Bacteria-bacteriophage cycles facilitate Cholera outbreak cycles: an indirect Susceptible - Infected - Bacteria - Phage (iSIBP) model-based mathematical study

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Mathematical Analysis

1 Existence of Equilibria with No shedding

Substituting $\xi = 0$ in system (1) and noticing that $R = N - S - I$, the third equation is then not necessary. Thus, system (1) reduces to

$$
\dot{S} = -\alpha(B_p)S - \mu S - \eta S + \mu N + \eta N - \eta I
$$

$$
\dot{I} = \alpha(B_p)S - \mu I - \delta I
$$

$$
\dot{B}_p = rB_p \left(1 - \frac{B_p + B_{np}}{K}\right) - \gamma_1 \frac{B_p}{K_1 + B_p} P + \theta(B_{np}, P)
$$

$$
\dot{B}_{np} = rB_{np} \left(1 - \frac{B_p + B_{np}}{K}\right) - \gamma_2 \frac{B_{np}}{K_1 + B_{np}} P - \theta(B_{np}, P)
$$

$$
\dot{P} = \beta \left(\gamma_1 \frac{B_p}{K_1 + B_p} + \gamma_2 \frac{B_{np}}{K_1 + B_{np}}\right) P - dP
$$

Then system (2) becomes:

**Case 1:** if the pathogenic bacteria level is below or equal to the minimum infectious dose, then $\alpha(B_p) = 0$. Hence system (2) becomes:

$$
\dot{S} = -\mu S - \eta S + \mu N + \eta N - \eta I
$$

$$
\dot{I} = -\mu I - \delta I
$$

$$
\dot{B}_p = rB_p \left(1 - \frac{B_p + B_{np}}{K}\right) - \gamma_1 \frac{B_p}{K_1 + B_p} P + \theta(B_{np}, P)
$$

$$
\dot{B}_{np} = rB_{np} \left(1 - \frac{B_p + B_{np}}{K}\right) - \gamma_2 \frac{B_{np}}{K_1 + B_{np}} P - \theta(B_{np}, P)
$$

$$
\dot{P} = \beta \left(\gamma_1 \frac{B_p}{K_1 + B_p} + \gamma_2 \frac{B_{np}}{K_1 + B_{np}}\right) P - dP
$$

Then system (3) has 4 equilibrium points which are listed below:

1. $E_0 = (N, 0, 0, 0, 0)$ always exists.

2. $E_1 = (N, 0, m, K - m, 0)$ always exists, where $m$ is a non-negative constant such that if $K \leq c$, then $m \leq K$, and if $K > c$, then $m \leq c$. Special cases of $E_1$ are:
   a. $E_{11} = (N, 0, 0, K, 0)$.
   b. $E_{12} = (N, 0, K, 0, 0)$, where $K \leq c$.  

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3. \( E_2 = (N, 0, B_{p2}, 0, P_2) \) exists if \( \beta \gamma_1 - d > 0 \). In this case, \( B_{p2} = \frac{dK_1}{\beta \gamma_1 - d} > 0 \) is such that if \( K \leq c \), then \( B_{p2} \leq K \), and if \( K > c \), then \( B_{p2} \leq c \), so that \( P_2 = \frac{r}{\gamma_1 K} (K_1 + B_{p2}) (K - B_{p2}) > 0 \).

4. The interior point \( E^* = (N, 0, B^*_p, B^*_{np}, P^*) \), exits if the following conditions hold:

(i) \( B^*_p < c \).

(ii) \( B^*_p \neq \frac{dK_1}{\beta \gamma_1 - d} = B_{p2} \). Note that the existence of \( B^*_p \) and \( B_{p2} \) is contrary.

(iii) \( B^*_p \neq \frac{K_1(d - \beta \gamma_2)}{\beta (\gamma_1 + \gamma_2) - d} \).

(iv) \( K > B^*_p + B^*_{np} \).

(v) \( B^*_p, B^*_{np} \) and \( P^* > 0 \).

The existence of the equilibria \( E_0 \) and \( E_1 \) is obvious. In the following we will show the existence of equilibria \( E_2 \) and \( E^* \).

