 doi:10.20944/preprints202102.0333.v1

On Group Property of Hassani Transforms and Relativity Principle in Hassani Kinematics

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УДК: 517.983 + 510.22 + 51-71

In the present paper, based on the ideas of Algerian physicist M.E. Hassani, the generalized Hassani spatio-temporal transformations in real Hilbert space are introduced. The original transformations, introduced by M.E. Hassani, are the particular cases of the transformations, introduced in this paper. It is proven that the classes of generalized Hassani transforms do not form a group of operators in the general case. Further, using these generalized Hassani transformations as well as the theory of changeable sets and universal kinematics, the mathematically strict models of Hassani kinematics are constructed and the performance of the relativity principle in these models is discussed.

Key words: groups of operators, universal kinematics, changeable sets, inertial reference frames, tachyons, the principle of relativity.

PACS 2010 classification numbers: 03.30.+p; 02.10.Ab
MSC 2020 classification numbers: 83A05; 70A05

1 Introduction

Subject of constructing the theory of super-light movement, had been initiated in the papers [1, 2] more than 55 years ago. Despite the fact that on today tachyons (ie objects moving at a velocity greater than the velocity of light) are not experimentally detected, this subject remains being actual. Initially, the theory of tachyons was considered in the framework of classical Lorentz transformations, and superlight speed for frames of reference was forbidden. But afterwards in the papers [3–5] and and later in the papers of S. Medvedev [6] as well as J. Hill and B. Cox [7] the generalized Lorentz transforms for superluminal reference frames were deduced in the case of three-dimension space of geometric variables. In the paper [8] it was proven, that the above generalized Lorentz transforms may be easy extend to the more general case of arbitrary (in particular infinity) dimension of the space of geometric variables. M.E. Hassani in the paper [9] have proposed the another, completely different and interesting, system of coordinate transforms for superluminal reference frames in the case of three-dimension space of geometric variables. In the present paper we introduce generalized Hassani (superluminal) spatio-temporal transformations for real Hilbert space. The main aim of this paper is to construct the (superluminal) universal kinematics, based on these generalized Hassani transforms and to show that these universal kinematics do not satisfy the relativity principle in the general case.

It should be noted that in the paper of A. Sfarti [10] author tries to refute the results of the Hassani original paper [9]. That is why to dispel all doubts concerning the main results of the Hassani's paper [9], we should give reply to the Sfarti's "rebuttal" [10]. The considerations of A. Sfarti in [10] are based on the fact that the Hassani coordinate transforms coincide with the Lorentz transforms on the subluminal diapason of reference frame velocities. That is why, using the elementary calculations of special relativity theory, A. Sfarti concludes that to accelerate the ordinary subluminal particle (tardyon) to the luminal (or superluminal) speed we need the infinitely many quantity of energy. From the point of view of A. Sfarti, the last fact, proves that tachyons (superluminal particles) can not exist and so the superluminal reference frames do not exist also. Note that using the A. Sfarti final conclusion we may try to "refute" not only the results of M.E. Hassani [9], but also all results of tachyon theory (this theory is a quite popular direction of theoretical physics). It should be emphasized that the considerations of A. Sfarti are based on the assumption that the velocity of particle should change by a continuous way. But A. Sfarti had not taken into account the following two hypotheses, which are also reasonable:

– Under some (unknown at the present time) conditions, the velocity of some particle may change instantly (skippingly) or (similar hypothesis) tachyons may be obtained from the tardyons during some (unknown at the present time) particle transformations.

– Tachyons may exist in Nature preassigned and they may interact with tardyons.

Thus, we have seen that the main conclusion of A. Sfarti does not rebut the existence of tachyons as well as superluminal reference frames. As we will see further, the Hassani transforms are, at least, mathematically correct objects. So we are not forbidden to investigate and to generalize them as well as to consider universal kinematics, generated by them.

Considerations similar to Safari's calculations are well known in physics. The founders of tachyon theory (O.-M. Bilaniuk, E. Recami, E. Sudarshan, etc) are famous physicists theorists, who knew the special relativity theory well. So, these considerations were well known to them. Hence to finish this discussion we cite one brilliant quote from the paper of E. Recami [5, p. 12] (see also [11, p. 65]):

> "... together with special relativity the conviction that the light speed c in vacuum was the upper limit of any speed started to spread over the scientific community the early-century physicists, being led by the evidence that ordinary bodies cannot overtake that speed. They behaved in a sense like Sudarshan's (1972) imaginary demographer studying the population patterns of the Indian subcontinent: ≪ Suppose a demographer calmly asserts that there are no people North of the Himalayas, since none could climb over the mountain ranges! That would be an absurd conclusion. People of central Asia are born there and live there: They did not have to be born in India and cross the mountain range. So with faster-than-light particles. ≫"

In Section 2 we introduce the generalized Hassani transforms over Hilbert space. In Section 3 we prove that the introduced classes of generalized Hassani transforms do not form a group of operators in the general case. In Section 4 we construct the generalized Hassani kinematics, based on generalized Hassani transforms and discuss the performance of the relativity principle in these kinematics.

