Bacteria-bacteriophage cycles facilitate Cholera outbreak cycles: an indirect Susceptible - Infected - Bacteria - Phage (iSIBP) model-based mathematical study

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Mathematical Analysis

1 Existence of Equilibria with No shedding

Substituting $\xi = 0$ in system (1) and noticing that $R = N - S - I$, the third equation is then not necessary. Thus, system (1) reduces to

$$\begin{aligned}
\dot{S} &= -\alpha(B_p)S - \mu S - \eta S + \mu N + \eta N - \eta I \\
\dot{I} &= \alpha(B_p)S - \mu I - \delta I \\
\dot{B}_p &= rB_p\left(1 - \frac{B_p + B_{np}}{K}\right) - \gamma_1 \frac{B_p}{K_1 + B_p} P + \theta(B_{np}, P) \\
\dot{B}_{np} &= rB_{np}\left(1 - \frac{B_p + B_{np}}{K}\right) - \gamma_2 \frac{B_{np}}{K_1 + B_{np}} P - \theta(B_{np}, P) \\
\dot{P} &= \beta\left(\gamma_1 \frac{B_p}{K_1 + B_p} + \gamma_2 \frac{B_{np}}{K_1 + B_{np}}\right) P - dP
\end{aligned} \tag{2}$$

By considering the values of $\alpha(B_p)$, we get two cases:

Case 1: if the pathogenic bacteria level is below or equal to the minimum infectious dose, then $\alpha(B_p) = 0$. Hence system (2) becomes:

$$\begin{aligned}
\dot{S} &= -\mu S - \eta S + \mu N + \eta N - \eta I \\
\dot{I} &= -\mu I - \delta I \\
\dot{B}_p &= rB_p\left(1 - \frac{B_p + B_{np}}{K}\right) - \gamma_1 \frac{B_p}{K_1 + B_p} P + \theta(B_{np}, P) \\
\dot{B}_{np} &= rB_{np}\left(1 - \frac{B_p + B_{np}}{K}\right) - \gamma_2 \frac{B_{np}}{K_1 + B_{np}} P - \theta(B_{np}, P) \\
\dot{P} &= \beta\left(\gamma_1 \frac{B_p}{K_1 + B_p} + \gamma_2 \frac{B_{np}}{K_1 + B_{np}}\right) P - dP
\end{aligned} \tag{3}$$

Then system (3) has 4 equilibrium points which are listed below:

1. $E_0 = (N, 0, 0, 0, 0)$ always exists.
2. $E_1 = (N, 0, m, K - m, 0)$ always exists, where m is a non-negative constant such that if $K \leq c$, then $m \leq K$, and if $K > c$, then $m \leq c$. Special cases of E_1 are:
   a. $E_{11} = (N, 0, 0, K, 0)$.
   b. $E_{12} = (N, 0, K, 0, 0)$, where $K \leq c$.

3. $E_2 = (N, 0, B_{p_2}, 0, P_2)$ exists if $\beta\gamma_1 - d > 0$. In this case, $B_{p_2} = \frac{dK_1}{\beta\gamma_1 - d} > 0$ is such that if $K \leq c$, then $B_{p_2} \leq K$, and if $K > c$, then $B_{p_2} \leq c$, so that $P_2 = \frac{r}{\gamma_1 K}(K_1 + B_{p_2})(K - B_{p_2}) > 0$.

4. The interior point $E^* = (N, 0, B_p^*, B_{np}^*, P^*)$, exits if the following conditions hold:

   (i) $B_p^* < c$.

   (ii) $B_p^* \neq \dfrac{dK_1}{\beta\gamma_1 - d} = B_{p_2}$. Note that the existence of B_p^* and B_{p_2} is contrary.

   (iii) $B_p^* \neq \dfrac{K_1(d - \beta\gamma_2)}{\beta(\gamma_1 + \gamma_2) - d}$.

   (iv) $K > B_p^* + B_{np}^*$.

   (v) B_p^*, B_{np}^* and $P^* > 0$.