Equation (4) of system (3) implies that at steady state either \( B_{np} = 0 \) or \( B_{np} \neq 0 \). Under the assumption that \( B_{np} = 0 \), one can obtain the equilibrium point \( E_2 \) by solving the following system of equations:

\[
0 = -\mu S - \eta S + \mu N + \eta N - \eta I \quad (1)
\]
\[
0 = -\mu I - \delta I \quad (2)
\]
\[
0 = r B_p \left(1 - \frac{B_p}{K}\right) - \gamma_1 \frac{B_p}{K_1 + B_p} P \quad (3)
\]
\[
0 = \beta \left(\frac{B_p}{K_1 + B_p}\right) - d \quad (4)
\]

From equation (2), we have \( I = 0 \) and then from equation (1), we have \( S = N \). From equation (4), we have \( d = \frac{\beta \gamma_1 B_p}{K_1 + B_p} \) so \( B_{p2} = \frac{dK_1}{\beta \gamma_1 - d} \) such that \( B_{p2} \leq c \) and \( \beta \gamma_1 > d \).

From equation (3), we get \( P_2 = \frac{r}{\gamma_1 K} (K_1 + B_{p2}) (K - B_{p2}) \). In order to have a positive value for \( P_2 \), the following conditions on \( B_{p2} \) must hold:

\( B_{p2} < K \) and if \( K \leq c \), then \( B_{p2} < K \), and if \( K > c \), then \( B_{p2} \leq c \).

If \( B_{np} \neq 0 \) and \( P \neq 0 \), then the equilibrium \( E^* \) is obtained by solving the following
system of equations:

\[
0 = -\mu S - \eta S + \mu N + \eta N - \eta I \\
0 = -\mu I - \delta I \\
0 = rB_p \left(1 - \frac{B_p + B_{np}}{K}\right) - \gamma_1 \frac{B_p}{K_1 + B_p} P + \theta B_{np} P \\
0 = r \left(1 - \frac{B_p + B_{np}}{K}\right) - \gamma_2 \frac{1}{K_1 + B_{np}} P - \theta P \\
0 = \beta \left(\gamma_1 \frac{B_p}{K_1 + B_p} + \gamma_2 \frac{B_{np}}{K_1 + B_{np}}\right) - d
\]

Clearly, \( I = 0 \) from equation (6). Thus, from equation (5), we get \( S = N \). From equation (9), we have

\[
B_{np} = \frac{x}{y} = \frac{K_1(-d(K_1 + B_p) + \beta \gamma_1 B_p)}{K_1(d - \beta \gamma_2) + B_p(d - \beta(\gamma_1 + \gamma_2))},
\]

where \( x = K_1[-d(K_1 + B_p) + \beta \gamma_1 B_p] \) and \( y = K_1(d - \beta \gamma_2) + B_p(d - \beta(\gamma_1 + \gamma_2)) \) are such that \( B_p \neq \frac{dK_1}{\beta \gamma_1} = B_{p2} \); otherwise \( x = 0 \) but \( B_{np} > 0 \). Clearly \( B_p \neq \frac{K_1(d - \beta \gamma_2)}{\beta(\gamma_1 + \gamma_2) - d} \). Thus, we get \( B_{np} = \frac{x}{y} \), where either \( x, y > 0 \) or \( x, y < 0 \).

In order to show that both \( x \) and \( y \) are negative, and by solving equation (8) above, we get

\[
P^* = \left(\frac{r}{K}\right)(K - B_p - B_{np}) \left(\frac{K_1 + B_{np}}{\gamma_2 + \theta(K_1 + B_{np})}\right)
\]

Since \( P^* > 0 \), then we must have \( K > B_p + B_{np} \). Consequently, and by substituting equation (10) in equation (11), we get

\[
P^* = \left(\frac{r}{K}\right)(K - B_p - \frac{x}{y}) \left(\frac{K_1 + \frac{x}{y}}{\gamma_2 \theta(K_1 + \frac{x}{y})}\right)
\]

\[
= \left(\frac{r}{K}\right) \left[\frac{yK - B_p - x}{y}\right] \left[\frac{(yK_1 + x)}{y\gamma_2 + \theta(yK_1 + x)}\right]
\]