2 Generalized Hassani Transforms over Hilbert Space

Generalized Hassani transforms for special case. In the works of M.E. Hassani (see, for example, [9]) it is proposed an interesting version of the generalized Lorentz transforms for a special case, when two inertial frames are moving along the x-axis in three-dimensional space and the directions of corresponding axises "y" and "z" are parallel:

$$t' = \frac{t - \frac{Vx}{\vartheta(V)^2}}{\sqrt{1 - \frac{V^2}{\vartheta(V)^2}}}; \quad x' = \frac{x - Vt}{\sqrt{1 - \frac{V^2}{\vartheta(V)^2}}}; \quad y' = y, \quad z' = z, \quad \text{where:} \tag{1}$$

- $V \in \mathbb{R}$ is the velocity of inertial reference frame $\mathfrak{l}'$, which moves relatively the fixed inertial reference frame $\mathfrak{l}$.
- (t, x, y, z) are the (space-time) coordinates of any point $\mathbf{M}$ in the fixed frame $\mathfrak{l}$,

- (t', x', y', z') are the coordinates of the same point $\mathbf{M}$ in the moving frame $\mathfrak{l}'$,
- $\vartheta(\cdot)$ is an arbitrary real function of real variable, possessing the following properties:

$$\left.\begin{array}{r} \vartheta(V) = c \quad \text{for} \quad 0 \leq V < c; \\ \vartheta(V) > V \quad \text{for} \quad c \leq V < \infty; \\ \vartheta(-V) = \vartheta(V) \quad (\forall V \in \mathbb{R}), \end{array}\right\} \tag{2}$$

where c is a positive real constant, which has the physical content of the speed of light in vacuum.

First of all, we note that is enough to restrict ourselves to the functions $\vartheta(\cdot)$ defined on $[0,\infty)$ and to consider the expression $\vartheta(|V|)$ instead of $\vartheta(V)$ in (1). Also instead of functions, which satisfy two first conditions (2) we will consider class of functions, satisfying the more weak conditions.

Denote by Υ the class of functions $\vartheta : [0,\infty) \to \mathbb{R}$, satisfying the following conditions:

$$\left.\begin{array}{r} \vartheta(\lambda) \geq \lambda \quad \text{for} \quad \lambda \in [0,\infty). \\ \exists\, \eta > 0 \quad \vartheta(\lambda) > \lambda \ \ (\forall \lambda \in [0,\eta)). \end{array}\right\} \tag{3}$$

For any function $\vartheta \in \Upsilon$ we use the following notation:

$$\mathfrak{D}_* [\vartheta] := \{\lambda \in [0,\infty) \,|\, \vartheta(\lambda) > \lambda\}. \tag{4}$$

According to the conditions (3), we have, $\mathfrak{D}_* [\vartheta] \neq \emptyset$, and moreover,

$$[0,\eta) \subseteq \mathfrak{D}_* [\vartheta] \quad \text{for some} \quad \eta > 0. \tag{5}$$

Then for $|V| \in \mathfrak{D}_* [\vartheta]$ we can introduce the following (space-temporally) coordinate transforms:

$$t' = \frac{t - \frac{Vx}{\vartheta(|V|)^2}}{\sqrt{1 - \frac{V^2}{\vartheta(|V|)^2}}}; \quad x' = \frac{x - Vt}{\sqrt{1 - \frac{V^2}{\vartheta(|V|)^2}}}; \quad y' = y, \quad z' = z.$$

Therefore, we have introduced the generalized Hassani transforms for the same special case as for transforms (1). In the case

$$\vartheta(\lambda) = \vartheta_c(\lambda) := \begin{cases} c, & 0 \leq \lambda < c \\ \lambda, & \lambda \geq c \end{cases} \tag{6}$$

we obtain the classical Lorentz transforms and in the case, where the function ϑ satisfies two first conditions (2) we obtain the Hassani transforms (1).