The existence of the equilibria E_0 and E_1 is obvious. In the following we will show the existence of equilibria E_2 and E^*.
Equation (4) of system (3) implies that at steady state either $B_{np} = 0$ or $B_{np} \neq 0$. Under the assumption that $B_{np} = 0$, one can obtain the equilibrium point E_2 by solving the following system of equations:

$$
\begin{aligned}
0 &= -\mu S - \eta S + \mu N + \eta N - \eta I && (1)\\
0 &= -\mu I - \delta I && (2)\\
0 &= rB_p\left(1 - \frac{B_p}{K}\right) - \gamma_1 \frac{B_p}{K_1 + B_p} P && (3)\\
0 &= \beta\left(\gamma_1 \frac{B_p}{K_1 + B_p}\right) - d && (4)
\end{aligned}
$$

From equation (2), we have $I = 0$ and then from equation (1), we have $S = N$. From equation (4), we have $d = \frac{\beta\gamma_1 B_p}{K_1 + B_p}$ so $B_{p_2} = \frac{dK_1}{\beta\gamma_1 - d}$ such that $B_{p_2} \leq c$ and $\beta\gamma_1 > d$.
From equation (3), we get $P_2 = \frac{r}{\gamma_1 K}(K_1 + B_{p_2})(K - B_{p_2})$. In order to have a positive value for P_2, the following conditions on B_{p_2} must hold:
$B_{p_2} < K$ and if $K \leq c$, then $B_{p_2} < K$, and if $K > c$, then $B_{p_2} \leq c$.
If $B_{np} \neq 0$ and $P \neq 0$, then the equilibrium E^* is obtained by solving the following

system of equations:

$$0 = -\mu S - \eta S + \mu N + \eta N - \eta I \tag{5}$$
$$0 = -\mu I - \delta I \tag{6}$$
$$0 = rB_p\left(1 - \frac{B_p + B_{np}}{K}\right) - \gamma_1\frac{B_p}{K_1 + B_p}P + \theta B_{np}P \tag{7}$$
$$0 = r\left(1 - \frac{B_p + B_{np}}{K}\right) - \gamma_2\frac{1}{K_1 + B_{np}}P - \theta P \tag{8}$$
$$0 = \beta\left(\gamma_1\frac{B_p}{K_1 + B_p} + \gamma_2\frac{B_{np}}{K_1 + B_{np}}\right) - d \tag{9}$$

Clearly, $I = 0$ from equation (6). Thus, from equation (5), we get $S = N$. From equation (9), we have

$$B_{np} = \frac{x}{y} = \frac{K_1(-d(K_1 + B_p) + \beta\gamma_1 B_p)}{K_1(d - \beta\gamma_2) + B_p(d - \beta(\gamma_1 + \gamma_2))}, \tag{10}$$

where $x = K_1[-d(K_1 + B_p) + \beta\gamma_1 B_p]$ and $y = K_1(d - \beta\gamma_2) + B_p(d - \beta(\gamma_1 + \gamma_2))$ are such that $B_p \neq \dfrac{dK_1}{\beta\gamma_1 - d} = B_{p2}$; otherwise $x = 0$ but $B_{np} > 0$. Clearly $B_p \neq \dfrac{K_1(d - \beta\gamma_2)}{\beta(\gamma_1 + \gamma_2) - d}$. Thus, we get $B_{np} = \dfrac{x}{y}$, where either $x, y > 0$ or $x, y < 0$. In order to show that both x and y are negative, and by solving equation (8) above, we get

$$P^* = \Big(\frac{r}{K}\Big)(K - B_p - B_{np})\Big(\frac{K_1 + B_{np}}{\gamma_2 + \theta(K_1 + B_{np})}\Big) \tag{11}$$

Since $P^* > 0$, then we must have $K > B_p + B_{np}$. Consequently, and by substituting equation (10) in equation (11), we get

$$P^* = \Big(\frac{r}{K}\Big)\Big(K - B_p - \frac{x}{y}\Big)\Big(\frac{K_1 + \frac{x}{y}}{\gamma_2\theta(K_1 + \frac{x}{y})}\Big)$$
$$= \Big(\frac{r}{K}\Big)\Big[\frac{yK - B_p - x}{y}\Big]\Big[\frac{\Big(\frac{yK_1 + x}{y}\Big)}{\Big(\frac{y\gamma_2 + \theta(yK_1 + x)}{y}\Big)}\Big]$$

Thus,

$$P^* = \Big(\frac{r}{K}\Big)\Big[\frac{yK - B_p - x}{y}\Big]\Big[\frac{yK_1 + x}{b\gamma_2 + \theta(yK_1 + x)}\Big], \tag{12}$$

where $yK - yB_p - x < 0$ in order for P^* to be positive. To see that, and by simplifying $yK_1 + x$, we get

$$yK_1 + x = (-K_1\beta\gamma_2)(K_1 + B_p) < 0 \tag{13}$$

Thus, $x, y < 0$. Hence, equation (12) becomes

$$P^* = \Big(\frac{r}{K}\Big)\Big[\frac{yK - B_p - x}{y}\Big]\Big[\frac{-K_1\beta\gamma_2(K_1 + B_p)}{y\gamma_2 - \theta K_1\beta\gamma_2(K_1 + B_p)}\Big] \tag{14}$$