Thus,

\[
P^* = \left(\frac{r}{K}\right) \left[\frac{yK - B_p - x}{y}\right] \left[\frac{yK_1 + x}{b\gamma_2 + \theta(yK_1 + x)}\right],
\]

(12)
where $yK - yB_p - x < 0$ in order for $P^*$ to be positive. To see that, and by simplifying $yK_1 + x$, we get

$$yK_1 + x = (-K_1\beta\gamma_2)(K_1 + B_p) < 0$$

(13)

Thus, $x, y < 0$. Hence, equation (12) becomes

$$P^* = \left(\frac{r}{K}\right)\left[\frac{yK - B_p - x}{y}\right]\left[\frac{-K_1\beta\gamma_2(K_1 + B_p)}{y\gamma_2 - \theta K_1\beta\gamma_2(K_1 + B_p)}\right]$$

(14)

Using equation (10) and equation (14) in equation (7), we get

$$0 = \left(\frac{r}{K}\right)\left[\frac{yK - B_p - x}{y}\right]\left[\frac{\gamma_1\gamma_2 K_1\beta B_p}{y\gamma_2 - \theta K_1\beta\gamma_2(K_1 + B_p)}\right] - \theta\left(\frac{x}{y}\right)\left(\frac{K_1\beta\gamma_2(K_1 + B_p)}{y\gamma_2 - \theta K_1\beta\gamma_2(K_1 + B_p)}\right)$$

Then

$$\left[\frac{yK - B_p - x}{y}\right] = 0$$

(15)

or

$$\left[\frac{\gamma_1\gamma_2 K_1\beta B_p}{y\gamma_2 - \theta K_1\beta\gamma_2(K_1 + B_p)}\right] - \theta\left(\frac{x}{y}\right)\left(\frac{K_1\beta\gamma_2(K_1 + B_p)}{y\gamma_2 - \theta K_1\beta\gamma_2(K_1 + B_p)}\right) = 0$$

(16)

Notice that equation (15) has no solution since $P^* > 0$. Thus, and by using Mathematica, equation (16) has three solutions, one of which is real, say $B^*_p$, and the other two are imaginary. Substituting $B^*_p$ in equation (14) to get $P^*$ and in equation (10) to get $B^*_np$, and this proves the existence of the interior point $E^*$.

**Case 2:** if the pathogenic bacteria level is above the minimum infectious dose, then $\alpha(B_p) \neq 0$. Leaving us with the following system:

$$\dot{S} = -\alpha(B_p)S - \mu S - \eta S + \mu N + \eta N - \eta I$$

$$\dot{I} = \alpha(B_p)S - \mu I - \delta I$$

$$\dot{B}_p = rB_p \left(1 - \frac{B_p + B_{np}}{K}\right) - \gamma_1\frac{B_p}{K_1 + B_p}P + \theta B_{np}P$$

$$\dot{B}_{np} = rB_{np} \left(1 - \frac{B_p + B_{np}}{K}\right) - \gamma_2\frac{B_{np}}{K_1 + B_{np}}P - \theta B_{np}P$$

$$\dot{P} = \beta\left(\frac{\gamma_1 B_p}{K_1 + B_p} + \frac{\gamma_2 B_{np}}{K_1 + B_{np}}\right)P - dP$$

Let $\Gamma_1 = a(\mu + \delta + \eta)(B_p - c) + (\mu + \eta)(\mu + \delta)(B_p - c + H)$. Then One can easily check that $\dot{I} = 0$ if $I_1 = \frac{\alpha(B_p)}{\mu + \delta}S$, and hence $\dot{S} = 0$ if
$S_1 = (\mu + \eta) (\mu + \delta) \left( \frac{(B_p-c)+h}{1} \right) N$. Consequently, $I_1 = \left( \frac{(\mu+\eta)(B_p-c)}{1} \right) N$.

The third, forth and fifth equations of (4) do not contain terms including $S$ and $I$ so the non-trivial values for $B_p$ and the values of $B_{np}$ and $P$ that satisfy these equations of system (4) are the same as third, fourth and fifth equations of system (3) with the condition that $B_p > c$ so that $\alpha(B_p) \neq 0$. That is, the equilibrium points of system (4) are:

1. $E_3 = (S_1, I_1, m, K-m, 0)$, where $m$ is a positive constant such that if $K > c$, then $c < m \leq K$ and the point does not exist if $K \leq c$.