Generalized Hassani transforms for the general case of Hilbert space. Let $(\mathfrak{H}, \|\cdot\|, \langle\cdot,\cdot\rangle)$ be a Hilbert space over the real field such, that $\dim(\mathfrak{H}) \geq 1$, where $\dim(\mathfrak{H})$ is dimension of the space $\mathfrak{H}$. Emphasize, that the condition $\dim(\mathfrak{H}) \geq 1$ should be interpreted in a way that the space $\mathfrak{H}$ may be infinite-dimensional. Let $\mathcal{L}(\mathfrak{H})$ be the space of (homogeneous) linear continuous operators over the space $\mathfrak{H}$. Denote by $\mathcal{L}^\times(\mathfrak{H})$ the space of all operators of affine transformations over the space $\mathfrak{H}$, that is $\mathcal{L}^\times(\mathfrak{H}) = \left\{\mathbf{A}_{[\mathbf{a}]} \,|\, \mathbf{A} \in \mathcal{L}(\mathfrak{H}),\, \mathbf{a} \in \mathfrak{H}\right\}$, where $\mathbf{A}_{[\mathbf{a}]}x = \mathbf{A}x + \mathbf{a}$, $x \in \mathfrak{H}$. The ***Minkowski space*** over the Hilbert space $\mathfrak{H}$ is defined as the Hilbert space $\mathcal{M}(\mathfrak{H}) = \mathbb{R} \times \mathfrak{H} = \{(t,x) \,|\, t \in \mathbb{R},\ x \in \mathfrak{H}\}$, equipped by the inner product and norm: $\langle \mathrm{w}_1, \mathrm{w}_2\rangle = \langle \mathrm{w}_1, \mathrm{w}_2\rangle_{\mathcal{M}(\mathfrak{H})} = t_1 t_2 + \langle x_1, x_2\rangle$, $\|\mathrm{w}_1\| = \|\mathrm{w}_1\|_{\mathcal{M}(\mathfrak{H})} = \left(t_1^2 + \|x_1\|^2\right)^{1/2}$ (where $\mathrm{w}_i = (t_i, x_i) \in \mathcal{M}(\mathfrak{H})$, $i \in \{1,2\}$) ([8,12]). In the space $\mathcal{M}(\mathfrak{H})$ we select the next subspaces: $\mathfrak{H}_0 := \{(t,\mathbf{0}) \,|\, t \in \mathbb{R}\}$, $\mathfrak{H}_1 := \{(0,x) \,|\, x \in \mathfrak{H}\}$ with $\mathbf{0}$ being zero vector. Then,

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$\mathcal{M}(\mathfrak{H}) = \mathfrak{H}_0 \oplus \mathfrak{H}_1$, where $\oplus$ means the orthogonal sum of subspaces. Denote: $\mathbf{e}_0 := (1, \mathbf{0}) \in \mathcal{M}(\mathfrak{H})$. Introduce the orthogonal projectors on the subspaces $\mathfrak{H}_1$ and $\mathfrak{H}_0$:

$$\mathbf{X}\mathrm{w} = (0, x) \in \mathfrak{H}_1; \; \widehat{\mathbf{T}}\mathrm{w} = (t, \mathbf{0}) = \mathcal{T}(\mathrm{w})\,\mathbf{e}_0 \in \mathfrak{H}_0,$$
$$\text{where } \mathcal{T}(\mathrm{w}) = t \quad (\mathrm{w} = (t, x) \in \mathcal{M}(\mathfrak{H})).$$

Definition 1 ([8, 12])**.** *The operator $S \in \mathcal{L}(\mathcal{M}(\mathfrak{H}))$ is referred to as* ***linear coordinate transform operator*** *if and only if there exist the continuous inverse operator $S^{-1} \in \mathcal{L}(\mathcal{M}(\mathfrak{H}))$.*

The linear coordinate transform operator $S \in \mathcal{L}(\mathcal{M}(\mathfrak{H}))$ is called ***v-determined*** *if and only if $\mathcal{T}(S^{-1}\mathbf{e}_0) \neq 0$. The vector*

$$\mathcal{V}(S) = \frac{\mathbf{X}S^{-1}\mathbf{e}_0}{\mathcal{T}(S^{-1}\mathbf{e}_0)} \in \mathfrak{H}_1$$

is named the ***velocity*** *of the v-determined coordinate transform operator S.*

Let $\mathbf{B}_1(\mathfrak{H}_1)$ be the unit sphere in the space $\mathfrak{H}_1$ ($\mathbf{B}_1(\mathfrak{H}_1) = \{x \in \mathfrak{H}_1 \mid \|x\| = 1\}$). Any vector $\mathbf{n} \in \mathbf{B}_1(\mathfrak{H}_1)$ generates the following orthogonal projectors, acting in $\mathcal{M}(\mathfrak{H})$:

$$\left.\begin{aligned} \mathbf{X}_1[\mathbf{n}]\,\mathrm{w} &= \langle \mathbf{n}, \mathrm{w}\rangle\,\mathbf{n} \quad (\mathrm{w} \in \mathcal{M}(\mathfrak{H})); \\ \mathbf{X}_1^{\perp}[\mathbf{n}] &= \mathbf{X} - \mathbf{X}_1[\mathbf{n}]\,. \end{aligned}\right] \tag{7}$$