Using equation (10) and equation (14) in equation (7), we get

$$0 = \Big(\frac{r}{K}\Big)\Big[\frac{yK - B_p - x}{y}\Big]\Big[B_p + \frac{\gamma_1\gamma_2 K_1\beta B_p}{(y\gamma_2 - \theta K_1\beta\gamma_2(K_1 + B_p))} - \theta(\frac{x}{y})\Big(\frac{K_1\beta\gamma_2(K_1 + B_p)}{y\gamma_2 - \theta K_1\beta\gamma_2(K_1 + B_p))}\Big)\Big].$$

Then

$$\Big[\frac{yK - B_p - x}{y}\Big] = 0 \tag{15}$$

or

$$\Big[B_p + \frac{\gamma_1\gamma_2 K_1\beta B_p}{(y\gamma_2 - \theta K_1\beta\gamma_2(K_1 + B_p))} - \theta(\frac{x}{y})\Big(\frac{K_1\beta\gamma_2(K_1 + B_p)}{y\gamma_2 - \theta K_1\beta\gamma_2(K_1 + B_p))}\Big)\Big] = 0 \tag{16}$$

Notice that equation (15) has no solution since $P^* > 0$. Thus, and by using Mathematica, equation (16) has three solutions, one of which is real, say B_p^*, and the other two are imaginary. Substituting B_p^* in equation (14) to get P^* and in equation (10) to get B_{np}^*, and this proves the existence of the interior point E^*.

Case 2: if the pathogenic bacteria level is above the minimum infectious dose, then $\alpha(B_p) \neq 0$, Leaving us with the following system:

$$\begin{aligned}
\dot{S} &= -\alpha(B_p)S - \mu S - \eta S + \mu N + \eta N - \eta I \\
\dot{I} &= \alpha(B_p)S - \mu I - \delta I \\
\dot{B}_p &= rB_p\left(1 - \frac{B_p + B_{np}}{K}\right) - \gamma_1\frac{B_p}{K_1 + B_p}P + \theta B_{np}P \\
\dot{B}_{np} &= rB_{np}\left(1 - \frac{B_p + B_{np}}{K}\right) - \gamma_2\frac{B_{np}}{K_1 + B_{np}}P - \theta B_{np}P \\
\dot{P} &= \beta\left(\gamma_1\frac{B_p}{K_1 + B_p} + \gamma_2\frac{B_{np}}{K_1 + B_{np}}\right)P - dP
\end{aligned} \tag{4}$$

Let $\Gamma_1 = a(\mu + \delta + \eta)(B_p - c) + (\mu + \eta)(\mu + \delta)(B_p - c + H)$. Then One can easily check that $\dot{I} = 0$ if $I_1 = \frac{\alpha(B_p)}{\mu+\delta}S$, and hence $\dot{S} = 0$ if

$S_1 = (\mu+\eta)(\mu+\delta)\left(\frac{(B_P-c)+H}{\Gamma_1}\right)N$. Consequently, $I_1 = \left(\frac{a(\mu+\eta)(B_p-c)}{\Gamma_1}\right)N$.
The third, forth and fifth equations of (4) do not contain terms including S and I so the non-trivial values for B_p and the values of B_{np} and P that satisfy these equations of system (4) are the same as third,fourth and fifth equations of system (3) with the condition that $B_p > c$ so that $\alpha(B_p) \neq 0$. That is, the equilibrium points of system (4) are:

1. $E_3 = (S_1, I_1, m, K-m, 0)$, where m is a positive constant such that if $K > c$, then $c < m \leq K$ and the point does not exist if $K \leq c$.
   Special case of E_3 is $E_{31} = (S_1, I_1, K, 0, 0)$, where $K > c$.

2. $E_4 = (S_1, I_1, B_{p_4}, 0, P_4)$, where $B_{p_4} = B_{p_2} = \frac{dK_1}{\beta\gamma_1 - d} > 0$ is such that if $K \leq c$, then E_4 does not exist, and if $K > c$, then $c < B_{p_2} \leq K$. Hence, $P_4 = P_2 = \frac{r}{\gamma_1 K}(K_1 + B_{p_2})(K - B_{p_2}) > 0$.