   Special case of $E_3$ is $E_{31} = (S_1, I_1, K, 0, 0)$, where $K > c$.

2. $E_4 = (S_1, I_1, B_{p4}, 0, P_4)$, where $B_{p4} = B_{p2} = \frac{dK_1}{\beta\gamma_1 - d} > 0$ is such that if $K \leq c$, then $E_4$ does not exist, and if $K > c$, then $c < B_{p2} \leq K$. Hence, $P_4 = P_2 = \frac{r_1}{\gamma_1}(K_1 + B_{p2})(K - B_{p2}) > 0$.

3. $E^{**} = (S_1, I_1, B_{p**}, B_{np**}, P^{**})$, exits if the following conditions are hold:
   
   (i) $B_{p**} > c$.
   
   (ii) $B_{p**} \neq \frac{dK_1}{\beta\gamma_1 - d} = B_{p4}$. Note that the existence of $B_{p**}$ and $B_{p4}$ is contrary.
   
   (iii) $B_{p**} \neq \frac{K_1(d - \beta \gamma_2)}{\beta(\gamma_1 + \gamma_2) - d}$.
   
   (iv) $K > B_{p**} + B_{np**}$.
   
   (v) $B_{p**}, B_{np**}$ and $P^{**} > 0$.

2 Linearization

Depending on the pathogenic bacteria level, the linearization of system (2) has two forms, one for system (3) when $\alpha(B_p) = 0$, denoted $J$, and one for system (4) when $\alpha(B_p) \neq 0$, denoted $J'$.

Define: $U = r - \frac{2rB_p}{K} - \frac{rB_{np}}{K} \frac{K_1\gamma_1 P}{(K_1 + B_p)^2}$ and $V = r - \frac{rB_p}{K} - \frac{2rB_{np}}{K} \frac{K_1\gamma_2 P}{(K_1 + B_{np})^2} - \theta P$. Then:

$$J_1 = \begin{bmatrix}
-\mu - \eta & 0 & 0 & 0 & 0 \\
0 & -(\mu + \delta) & 0 & 0 & 0 \\
0 & 0 & -\frac{rB_p}{K} & + \theta P & 0 \\
0 & 0 & -\frac{rB_{np}}{K} & V & 0 \\
0 & 0 & -\frac{\beta\gamma_1 K_p}{(K_1 + B_p)^2} & -\frac{\beta\gamma_2 K_p}{(K_1 + B_{np})^2} & \beta(\frac{\gamma_1 + B_p}{K_1 + B_p} + \frac{\gamma_2 + B_{np}}{K_1 + B_{np}}) - d
\end{bmatrix}$$

and
2.1 Stability of the equilibrium $E_0$

The Jacobin matrix for $E_0$ is:

$$
J_0 = \begin{bmatrix}
-\mu - \eta & 0 & 0 & 0 \\
0 & -(\mu + \delta) & 0 & 0 \\
0 & 0 & r & 0 \\
0 & 0 & 0 & -d
\end{bmatrix}
$$

The eigenvalues corresponding to $E_0$ are: $-\mu - \eta, -(\mu + \delta), -d < 0$ and $r, r > 0$. Since $r > 0$, then $E_0$ is unstable.

2.2 Stability of the equilibrium $E_1$

The Jacobin matrix for $E_1$ is:

$$
J_1 = \begin{bmatrix}
-\mu - \eta & 0 & 0 & 0 & 0 \\
0 & -(\mu + \delta) & 0 & 0 & 0 \\
0 & 0 & \frac{-\gamma_1 m}{K_1 + m} + \theta(K - m) & -\frac{r(K-m)}{K} & \frac{\gamma_1 m}{K_1 + m} + \theta(K - m) \\
0 & 0 & 0 & 0 & \beta\left(\frac{\gamma_1 m}{K_1 + m} + \frac{\gamma_2 (K-m)}{K_1 + K - m}\right) - d
\end{bmatrix}
$$

The eigenvalues corresponding to $E_1$ are: $-\mu - \eta, -(\mu + \delta), -r, 0$ and $\beta\left(\frac{\gamma_1 m}{K_1 + m} + \frac{\gamma_2 (K-m)}{K_1 + K - m}\right) - d$.