Recall, that an operator $U \in \mathcal{L}(\mathfrak{H})$ is referred to as ***unitary*** on $\mathfrak{H}$, if and only if $\exists\, U^{-1} \in \mathcal{L}(\mathfrak{H})$ and $\forall\, x \in \mathfrak{H}$ $\|Ux\| = \|x\|$. Let $\mathfrak{U}(\mathfrak{H}_1)$ be the set of all ***unitary*** operators over the space $\mathfrak{H}_1$. Fix some real number c such, that $0 < c < \infty$. Then for every $\lambda \in [0, c)$, $s \in \{-1, 1\}$, $J \in \mathfrak{U}(\mathfrak{H}_1)$, $\mathbf{n} \in \mathbf{B}_1(\mathfrak{H}_1)$ and $\mathbf{a} \in \mathcal{M}(\mathfrak{H})$ we introduce the following operators, acting in $\mathcal{M}(\mathfrak{H})$:

$$\mathbf{W}_{\lambda,c}[s, \mathbf{n}, J]\,\mathrm{w} := \frac{\left(s\mathcal{T}(\mathrm{w}) - \frac{\lambda}{c^2}\langle \mathbf{n}, \mathrm{w}\rangle\right)}{\sqrt{1 - \frac{\lambda^2}{c^2}}}\mathbf{e}_0 + J\left(\frac{\lambda\mathcal{T}(\mathrm{w}) - s\langle \mathbf{n}, \mathrm{w}\rangle}{\sqrt{1 - \frac{\lambda^2}{c^2}}}\mathbf{n} + \mathbf{X}_1^{\perp}[\mathbf{n}]\,\mathrm{w}\right); \tag{8}$$

$$\mathbf{W}_{\lambda,c}[s, \mathbf{n}, J; \mathbf{a}]\,\mathrm{w} := \mathbf{W}_{\lambda,c}[s, \mathbf{n}, J](\mathrm{w} + \mathbf{a}) \qquad (\mathrm{w} \in \mathcal{M}(\mathfrak{H})). \tag{9}$$

Under the additional conditions $\dim(\mathfrak{H}) = 3$, $s = 1$ the right-hand part of the formula (8) is equivalent to the same part of the formula (28b) from [13, page 43]. That is why, in this case we obtain the classical Lorentz transforms for inertial reference frame in the most general form (with arbitrary orientation of axes). Now we introduce the following classes of operators:

$$\begin{aligned} \mathfrak{O}(\mathfrak{H}, c) &:= \{\mathbf{W}_{\lambda,c}[s, \mathbf{n}, J] \mid s \in \{-1, 1\},\, \lambda \in [0, c),\, \mathbf{n} \in \mathbf{B}_1(\mathfrak{H}_1),\, J \in \mathfrak{U}(\mathfrak{H}_1)\}; \\ \mathfrak{O}_+(\mathfrak{H}, c) &:= \{\mathbf{W}_{\lambda,c}[s, \mathbf{n}, J] \in \mathfrak{O}(\mathfrak{H}, c) \mid s = 1\} = \\ &= \{\mathbf{W}_{\lambda,c}[1, \mathbf{n}, J] \mid \lambda \in [0, c),\, \mathbf{n} \in \mathbf{B}_1(\mathfrak{H}_1),\, J \in \mathfrak{U}(\mathfrak{H}_1)\}; \\ \mathfrak{P}(\mathfrak{H}, c) &:= \{\mathbf{W}_{\lambda,c}[s, \mathbf{n}, J; \mathbf{a}] \mid \mathbf{W}_{\lambda,c}[s, \mathbf{n}, J] \in \mathfrak{O}(\mathfrak{H}, c),\, \mathbf{a} \in \mathcal{M}(\mathfrak{H})\}; \\ \mathfrak{P}_+(\mathfrak{H}, c) &:= \{\mathbf{W}_{\lambda,c}[s, \mathbf{n}, J; \mathbf{a}] \mid \mathbf{W}_{\lambda,c}[s, \mathbf{n}, J] \in \mathfrak{O}_+(\mathfrak{H}, c),\, \mathbf{a} \in \mathcal{M}(\mathfrak{H})\}. \end{aligned} \tag{10}$$

It is clearly that $\mathfrak{O}(\mathfrak{H}, c), \mathfrak{O}_+(\mathfrak{H}, c) \subseteq \mathcal{L}(\mathcal{M}(\mathfrak{H}))$ and $\mathfrak{P}(\mathfrak{H}, c), \mathfrak{P}_+(\mathfrak{H}, c) \subseteq \mathcal{L}^{\times}(\mathcal{M}(\mathfrak{H}))$.