3. $E^{**} = (S_1, I_1, B_p^{**}, B_{np}^{**}, P^{**})$, exits if the following conditions are hold:
   (i) $B_p^{**} > c$.
   (ii) $B_p^{**} \neq \dfrac{dK_1}{\beta\gamma_1 - d} = B_{p_4}$. Note that the existence ofB_p^{**} and B_{p_4} is contrary.
   (iii) $B_p^{**} \neq \dfrac{K_1(d-\beta\gamma_2)}{\beta(\gamma_1+\gamma_2)-d}$.
   (iv) $K > B_p^{**} + B_{np}^{**}$.
   (v) B_p^{**}, B_{np}^{**} and $P^{**} > 0$.

2 Linearization

Depending on the pathogenic bacteria level, the linearization of system (2) has two forms, one for system (3) when $\alpha(B_p) = 0$, denoted J, and one for system (4) when $\alpha(B_p) \neq 0$, denoted $\hat{J}$.
Define: $U = r - \frac{2rB_p}{K} - \frac{rB_{np}}{K} - \frac{K_1\gamma_1 P}{(K_1+B_p)^2}$ and $V = r - \frac{rB_p}{K} - \frac{2rB_{np}}{K} - \frac{K_1\gamma_2 P}{(K_1+B_{np})^2} - \theta P$.Then:

$$J_1 = \begin{bmatrix} -\mu-\eta & 0 & 0 & 0 & 0 \\ 0 & -(\mu+\delta) & 0 & 0 & 0 \\ 0 & 0 & U & \frac{-rB_P}{K} + \theta P & \frac{-\gamma_1 B_P}{K_1+B_P} + \theta B_{np} \\ 0 & 0 & -\frac{rB_{np}}{K} & V & \frac{-\gamma_2 B_{np}}{K_1+B_{np}} - \theta B_{np} \\ 0 & 0 & \frac{\beta\gamma_1 K_1 P}{(K_1+B_P)^2} & \frac{\beta\gamma_2 K_1 P}{(K_1+B_{np})^2} & \beta\left(\frac{\gamma_1+B_p}{K_1+B_p} + \frac{\gamma_2+B_{np}}{K_1+B_{np}}\right) - d \end{bmatrix}$$

and

$$J_2 = \begin{bmatrix} -\alpha(B_p)-\mu-\eta & 0 & \frac{-aS_1H}{(B_p-c+H)^2} & 0 & 0 \\ \alpha(B_p) & -(\mu+\delta) & \frac{-aS_1H}{(B_p-c+H)^2} & 0 & 0 \\ 0 & 0 & U & \frac{-rB_P}{K}+\theta P & \frac{-\gamma_1 B_P}{K_1+B_P}+\theta B_{np} \\ 0 & 0 & -\frac{rB_{np}}{K} & V & \frac{-\gamma_2 B_{np}}{K_1+B_{np}}-\theta B_{np} \\ 0 & 0 & \frac{\beta\gamma_1 K_1 P}{(K_1+B_P)^2} & \frac{\beta\gamma_2 K_1 P}{(K_1+B_{np})^2} & \beta(\frac{\gamma_1 B_p}{K_1+B_p}+\frac{\gamma_2 B_{np}}{K_1+B_{np}})-d \end{bmatrix}$$

2.1 Stability of the equilibrium E_0

The Jacobin matrix for E_0 is:

$$J_0 = \begin{bmatrix} -\mu-\eta & 0 & 0 & 0 & 0 \\ 0 & -(\mu+\delta) & 0 & 0 & 0 \\ 0 & 0 & r & 0 & 0 \\ 0 & 0 & 0 & r & 0 \\ 0 & 0 & 0 & 0 & -d \end{bmatrix}$$

The eigenvalues corresponding to E_0 are: $-\mu-\eta, -(\mu+\delta), -d < 0$ and $r, r > 0$. Since $r > 0$, then E_0 is unstable.

2.2 Stability of the equilibrium E_1

The Jacobin matrix foe E_1 is:

$$J_1 = \begin{bmatrix} -\mu-\eta & 0 & 0 & 0 & 0 \\ 0 & -(\mu+\delta) & 0 & 0 & 0 \\ 0 & 0 & -\frac{r}{K} & -\frac{r}{K} & \frac{-\gamma_1 m}{K_1+m}+\theta(K-m) \\ 0 & 0 & -\frac{r(K-m)}{K} & -\frac{r(K-m)}{K} & \frac{-\gamma_2(K-m)}{K_1+(K-m)}-\theta(K-m) \\ 0 & 0 & 0 & 0 & \beta(\frac{\gamma_1+B_p}{K_1+B_p}+\frac{\gamma_2+B_{np}}{K_1+B_{np}})-d \end{bmatrix}$$