Hence, $E_1$ might be stable if the following condition holds:

$$
R_B = \frac{\beta}{d}\left(\frac{\gamma_1 m}{K_1 + m} + \frac{\gamma_2 (K-m)}{K_1 + K - m}\right) < 1.
$$

Considering the special case $E_{11}$, we get the following eigenvalues: $-(\mu + \eta), -(\mu + \delta), -r, 0$ and $\frac{\beta \gamma_1 K}{K_1 + K - d} - d$. Thus, if $\frac{\beta \gamma_1 K}{d(K_1 + K)} < 1$, then $E_{11}$ might be stable, and if $\frac{\beta \gamma_1 K}{d(K_1 + K)} > 1$, then $E_{11}$ might be unstable.

When considering the equilibrium point $E_{12}$, we found that the eigenvalues corresponding to $J_1$ are: $-(\mu + \eta), -(\mu + \delta), -r, 0$ and $\frac{\beta \gamma_1 K}{K_1 + K - d} - d$. So, $E_{12}$ might be stable if $\frac{\beta \gamma_1 K}{d(K_1 + K)} < 1$, which is equivalent to say that $E_{12}$ might be stable if $K < \frac{dK_1}{\beta \gamma_1 - d} = B_{p_2}$. But if $E_2$ exists, then $K > \frac{dK_1}{\beta \gamma_1 - d} = B_{p_2}$ and hence, $E_{12}$ might be unstable whenever $E_2$ exists.
2.3 Stability of $E_2$

Let $O = r - \frac{2rB_p}{K} - \frac{K_1\gamma_1 P_2}{(K_1 + B_p)^2}$, $Q = r - \frac{rB_p}{K} - \frac{\gamma_2 P_2}{K_1} - \theta P_2$ and $L = \frac{\beta_2 K_1 P_2}{(K_1)^2}$, then the Jacobin matrix is:

$$J_2 = \begin{bmatrix}
-\mu - \eta & 0 & 0 & 0 & 0 \\
0 & -(\mu + \delta) & 0 & 0 & 0 \\
0 & 0 & O & -\frac{rB_p}{K} + \theta P_2 & -\frac{d}{\beta} \\
0 & 0 & 0 & Q & 0 \\
0 & 0 & L & \frac{\beta_2 K_1 P_2}{(K_1)^2} & 0
\end{bmatrix}$$

Now,

$$\det(J_2 - \lambda I) = (-\mu - \eta - \lambda)(-\mu - \delta - \lambda)(Q - \lambda)\left[\frac{d}{\beta} L - \lambda(O - \lambda)\right]$$

$$= (-\mu - \eta - \lambda)(-\mu - \delta - \lambda)\frac{d}{\beta} (Q - \lambda)(\lambda^2 - \lambda O + \frac{d}{\beta} L)$$

Note that the equation $(\lambda^2 - \lambda O + \frac{d}{\beta} L) = 0$ has two real solutions non of which is zero since $\frac{d}{\beta} L > 0$. Hence, the stability of $E_2$ is determined by the sign of the eigenvalue $\lambda = Q$.

If $Q \leq 0$, then

$$r - \frac{rB_p}{K} - \frac{\gamma_2 P_2}{K_1} - \theta P_2 \leq 0$$

$$K \leq \frac{dK_1}{\beta_1 - d} + \left(\frac{\gamma_2}{\gamma_1 K_1} + \frac{\theta}{\gamma_1}\right)(K_1 + \frac{dK_1}{\beta_1 - d})(K - \frac{dK_1}{\beta_1 - d})$$

$$(1 - \frac{\gamma_2\beta + \theta K_1\beta}{\beta_1 - d})K \leq \frac{dK_1}{\beta_1 - d} - \frac{dK_1\beta(\gamma_2 + \theta K_1)}{(\beta_1 - d)^2}$$

$$K \leq \frac{dK_1}{\beta_1 - d} = B_{p_2}$$

which is not the case since we must have $B_{p_2} < K$. Thus, $Q > 0$, and hence $E_2$ is unstable.

