S^{12})(x, y, z) &= S^{12}(S^{23}(S^{12}(x, y, z))) \\ &= S^{12}(S^{23}(((y \rightarrow 0) \rightarrow (x \rightarrow 0)) \rightarrow 0, x, z)) \\ &= S^{12}(((y \rightarrow 0) \rightarrow (x \rightarrow 0)) \rightarrow 0, ((z \rightarrow 0) \rightarrow \\ &\quad (x \rightarrow 0)) \rightarrow 0, x) \\ &= (((((z \rightarrow 0) \rightarrow (x \rightarrow 0)) \rightarrow 0) \rightarrow 0) \rightarrow (((y \rightarrow 0) \rightarrow \\ &\quad (x \rightarrow 0)) \rightarrow 0) \rightarrow 0, ((y \rightarrow 0) \rightarrow (x \rightarrow 0)) \rightarrow 0, x) \end{aligned}$$

and

$$\begin{aligned} (S^{23} \circ S^{12} \circ S^{23})(x, y, z) &= S^{23}(S^{12}(S^{23}(x, y, z))) \\ &= S^{23}(S^{12}(x, ((z \rightarrow 0) \rightarrow (y \rightarrow 0)) \rightarrow 0, y)) \\ &= S^{23}((((z \rightarrow 0) \rightarrow (y \rightarrow 0)) \rightarrow 0) \rightarrow 0 \rightarrow (x \rightarrow 0))) \\ &\quad \rightarrow 0, x, y) \\ &= (((((z \rightarrow 0) \rightarrow (y \rightarrow 0)) \rightarrow 0) \rightarrow 0) \rightarrow (x \rightarrow 0))) \rightarrow 0, \\ &\quad ((y \rightarrow 0) \rightarrow (x \rightarrow 0)) \rightarrow 0, x). \end{aligned}$$

Then, the Yang-Baxter equation is satisfied in negative implicative BCK-algebras. The mapping $S(x, y) = (((y \rightarrow 0) \rightarrow (x \rightarrow 0)) \rightarrow 0, x)$ is a set-theoretical solution of it. \square

Lemma 3.15. *Let $(A, \rightarrow, 0, 1)$ be a bounded BCK-algebra. The mapping $S(x, y) = (N(y), N(x))$ is a set-theoretical solution of the Yang-Baxter equation in BCK-algebras. Moreover, for each $k \in A$ and for every $x, y \in [k, 1]$, $S(x, y) = (N_k(y), N_k(x))$ also verifies the braid condition on this structure. Therefore, Yang-Baxter equation has a set-theoretical solution in bounded BCK-algebras.*

Proof. It follows from the Definition 2.5, Lemma 3.2 and Theorem 2.9. \square

Lemma 3.16. *Let $\mathcal{A} = (A; \rightarrow, 0, 1)$ be a bounded BCK-algebra. Then the following identity*

$$x \rightarrow y = N(y) \rightarrow N(x)$$

holds for each $x, y \in A$.

Proof. Assume that $x, y \in A$. By using the Definition 2.5 and Lemma 2.4 (c), we obtain

$$\begin{aligned} N(y) \rightarrow N(x) &= (y \rightarrow 0) \rightarrow (x \rightarrow 0) \\ &= x \rightarrow ((y \rightarrow 0) \rightarrow 0) \\ &= x \rightarrow y. \end{aligned}$$

\square

Lemma 3.17. *Let $\mathcal{A} = (A; \rightarrow, 0, 1)$ be a bounded commutative BCK-algebra. Then the identity*

$$N(x \rightarrow N(x \rightarrow y)) = N((y \rightarrow x) \rightarrow N(y))$$

holds for each $x, y \in A$.

Proof. By using commutativity and the Lemma 3.16, we obtain

$$\begin{aligned} N((y \rightarrow x) \rightarrow N(y)) &= N((N(x) \rightarrow N(y)) \rightarrow N(y)) \\ &= N((N(y) \rightarrow N(x)) \rightarrow N(x)) \\ &= N((x \rightarrow y) \rightarrow N(x)) \\ &= N(x \rightarrow N(x \rightarrow y)) \end{aligned}$$

for each $x, y \in A$. □

Lemma 3.18. *Let $(A, \rightarrow, 0, 1)$ be a bounded commutative BCK-algebra. The mapping $S(x, y) = (x \sqcup y, x \sqcap y)$ verifies the braid condition on this structure. Therefore, the Yang-Baxter equation has a set-theoretical solution in bounded BCK-algebras.*

Proof. It follows from Definition 2.7, 3.4 and Lemma 3.17. □

Lemma 3.19. *Let $\mathcal{C} = (C, \vee, \Delta, N, 0, 1)$, be a Boolean coalgebra, then the mapping $S(x, y) = (x_1, x_2)$ verifies the braid condition.*

Proof. It follows from the coassociativity of Δ . □

One can ask about the relationship between the maps from Example 3.12 and Lemma 3.19, but we will leave our proposed problems for the future. The next section is about related poetry.

4. POETRY AND THE YANG-BAXTER EQUATION

In this final section, we will present poetry related to the Yang-Baxter equation.

When B. Karlgren, a member of the Royal Academy of Sciences, addressed the Nobel laureates, Mr Lee and Mr Yang, he recalled some pieces of poetry ([19]):

“... your culture of 3 000 years is really as it is said in the sacred ancient hymn:

*like the Kiang river,
like the Han river,
massive like the mountains,
voluminously flowing like the rivers.”*

“... in the words of a famous Tang poet:

*how can we for a single day
do without these men”*

When Chen Ning Yang, received a birthday present, at the age of 90, lines from the poet Tu Fu were engraved on the top of it:

*“A piece of literature
Is meant for the millennium
But its ups and downs are known
Already in the authors heart.”*

In THOUGHTS ON MY FIRST THEOREM, F.F. Nichita describes in a poetic manner the beginning of the unification theory of algebras and coalgebras structures in the framework of Yang-Baxter equations ([20]):

(*...*)
*The small particle
was captured...
The common piece of information...
The two streams
arrived on my table from overseas
were unified...*

Returning from a daily walk, the above author finds out that his office is full with literature works, but there is no contradiction in this mixture:

A POST-MODERN MANIFEST

*Once... after a promenade,
I gorged a "pomegranate":*

*Abstract cocktail of notes,
mixed with books,
flowers and clothes,
resting on my desks,
falling from the shelves,
rolling on the chair,
flying in the air...*

*From the open volumes,
on inevitable social inequalities,
to the open problems,
on classical means inequalities...*

*Subtle metaphors,
musical measures,
philosophical concepts,
mathematical models,
entwined structures,
historical phrases...*

Amalgamated groups...

Kaleidoscopic traces...

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