The eigenvalues corresponding to E_1 are: $-\mu-\eta, -(\mu+\delta), -r, 0$ and $\beta(\frac{\gamma_1 m}{K_1+m}+\frac{\gamma_2(K-m)}{K_1+K-m})-d$.
Hence, E_1 might be stable if the following condition holds:

$$R_B = \frac{\beta}{d}\Big(\frac{\gamma_1 m}{K_1+m}+\frac{\gamma_2(K-m)}{K_1+K-m}\Big) < 1.$$

Considering the special case E_{11}, we get the following eigenvalues: $-(\mu+\eta), -(\mu+\delta), -r, 0$ and $\frac{\beta\gamma_2 K}{K+K_1}-d$. Thus, if $\frac{\beta\gamma_2 K}{d(K_1+K)} < 1$, then E_{11} might be stable, and if $\frac{\beta\gamma_2 K}{d(K_1+K)} > 1$, then E_{11} might be unstable.
When considering the equilibrium point E_{12}, we found that the eigenvalues corresponding to J_1 are: $-(\mu+\eta), -(\mu+\delta), -r, 0$ and $\frac{\beta\gamma_1 K}{K+K_1}-d$. So, E_{12} might be stable if $\frac{\beta\gamma_1 K}{d(K+K_1)} < 1$, which is equivalent to say that E_{12} might be stable if $K < \frac{dK_1}{\beta\gamma_1-d} = B_{p_2}$. But if E_2 exists, then $K > \frac{dK_1}{\beta\gamma_1-d} = B_{p_2}$ and hence, E_{12} might be unstable whenever E_2 exists.

2.3 Stability of E_2

Let $O = r - \frac{2rB_{p_2}}{K} - \frac{K_1\gamma_1 P_2}{(K_1+B_{p_2})^2}$, $Q = r - \frac{rB_{p_2}}{K} - \frac{\gamma_2 P_2}{K_1} - \theta P_2$ and $L = \frac{\beta\gamma_1 K_1 P_2}{(K_1+B_{p_2})^2}$, then the Jacobin matrix is:

$$J_2 = \begin{bmatrix} -\mu-\eta & 0 & 0 & 0 & 0 \\ 0 & -(\mu+\delta) & 0 & 0 & 0 \\ 0 & 0 & O & \frac{-rB_{p_2}}{K}+\theta P_2 & \frac{-d}{\beta} \\ 0 & 0 & 0 & Q & 0 \\ 0 & 0 & L & \frac{\beta\gamma_2 K_1 P_2}{(K_1)^2} & 0 \end{bmatrix}$$

Now,

$$\begin{aligned} \det(J_2 - \lambda I) &= (-\mu-\eta-\lambda)(-\mu-\delta-\lambda)(Q-\lambda)\Big[\frac{d}{\beta}L - \lambda(O-\lambda)\Big] \\ &= (-\mu-\eta-\lambda)(-\mu-\delta-\lambda)\frac{d}{\beta}(Q-\lambda)(\lambda^2 - \lambda O + \frac{d}{\beta}L) \end{aligned}$$

Note that the equation $(\lambda^2 - \lambda O + \frac{d}{\beta}L) = 0$ has two real solutions non of which is zero since $\frac{d}{\beta}L > 0$. Hence, the stability of E_2 is determined by the sign of the eigenvalue $\lambda = Q$.

If $Q \leq 0$, then

$r - \frac{rB_{p_2}}{K} - \frac{\gamma_2 P_2}{K_1} - \theta P_2 \leq 0$

$K \leq \frac{dK_1}{\beta\gamma_1 - d} + (\frac{\gamma_2}{\gamma_1 K_1} + \frac{\theta}{\gamma_1})(K_1 + \frac{dK_1}{\beta\gamma_1 - d})(K - \frac{dK_1}{\beta\gamma_1 - d})$

$(1 - \frac{\gamma_2\beta + \theta K_1\beta}{\beta\gamma_1 - d})K \leq \frac{dK_1}{\beta\gamma_1 - d} - \frac{dK_1\beta(\gamma_2+\theta K_1)}{(\beta\gamma_1 - d)^2}$

$K \leq \frac{dK_1}{\beta\gamma_1 - d} = B_{p_2}$

which is not the case since we must have $B_{p_2} < K$. Thus, $Q > 0$, and hence E_2 is unstable.