2.4 Stability of $E_3$

We have the following Jacobin matrix for $E_3$:

$$J_3 = \begin{bmatrix}
-\alpha(m) - \mu - \eta & 0 & -\frac{aS_1 H}{(m-c+H)^2} & 0 & 0 \\
\alpha(m) & -(\mu + \delta) & -\frac{aS_1 H}{(m-c+H)^2} & 0 & 0 \\
0 & 0 & -\frac{m}{K} & 0 & 0 \\
0 & 0 & -\frac{r(K-m)}{K} & -\frac{\gamma_1 m}{K_1 + K - m} + \theta(K - m) & 0 \\
0 & 0 & 0 & \beta(\frac{\gamma_1 m}{K_1 + K - m} + \frac{\gamma_2(K-m)}{K_1 + K - m}) - d & 0
\end{bmatrix}$$

The eigenvalues of $J_3$ are: $-\alpha(m) - \mu - \eta, -(\mu + \delta), -r, 0$ and $\beta(\frac{\gamma_1 m}{K_1 + K - m} + \frac{\gamma_2(K-m)}{K_1 + K - m}) - d$.

Hence, $E_3$ might be stable if $R_B < 1$. 

8
Considering the equilibrium point $E_{31} = (S_1, I_1, K, 0, 0)$, $K > c$, we get the following eigenvalues: $-(\alpha(K) + \mu + \eta), -(\mu + \delta), -r, 0$ and $\frac{\beta\gamma_1 K}{K_1 + K} - d$. Hence, $E_{31}$ might be stable if $\frac{\beta\gamma_1 K}{d(K_1 + K)} < 1$.

2.5 Stability of $E_4$

We recall that $E_4 = (S_1, I_1, B_{p4}, 0, P_4)$, where $B_{p4} = B_{p2} = \frac{dK_1}{\beta\gamma_1 - d} > 0$ is such that if $K \leq c$, then $E_4$ does not exist, and if $K > c$, then $c < B_{p4} \leq K$, so that $P_4 = P_2 = \frac{r}{\beta K} (K_1 + B_{p2}) (K - B_{p2}) > 0$.

Let $O = r - \frac{2rB_{p2}}{K} - \frac{K_1 \gamma_2 p_2}{(K_1 + B_{p2})^2}$, $Q = r - \frac{rB_{p2}}{K_1} - \frac{\gamma_2 p_1}{K_1} - \theta P_2$ and $L = \frac{\beta\gamma_1 K K_1 P_2}{(K_1 + B_{p2})^2}$, then the Jacobin matrix corresponding to $E_4$ is:

$$\hat{J}_4 = \begin{bmatrix}
-\alpha(B_{p4}) - \mu - \eta & 0 & 0 \\
\alpha(B_{p4}) & -(\mu + \delta) & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & \frac{\beta\gamma_1 K}{K_1}
\end{bmatrix}$$

Then, $\det(\hat{J}_4 - \lambda I) = (-\alpha(B_{p4}) - \mu - \eta - \lambda)(-\mu - \delta - \lambda)(Q - \lambda)\left[\frac{d}{\beta} L - \lambda(O - \lambda)\right] = (-\mu - \lambda)(-\mu - \delta - \lambda)\frac{d}{\beta}(Q - \lambda)(\lambda^2 - \lambda O + \frac{d}{\beta} L)$

Hence, and is shown in Section 2.3, none of the eigenvalues is zero, and one of the eigenvalue, namely $Q$, is positive. So, $E_4$ is unstable.

For the equilibrium points $E^*, E^{**}$ we lack exact expressions for the equilibrium quantities, and so the local stability is difficult to find analytically.
3 Existence of Equilibria with shedding