Remark 1. It can be proven that all four classes of operators are groups of operators in the space $\mathcal{M}(\mathfrak{H})$ (see [14, Remark 4.1, Corollary 4.1]; see also [12, Assertion 2.17.1 and formula (2.94), Assertion 2.17.6, Corollary 2.19.5]). In particular $\mathfrak{O}(\mathfrak{H}, c)$ coincides with the group of all linear coordinate transform operators over the space $\mathcal{M}(\mathfrak{H})$, leaving unchanged values of the functional:

$$\mathsf{M}_c(\mathrm{w}) = \|\mathbf{X}\mathrm{w}\|^2 - c^2\mathcal{T}^2(\mathrm{w}) \qquad (\mathrm{w} \in \mathcal{M}(\mathfrak{H})), \tag{11}$$

that is the set of all bijective operators $L \in \mathcal{L}(\mathcal{M}(\mathfrak{H}))$ such, that:

$$\mathsf{M}_c(L\mathrm{w}) = \mathsf{M}_c(\mathrm{w}) \qquad (\forall \mathrm{w} \in \mathcal{M}(\mathfrak{H})). \tag{12}$$

In the case $\mathfrak{H} = \mathbb{R}^3$ the group of operators $\mathfrak{O}_+(\mathfrak{H}, c)$ coincides with with the full Lorentz group, being considered in [15]. In the case $\mathfrak{H} = \mathbb{R}^3$ the group of operators $\mathfrak{P}_+(\mathfrak{H}, c)$ coincides with the famous Poincare group [12, Remark 2.19.1].

Remark 2. It should be emphasized that for every $c \in (0, \infty)$, $\lambda \in [0, c)$, $s \in \{-1, 1\}$, $J \in \mathfrak{U}(\mathfrak{H}_1)$ and $\mathbf{n} \in \mathbf{B}_1(\mathfrak{H}_1)$ the operator $\mathbf{W}_{\lambda,c}[s, \mathbf{n}, J]$, defined in (8) is v-determined, moreover

$$\mathcal{V}(\mathbf{W}_{\lambda,c}[s, \mathbf{n}, J]) = \lambda s \mathbf{n}. \tag{13}$$

Indeed, consider the vector:

$$\mathbf{n}_{t_0,\mu_0} = t_0\mathbf{e}_0 + \mu_0\mathbf{n} \in \mathcal{M}(\mathfrak{H}), \quad \text{where} \quad t_0 := \frac{s}{\sqrt{1-\frac{\lambda^2}{c^2}}}, \quad \mu_0 := \frac{\lambda}{\sqrt{1-\frac{\lambda^2}{c^2}}},$$

$$\mathcal{T}(\mathbf{n}_{t_0,\mu_0}) = t_0, \quad \langle \mathbf{n}, \mathbf{n}_{t_0,\mu_0}\rangle = \mu_0, \quad \mathbf{X}\mathbf{n}_{t_0,\mu_0} = \mu_0\mathbf{n}, \quad \mathbf{X}_1^{\perp}[\mathbf{n}]\,\mathbf{n}_{t_0,\mu_0} = \mathbf{0}.$$

Using formula (8), we obtain:

$$\mathbf{W}_{\lambda,c}[s, \mathbf{n}, J]\,\mathbf{n}_{t_0,\mu_0} = \frac{1}{\sqrt{1-\frac{\lambda^2}{c^2}}}\left(\left(st_0 - \frac{\lambda}{c^2}\mu_0\right)\mathbf{e}_0 + (\lambda t_0 - s\mu_0)\,J\mathbf{n}\right) = \mathbf{e}_0.$$

Hence, $\mathbf{W}_{\lambda,c}[s, \mathbf{n}, J]^{-1}\mathbf{e}_0 = \mathbf{n}_{t_0,\mu_0}$, $\mathcal{T}\left(\mathbf{W}_{\lambda,c}[s, \mathbf{n}, J]^{-1}\mathbf{e}_0\right) = \mathcal{T}(\mathbf{n}_{t_0,\mu_0}) = t_0 = \frac{s}{\sqrt{1-\frac{\lambda^2}{c^2}}} \neq 0$ and $\mathcal{V}(\mathbf{W}_{\lambda,c}[s, \mathbf{n}, J]) = \frac{\mathbf{X}\mathbf{W}_{\lambda,c}[s,\mathbf{n},J]^{-1}\mathbf{e}_0}{\mathcal{T}\left(\mathbf{W}_{\lambda,c}[s,\mathbf{n},J]^{-1}\mathbf{e}_0\right)} = \frac{\mu_0\mathbf{n}}{t_0} = \lambda s\mathbf{n}$.