2.4 Stability of E_3

We have the following Jacobin matrix for E_3:

$$\hat{J}_3 = \begin{bmatrix} -\alpha(m)-\mu-\eta & 0 & \frac{-aS_1H}{(m-c+H)^2} & 0 & 0 \\ \alpha(m) & -(\mu+\delta) & \frac{-aS_1H}{(m-c+H)^2} & 0 & 0 \\ 0 & 0 & r - \frac{mr}{K} & \frac{-rm}{K} & \frac{-\gamma_1 m}{K_1+m} + \theta(K-m) \\ 0 & 0 & -\frac{r(K-m)}{K} & -\frac{r(K-m)}{K} & \frac{-\gamma_2(K-m)}{K_1+K-m} - \theta(K-m) \\ 0 & 0 & 0 & 0 & \beta(\frac{\gamma_1 m}{K_1+m} + \frac{\gamma_2(K-m)}{K_1+K-m}) - d \end{bmatrix}$$

The eigenvalues of $\hat{J}_3$ are:$-\alpha(m)-\mu-\eta, -(\mu+\delta), -r, 0$ and $\beta(\frac{\gamma_1 m}{K_1+m} + \frac{\gamma_2(K-m)}{K_1+K-m}) - d$. Hence, E_3 might be stable if $R_B < 1$.

Considering the equilibrium point $E_{31} = (S_1, I_1, K, 0, 0), K > c$, we get the following eigenvalues: $-(\alpha(K)+\mu+\eta), -(\mu+\delta), -r, 0$ and $\frac{\beta\gamma_1 K}{K_1+K} - d$. Hence, E_{31} might be stable if $\frac{\beta\,\gamma_1\,K}{d(K_1+K)} < 1$.

2.5 Stability of E_4

We recall that $E_4 = (S_1, I_1, B_{p_4}, 0, P_4)$, where $B_{p_4} = B_{p_2} = \frac{dK_1}{\beta\gamma_1 - d} > 0$ is such that if $K \leq c$, then E_4 does not exist, and if $K > c$, then $c < B_{p_4} \leq K$, so that $P_4 = P_2 = \frac{r}{\gamma_1 K}\left(K_1 + B_{p_2}\right)\left(K - B_{p_2}\right) > 0$.
Let $O = r - \frac{2rB_{p_2}}{K} - \frac{K_1\gamma_1 P_2}{(K_1+B_{p_2})^2}$, $Q = r - \frac{rB_{p_2}}{K} - \frac{\gamma_2 P_2}{K_1} - \theta P_2$ and $L = \frac{\beta\gamma_1 K_1 P_2}{(K_1+B_{p_2})^2}$,then the Jacobin matrix corresponding to E_4 is:

$$\hat{J}_4 = \begin{bmatrix} -\alpha(B_{p_4}) - \mu - \eta & 0 & \frac{-aS_1H}{(B_{p_4}-c+H)^2} & 0 & 0 \\ \alpha(B_{p_4}) & -(\mu+\delta) & \frac{-aS_1H}{(B_{p_4}-c+H)^2} & 0 & 0 \\ 0 & 0 & O & \frac{-rB_{p_4}}{K} + \theta P_4 & \frac{-d}{\beta} \\ 0 & 0 & 0 & Q & 0 \\ 0 & 0 & L & \frac{\beta\gamma_2 P_4}{K_1} & 0 \end{bmatrix}$$

Then,

$$\begin{aligned} \det(\hat{J}_4) - \lambda I) &= (-\alpha(B_{p_4}) - \mu - \eta - \lambda)(-\mu - \delta - \lambda)(Q - \lambda)\left[\frac{d}{\beta}L - \lambda(O - \lambda)\right] \\ &= (-\mu - \lambda)(-\mu - \delta - \lambda)\frac{d}{\beta}(Q - \lambda)(\lambda^2 - \lambda O + \frac{d}{\beta}L) \\ &= \det(J_2 - \lambda I) \end{aligned}$$

Hence, and is shown in Section 2.3, none of the eigenvalues is zero, and one of the eigenvalue, namely Q, is positive. So, E_4 is unstable.

For the equilibrium points E^*, E^{**} we lack exact expressions for the equilibrium quantities, and so the local stability is difficult to find analytically.