In this section, we determine the equilibrium points of the model system when \( \xi \neq 0 \), and then perform stability analysis of the equilibria. Since \( R = N - S - I \), then the third equation of system (1) is not necessary, leaving us with the following system:

\[
\begin{align*}
\dot{S} &= -\alpha(B_p)S - \mu S + \mu N + \eta R \\
\dot{I} &= \alpha(B_p)S - \mu I - \delta I \\
\dot{B}_p &= rB_p \left( 1 - \frac{B_p + B_{np}}{K} \right) - \gamma_1 \frac{B_p}{K_1 + B_p} P + \xi I + \theta(B_{np}, P) \\
\dot{B}_{np} &= rB_{np} \left( 1 - \frac{B_p + B_{np}}{K} \right) - \gamma_2 \frac{B_{np}}{K_1 + B_{np}} P + \xi I - \theta(B_{np}, P) \\
\dot{P} &= \beta \left( \gamma_1 \frac{B_p}{K_1 + B_p} + \gamma_2 \frac{B_{np}}{K_1 + B_{np}} \right) P - dP + \phi \xi I \\
\end{align*}
\]

(5)

If \( \alpha(B_p) = 0 \), then we will have the same equilibrium points \( E_0, E_1, E_2 \) and \( E^* \) as in Section 1, with the same conditions.

If \( B_P > c \), then \( \alpha(B_p) \neq 0 \), and hence at steady state system (5) reduces to:

\[
\begin{align*}
0 &= -\alpha(B_p)S - \mu S - \eta S + \mu N + \eta N - \eta I \\
0 &= \alpha(B_p)S - \mu I - \delta I \\
0 &= rB_p \left( 1 - \frac{B_p + B_{np}}{K} \right) - \gamma_1 \frac{B_p}{K_1 + B_p} P + \xi I + \theta(B_{np}, P) \\
0 &= rB_{np} \left( 1 - \frac{B_p + B_{np}}{K} \right) - \gamma_2 \frac{B_{np}}{K_1 + B_{np}} P + \xi I - \theta(B_{np}, P) \\
0 &= \beta \left( \gamma_1 \frac{B_p}{K_1 + B_p} + \gamma_2 \frac{B_{np}}{K_1 + B_{np}} \right) P - dP + \phi \xi I \\
\end{align*}
\]

(6)

Then solving equation (2) of (6) for \( I \), one has \( I = \frac{\alpha(B_p)S}{\mu + \eta} \), and hence, and by solving equation (1) of (6), one can easily check that \( S = S_1 = (\mu + \eta) (\mu + \delta) \frac{(B_{p**} - c) + H}{\Gamma_1} N \).

Thus, \( I = I_1 = \frac{(a(\mu + \delta + \eta)(B_{p**} - c) + (\mu + \eta)(\mu + \delta)(B_{p**} - c + H)}{\Gamma_1} N \), where 
\( \Gamma_1 = a(\mu + \delta + \eta)(B_{p**} - c) + (\mu + \eta)(\mu + \delta)(B_{p**} - c + H) \). Noting that these are the same formulas for \( S \) and \( I \) in system (4) at steady state.

By solving the other three equation of system (6), we will prove that system (6) has only one equilibrium point, namely \( E^{**} = (S_1, I_1, B_{p**}, B_{np**}, P^{**}) \) which exists if the following condition holds:

\[
\frac{B_{p**}}{K_1 + B_{p**}} + \frac{B_{np**}}{K_1 + B_{np**}} < \frac{d}{\beta},
\]

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where \( P^{***} = -\phi \xi I_1 \left[ \frac{(K_1 + B_p^{***})(K_1 + B_{np}^{***})}{l_2} \right] \), where

\[
\Gamma_2 = \beta \gamma_1 B_p^{***}(K_1 + B_{np}^{***}) + \beta \gamma_2 B_{np}^{***}(K_1 + B_p^{**}) - d(K_1 + B_p^{**})(K_1 + B_{np}^{**}).
\]

In order to have a well-defined and positive value for \( P^{***} \), \( \Gamma_2 \) must be negative. That is, the value of \( P \) provides a condition for \( B_p^{***} \) and \( B_{np}^{***} \).