If we use the function parameter $\vartheta \in \Upsilon$ (where Υ is the class of functions, satisfying (3)) instead of the constant speed c, then we obtain the following classes of operators (for each $\vartheta \in \Upsilon$):

$$\mathfrak{O}(\mathfrak{H}, [\vartheta]) := \left\{\mathbf{W}_{\lambda,\vartheta(\lambda)}[s, \mathbf{n}, J] \mid s \in \{-1, 1\},\ \lambda \in \mathfrak{D}_*[\vartheta],\ \mathbf{n} \in \mathbf{B}_1(\mathfrak{H}_1),\ J \in \mathfrak{U}(\mathfrak{H}_1)\right\}; \tag{14}$$

$$\begin{aligned}\mathfrak{O}_+(\mathfrak{H}, [\vartheta]) &:= \left\{\mathbf{W}_{\lambda,\vartheta(\lambda)}[s, \mathbf{n}, J] \in \mathfrak{O}(\mathfrak{H}, [\vartheta]) \mid s = 1\right\} = \\ &= \left\{\mathbf{W}_{\lambda,\vartheta(\lambda)}[1, \mathbf{n}, J] \mid \lambda \in \mathfrak{D}_*[\vartheta],\ \mathbf{n} \in \mathbf{B}_1(\mathfrak{H}_1),\ J \in \mathfrak{U}(\mathfrak{H}_1)\right\}; \quad (15)\\ \mathfrak{P}(\mathfrak{H}, [\vartheta]) &:= \left\{\mathbf{W}_{\lambda,\vartheta(\lambda)}[s, \mathbf{n}, J; \mathbf{a}] \mid \mathbf{W}_{\lambda,\vartheta(\lambda)}[s, \mathbf{n}, J] \in \mathfrak{O}(\mathfrak{H}, [\vartheta]),\ \mathbf{a} \in \mathcal{M}(\mathfrak{H})\right\}; \quad (16)\\ \mathfrak{P}_+(\mathfrak{H}, [\vartheta]) &:= \left\{\mathbf{W}_{\lambda,\vartheta(\lambda)}[s, \mathbf{n}, J; \mathbf{a}] \mid \mathbf{W}_{\lambda,\vartheta(\lambda)}[s, \mathbf{n}, J] \in \mathfrak{O}_+(\mathfrak{H}, [\vartheta]),\ \mathbf{a} \in \mathcal{M}(\mathfrak{H})\right\}, \quad (17)\end{aligned}$$

where $\mathfrak{D}_*[\vartheta]$ is the set of ***ϑ-allowed*** velocities, defined by (4). It is easy to see that for each $\vartheta \in \Upsilon$ the following set-theoretic inclusions are performed:

$$\mathfrak{O}_+(\mathfrak{H}, [\vartheta]) \subseteq \mathfrak{O}(\mathfrak{H}, [\vartheta]) \subseteq \mathfrak{P}(\mathfrak{H}, [\vartheta]); \tag{18}$$

$$\mathfrak{O}_+(\mathfrak{H}, [\vartheta]) \subseteq \mathfrak{P}_+(\mathfrak{H}, [\vartheta]) \subseteq \mathfrak{P}(\mathfrak{H}, [\vartheta]). \tag{19}$$

- $\mathfrak{O}(\mathfrak{H}, [\vartheta])$ we name by class of ***generalized Hassani transforms*** over Hilbert space $\mathfrak{H}$;
- $\mathfrak{O}_+(\mathfrak{H}, [\vartheta])$ we name by class of ***time-positive generalized Hassani transforms*** over Hilbert space $\mathfrak{H}$;
- $\mathfrak{P}(\mathfrak{H}, [\vartheta])$ we name by class of ***Poincare-Hassani transforms*** over Hilbert space $\mathfrak{H}$;
- $\mathfrak{P}_+(\mathfrak{H}, [\vartheta])$ we name by class of ***time-positive Poincare-Hassani transforms*** over Hilbert space $\mathfrak{H}$.

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3 On the Group Property of Hassani Transforms

We have seen above that in the case of constant speed of light c all classes of operators $\mathfrak{O}(\mathfrak{H},c)$, $\mathfrak{O}_+(\mathfrak{H},c)$, $\mathfrak{P}(\mathfrak{H},c)$, $\mathfrak{P}_+(\mathfrak{H},c)$ are groups in the space $\mathcal{M}(\mathfrak{H})$.