3 Existence of Equilibria with shedding

In this section, we determine the equilibrium points of the model system when $\xi \neq 0$, and then perform stability analysis of the equilibria. Since $R = N - S - I$, then the third equation of system (1) is not necessary, leaving us with the following system:

$$
\begin{aligned}
\dot{S} &= -\alpha(B_p)S - \mu S + \mu N + \eta R \\
\dot{I} &= \alpha(B_p)S - \mu I - \delta I \\
\dot{B}_p &= rB_p\left(1 - \frac{B_p + B_{np}}{K}\right) - \gamma_1 \frac{B_p}{K_1 + B_p} P + \xi I + \theta(B_{np}, P) \\
\dot{B}_{np} &= rB_{np}\left(1 - \frac{B_p + B_{np}}{K}\right) - \gamma_2 \frac{B_{np}}{K_1 + B_{np}} P + \xi I - \theta(B_{np}, P) \\
\dot{P} &= \beta\left(\gamma_1 \frac{B_p}{K_1 + B_p} + \gamma_2 \frac{B_{np}}{K_1 + B_{np}}\right) P - dP + \phi\xi I
\end{aligned}
\tag{5}
$$

If $\alpha(B_p) = 0$, then we will have the same equilibrium points E_0, E_1, E_2 and E^* as in Section 1, with the same conditions.
If $B_P > c$, then $\alpha(B_p) \neq 0$, and hence at steady state system (5) reduces to:

$$
\begin{aligned}
0 &= -\alpha(B_p)S - \mu S - \eta S + \mu N + \eta N - \eta I \\
0 &= \alpha(B_p)S - \mu I - \delta I \\
0 &= rB_p\left(1 - \frac{B_p + B_{np}}{K}\right) - \gamma_1 \frac{B_p}{K_1 + B_p} P + \xi I + \theta B_{np} P \\
0 &= rB_{np}\left(1 - \frac{B_p + B_{np}}{K}\right) - \gamma_2 \frac{B_{np}}{K_1 + B_{np}} P + \xi I - \theta B_{np} P \\
0 &= \beta\left(\gamma_1 \frac{B_p}{K_1 + B_p} + \gamma_2 \frac{B_{np}}{K_1 + B_{np}}\right) P - dP + \phi\xi I
\end{aligned}
\tag{6}
$$

Then solving equation (2) of (6) for I, one has $I = \frac{\alpha(B_p)S}{\mu+\eta}$, and hence, and by solving equation (1) of (6), one can easily check that $S = S_1 = (\mu + \eta)\,(\mu + \delta)\,\frac{\left(B_p^{***} - c\right) + H}{\Gamma_1} N$. Thus, $I = I_1 = \left(\frac{a(\mu+\eta)(B_p^{***} - c)}{\Gamma_1}\right) N$, where
$\Gamma_1 = a(\mu + \delta + \eta)(B_p^{***} - c) + (\mu + \eta)(\mu + \delta)(B_p^{***} - c + H)$. Noting that these are the same formulas for S and I in system (4) at steady state.
By solving the other three equation of system (6), we will prove that system (6) has only one equilibrium point, namely $E^{***} = \left(S_1, I_1, B_p^{***}, B_{np}^{***}, P^{***}\right)$ which exists if the following condition holds:

$$
\gamma_1 \frac{B_p^{***}}{K_1 + B_p^{***}} + \gamma_2 \frac{B_{np}^{***}}{K_1 + B_{np}^{***}} < \frac{d}{\beta},
$$

where $P^{***} = -\phi\xi I_1\Big[\frac{(K_1+B_p^{***})(K_1+B_{np}^{***})}{\Gamma_2}\Big]$, where

$$\Gamma_2 = \beta\gamma_1 B_p^{***}(K_1 + B_{np}^{***}) + \beta\gamma_2 B_{np}^{***}(K_1 + B_p^{***}) - d(K_1 + B_p^{***})(K_1 + B_{np}^{***}).$$

In order to have a well-defined and positive value for P^{***}, Γ_2 must be negative. That is, the value of P provides a condition for B_P^{***} and B_{np}^{***}.
Now, $\Gamma_2 < 0$ if
$\beta\gamma_1 B_p(K_1 + B_{np}) + \beta\gamma_2 B_{np}(K_1 + B_p) < d(K_1 + B_p)(K_1 + B_{np})$.
Dividing both sides by: $(K_1 + B_p)(K_1 + B_{np})$, to get: $\gamma_1\frac{B_p}{K_1+B_p} + \gamma_2\frac{B_{np}}{K_1+B_{np}} < \frac{d}{\beta}$.