Now, \( \Gamma_2 < 0 \) if

\[
\beta \gamma_1 B_p(K_1 + B_{np}) + \beta \gamma_2 B_{np}(K_1 + B_p) < d(K_1 + B_p)(K_1 + B_{np}).
\]

Dividing both sides by: \((K_1 + B_p)(K_1 + B_{np})\), to get:

\[
\frac{\beta \gamma_1 B_p}{K_1 + B_p} + \frac{\beta \gamma_2 B_{np}}{K_1 + B_{np}} < \frac{d}{\beta}.
\]

Consequently,

\[
0 = K \phi \xi \mu N a (B_p - c)(1 + B_p) - \frac{\xi \mu N a (B_p - c)}{(\mu + \delta)(a + \mu)(B_p - c) + \mu H} + \frac{\phi \xi \mu N a \left[ \frac{\gamma_2 B_{np}(B_p - c)(K_1 + B_p)}{(\mu + \delta)(a + \mu)(B_p - c) + \mu H} \frac{1}{\Gamma_2} \right]}{B_{np}^{**} + \frac{\theta}{\Gamma_2} \left( K \phi \xi \mu N a (B_p - c)(K_1 + B_p) \right)}
\]

Using Mathematica to solve equation (18), we found out that it is a cubic equation of \( B_p \), which has three solutions. Only one of these solutions is a real solution, say \( B_p^{***} \), which is in-terms of \( B_{np} \).

Substitute \( B_p^{***} \), \( I \) and \( P^{***} \) in equation (3) of system (6) to get an equation in terms of \( B_{np} \) only, then solve the resulting equation to get the value of \( B_{np}^{***} \). The exact formula of \( B_{np}^{***} \) is so complicated, and since the other four coordinates of \( E^{***} \) depend on \( B_{np}^{***} \), exact formulas of these variables are not given.

### 4 Linearization

According to the level of pathogenic bacteria, we will have two linearizations of system (5). First when \( \alpha(B_p) = 0 \), denoted by \( \mathcal{J} \) and the other when \( \alpha(B_p) \neq 0 \),
denoted by $\hat{J}$. Set: $U = r - \frac{2rB_p}{K} - \frac{rB_{np}}{K} - \frac{K_1\gamma_1P}{(K_1+B_p)^2}$ and $V = r - \frac{rB_p}{K} - \frac{2rB_{np}}{K} - \frac{K_1\gamma_2P}{(K_1+B_{np})^2} - \theta P$. Then:

$$J = \begin{bmatrix}
-\mu & 0 & 0 & 0 & 0 \\
0 & -\mu - \delta & 0 & 0 & 0 \\
0 & \xi & -\frac{rB_p}{K} + \theta P & \frac{-\gamma_1B_p}{K_1+B_p} + \theta B_{np} & 0 \\
0 & \xi & -\frac{rB_{np}}{K} & \frac{-\gamma_2B_{np}}{K_1+B_{np}} - \theta B_{np} & 0 \\
0 & \phi \xi & \frac{\beta_1K_1P}{(K_1+B_p)^2} & \frac{\beta_2K_1P}{(K_1+B_{np})^2} & \beta \left(\frac{\gamma_1+B_p}{K_1+B_p} + \frac{\gamma_2+B_{np}}{K_1+B_{np}}\right) - d
\end{bmatrix},$$

Letting $J_i$ be the Jacobian matrix corresponding to the equilibrium point $E_i$, $i = 0, 1$ or 2, one can easily check that the eigenvalues of $J_i$ are the same as those for $J$. Consequently, the stability of the equilibrium points of system (5) has been already stated in section 2.

The Jacobian matrix corresponding to system (6) is

$$\hat{J} = \begin{bmatrix}
-\alpha(B_p) - \mu & 0 & 0 & 0 & 0 \\
\alpha(B_p) & -\mu - \delta & 0 & 0 & 0 \\
0 & \xi & -\frac{rB_p}{K} + \theta P & \frac{-\gamma_1B_p}{K_1+B_p} + \theta B_{np} & 0 \\
0 & \xi & -\frac{rB_{np}}{K} & \frac{-\gamma_2B_{np}}{K_1+B_{np}} - \theta B_{np} & 0 \\
0 & \phi \xi & \frac{\beta_1K_1P}{(K_1+B_p)^2} & \frac{\beta_2K_1P}{(K_1+B_{np})^2} & \beta \left(\frac{\gamma_1+B_p}{K_1+B_p} + \frac{\gamma_2+B_{np}}{K_1+B_{np}}\right) - d
\end{bmatrix},$$

Since we lack exact expressions for the coordinates of $E^{***}$, the stability of this equilibrium point is difficult to find analytically.