Theorem 1. *Let $\vartheta\in\Upsilon$ and $\mathcal{Q}\in\{\mathfrak{O}(\mathfrak{H},[\vartheta]),\ \mathfrak{O}_+(\mathfrak{H},[\vartheta]),\ \mathfrak{P}(\mathfrak{H},[\vartheta]),\ \mathfrak{P}_+(\mathfrak{H},[\vartheta])\}$ be one of the four classes of operators, introduced above. Then the class $\mathcal{Q}$ is a group of operators in the space $\mathcal{M}(\mathfrak{H})$ if and only if a number $c\in(0,\infty)$ exists such, that $\vartheta(\lambda)=\vartheta_c(\lambda)$ for every $\lambda\in(0,\infty)$, where $\vartheta_c(\lambda)$ is the function, defined by (6).*

Remark 3. Since (by (8), (9)) $\mathbf{W}_{0,\chi_1}[s,\mathbf{n},J]=\mathbf{W}_{0,\chi_2}[s,\mathbf{n},J]$ and $\mathbf{W}_{0,\chi_1}[s,\mathbf{n},J;\mathbf{a}]=\mathbf{W}_{0,\chi_2}[s,\mathbf{n},J;\mathbf{a}]$ ($\forall s\in\{-1,1\}\ \forall J\in\mathfrak{U}(\mathfrak{H}_1)\ \forall\chi_1,\chi_2>0\ \forall\mathbf{n}\in\mathbf{B}_1(\mathfrak{H}_1)\ \forall\mathbf{a}\in\mathcal{M}(\mathfrak{H})$), in the case where there exists a number $c\in(0,\infty)$ such, that $\vartheta(\lambda)=\vartheta_c(\lambda)$ ($\forall\,\lambda\in(0,\infty)$), we have

$$\left.\begin{array}{ll}\mathfrak{O}(\mathfrak{H},[\vartheta])=\mathfrak{O}(\mathfrak{H},[\vartheta_c])=\mathfrak{O}(\mathfrak{H},c)\,; & \mathfrak{P}(\mathfrak{H},[\vartheta])=\mathfrak{P}(\mathfrak{H},c)\,;\\ \mathfrak{O}_+(\mathfrak{H},[\vartheta])=\mathfrak{O}_+(\mathfrak{H},c)\,; & \mathfrak{P}_+(\mathfrak{H},[\vartheta])=\mathfrak{P}_+(\mathfrak{H},c)\,.\end{array}\right\}\tag{20}$$

Hence, Theorem 1 asserts that the class $\mathcal{Q}\in\{\mathfrak{O}(\mathfrak{H},[\vartheta]),\ \mathfrak{O}_+(\mathfrak{H},[\vartheta])\,,\ \mathfrak{P}(\mathfrak{H},[\vartheta])\,,\ \mathfrak{P}_+(\mathfrak{H},[\vartheta])\}$ is a group of operators in the space $\mathcal{M}(\mathfrak{H})$ if and only if $\mathcal{Q}$ coincides with the some ordinary Lorentz or Poincare group (with the constant velocity of light).

Proof of Theorem 1. Let $\vartheta\in\Upsilon$. Assume that the class $\mathcal{Q}\in\{\mathfrak{O}(\mathfrak{H},[\vartheta]),\ \mathfrak{O}_+(\mathfrak{H},[\vartheta]),\ \mathfrak{P}(\mathfrak{H},[\vartheta]),\ \mathfrak{P}_+(\mathfrak{H},[\vartheta])\}$ is a group. Since $\vartheta\in\Upsilon$ then, according to (5), we have $\mathfrak{D}_*[\vartheta]\setminus\{0\}\neq\emptyset$. First of all we will prove that

$$\vartheta(\lambda)\equiv\mathbf{const}\quad(\forall\lambda\in\mathfrak{D}_*[\vartheta]\setminus\{0\})\,.\tag{21}$$

Consider any numbers $\lambda_1,\lambda_2\in\mathfrak{D}_*[\vartheta]\setminus\{0\}$ ($\lambda_1,\lambda_2>0$). Chose any vector $\mathbf{n}\in\mathbf{B}_1(\mathfrak{H}_1)$. It is apparently that $\mathbb{I}_{\sigma,\mu}[\mathbf{n}]\in\mathfrak{U}(\mathfrak{H}_1)$ ($\forall\,\sigma,\mu\in\{-1,1\}$), where

$$\mathbb{I}_{\sigma,\mu}[\mathbf{n}]\ x:=\sigma\mathbf{X}_1[\mathbf{n}]\,x+\mu\mathbf{X}_1^{\perp}[\mathbf{n}]\,x=\sigma\langle\mathbf{n},x\rangle\,\mathbf{n}+\mu\mathbf{X}_1^{\perp}[\mathbf{n}]\,x,\qquad (x\in\mathfrak{H}_1,\ \sigma,\mu\in\{-1,1\})\,.$$