Then, and by substituting the values of S, I and P in equation (4) of system (6), we get

$$\begin{aligned}0 = \frac{r}{K}B_{np}(K - B_p - B_{np}) &+ \frac{\xi\,\mu N a(B_p - c)}{(\mu+\delta)((a+\mu)(B_p - c) + \mu H)}\\ &+ \phi\,\xi\,\mu\,N\,a\Big[\frac{\gamma_2\,B_{np}(B_p - c)(K_1 + B_p)}{(\mu+\delta)(a+\mu)(B_p - c) + \mu H}\Big]\frac{1}{\Gamma_2}\\ &+ \theta\,K\,\phi\,\xi\,\mu\,N\,a\Big[\frac{B_{np}(B_p - c)(K_1 + B_p)\,(K_1 + B_{np})}{(\mu+\delta)((a+\mu)(B_p - c) + \mu H\Big)}\Big]\frac{1}{\Gamma_2}\end{aligned} \tag{17}$$

Consequently,

$$\begin{aligned}0 = &K\phi\xi\mu N a(B_p - c)(K_1 + B_p)[\gamma_2 B_{np} + \theta\,(K_1 + B_{np})]\\ &+ \Gamma_2\Big(rB_{np}(K - B_p - B_{np})(\mu+\delta)[(a+\mu)(B_p - c) + \mu H] + \xi\,K\mu N a(B_p - c)\Big)\end{aligned} \tag{18}$$

Using Mathematica to solve equation (18), we found out that it is a cubic equation of B_p which has three solutions. Only one of these solutions is a real solution, say B_p^{***}, which is in-terms of B_{np}.
Substitute B_p^{***}, I and P^{***} in equation (3) of system (6) to get an equation in terms of B_{np} only, then solve the resulting equation to get the value of B_{np}^{***}. The exact formula of B_{np}^{***} is so complicated, and since the other four coordinates of E^{***} depend on B_{np}^{***}, exact formulas of these variables are not given.

4 Linearization

According to the level of pathogenic bacteria, we will have two linearizations of system (5). First when $\alpha(B_p) = 0$, denoted by $\mathcal{J}$ and the other when $\alpha(B_p) \neq 0$,

denoted by $\hat{\mathcal{J}}$.
Set: $U = r - \frac{2rB_p}{K} - \frac{rB_{np}}{K} - \frac{K_1\gamma_1 P}{(K_1+B_p)^2}$ and $V = r - \frac{rB_p}{K} - \frac{2rB_{np}}{K} - \frac{K_1\gamma_2 P}{(K_1+B_{np})^2} - \theta P$.
Then:

$$\mathcal{J} = \begin{bmatrix} -\mu & 0 & 0 & 0 & 0 \\ 0 & -(\mu+\delta) & 0 & 0 & 0 \\ 0 & \xi & U & \frac{-rB_P}{K} + \theta P & \frac{-\gamma_1 B_P}{K_1+B_P} + \theta B_{np} \\ 0 & \xi & -\frac{rB_{np}}{K} & V & \frac{-\gamma_2 B_{np}}{K_1+B_{np}} - \theta B_{np} \\ 0 & \phi\xi & \frac{\beta\gamma_1 K_1 P}{(K_1+B_P)^2} & \frac{\beta\gamma_2 K_1 P}{(K_1+B_{np})^2} & \beta(\frac{\gamma_1+B_p}{K_1+B_p} + \frac{\gamma_2+B_{np}}{K_1+B_{np}}) - d \end{bmatrix},$$

Letting $\mathcal{J}_i$ be the Jacobian matrix corresponding to the equilibrium point E_i, $i = 0, 1$ or 2, one can easily check that the eigenvalues of $\mathcal{J}_i$ are the same as those for J_i. Consequently, the stability of the equilibrium points of system (5) has been already stated in section 2.
The Jacobian matrix corresponding to system (6) is

$$\hat{\mathcal{J}} = \begin{bmatrix} -\alpha(B_p) - \mu & 0 & \frac{-aS_1 H}{(B_p-c+H)^2} & 0 & 0 \\ \alpha(B_p) & -(\mu+\delta) & \frac{-aS_1 H}{(B_p-c+H)^2} & 0 & 0 \\ 0 & \xi & U & \frac{-rB_P}{K} + \theta P & \frac{-\gamma_1 B_P}{K_1+B_P} + \theta B_{np} \\ 0 & \xi & -\frac{rB_{np}}{K} & V & \frac{-\gamma_2 B_{np}}{K_1+B_{np}} - \theta B_{np} \\ 0 & \phi\xi & \frac{\beta\gamma_1 K_1 P}{(K_1+B_P)^2} & \frac{\beta\gamma_2 K_1 P}{(K_1+B_{np})^2} & \beta(\frac{\gamma_1 B_p}{K_1+B_p} + \frac{\gamma_2 B_{np}}{K_1+B_{np}}) - d \end{bmatrix},$$

Since we lack exact expressions for the coordinates of E^{***}, the stability of this equilibrium point is difficult to find analytically.