Denote:

$$c_i:=\vartheta(\lambda_i)\qquad\left(i\in\overline{1,2}\right),\tag{22}$$

where $\overline{1,n}=\{1,\dots,n\}$ ($n\in\mathbb{N}$). Then we obtain $0<\lambda_i<c_i$ $\left(i\in\overline{1,2}\right)$ (according to (4)) and $\mathbf{W}_{\lambda_i,c_i}[1,\mathbf{n},\mathbb{I}_{-1,1}[\mathbf{n}]]\in\mathfrak{O}_+(\mathfrak{H},[\vartheta])$ $\left(i\in\overline{1,2}\right)$. So, according to inclusions (18), (19), we have:

$$\mathbf{W}_{\lambda_i,c_i}[1,\mathbf{n},\mathbb{I}_{-1,1}[\mathbf{n}]]\in\mathcal{Q}\quad\left(i\in\overline{1,2}\right).$$

Since, according to the above assumption, the class $\mathcal{Q}$ is a group, the operator $\mathbf{L}^{(c_1,c_2)}_{(\lambda_1,\lambda_2)}:=\mathbf{W}_{\lambda_2,c_2}[1,\mathbf{n},\mathbb{I}_{-1,1}[\mathbf{n}]]\,\mathbf{W}_{\lambda_1,c_1}[1,\mathbf{n},\mathbb{I}_{-1,1}[\mathbf{n}]]$ must belong to $\mathcal{Q}$. Consider any numbers $t,\mu\in\mathbb{R}$. For the vector $\mathbf{n}_{t,\mu}=t\mathbf{e}_0+\mu\mathbf{n}\in\mathcal{M}(\mathfrak{H})$ we deliver $\mathbf{L}^{(c_1,c_2)}_{(\lambda_1,\lambda_2)}\mathbf{n}_{t,\mu}=\mathbf{W}_{\lambda_2,c_2}[1,\mathbf{n},\mathbb{I}_{-1,1}[\mathbf{n}]]\,\mathrm{w}_{t,\mu}$, where $\mathrm{w}_{t,\mu}=\mathbf{W}_{\lambda_1,c_1}[1,\mathbf{n},\mathbb{I}_{-1,1}[\mathbf{n}]]\,\mathbf{n}_{t,\mu}$. Hence, according to (8) and (7), we deduce:

$$\mathrm{w}_{t,\mu}=\mathbf{W}_{\lambda_1,c_1}[1,\mathbf{n},\mathbb{I}_{-1,1}[\mathbf{n}]]\,\mathbf{n}_{t,\mu}=\frac{1}{\sqrt{1-\frac{\lambda_1^2}{c_1^2}}}\left(\left(t-\frac{\lambda_1\mu}{c_1^2}\right)\mathbf{e}_0+(\mu-\lambda_1 t)\,\mathbf{n}\right);$$

$$\mathcal{T}(\mathrm{w}_{t,\mu})=\frac{t-\frac{\lambda_1\mu}{c_1^2}}{\sqrt{1-\frac{\lambda_1^2}{c_1^2}}},\quad\langle\mathbf{n},\mathrm{w}_{t,\mu}\rangle=\frac{\mu-\lambda_1 t}{\sqrt{1-\frac{\lambda_1^2}{c_1^2}}},\quad\mathbf{X}_1^{\perp}[\mathbf{n}]\,\mathrm{w}_{t,\mu}=\mathbf{0};$$

$$\mathbf{L}^{(c_1,c_2)}_{(\lambda_1,\lambda_2)}\mathbf{n}_{t,\mu}=\mathbf{W}_{\lambda_2,c_2}[1,\mathbf{n},\mathbb{I}_{-1,1}[\mathbf{n}]]\,\mathrm{w}_{t,\mu}=$$

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$$= \frac{1}{\sqrt{\left(1-\frac{\lambda_1^2}{c_1^2}\right)\left(1-\frac{\lambda_2^2}{c_2^2}\right)}}\left(\left(t-\frac{\lambda_1\mu}{c_1^2}-\frac{(\mu-\lambda_1 t)\lambda_2}{c_2^2}\right)\mathbf{e}_0+\right.$$

$$\left.+\left((\mu-\lambda_1 t)-\lambda_2\left(t-\frac{\lambda_1\mu}{c_1^2}\right)\right)\mathbf{n}\right)=$$

$$= \frac{1}{\sqrt{\left(1-\frac{\lambda_1^2}{c_1^2}\right)\left(1-\frac{\lambda_2^2}{c_2^2}\right)}}\left(\left(t\left(1+\frac{\lambda_1\lambda_2}{c_2^2}\right)-\mu\left(\frac{\lambda_2}{c_2^2}+\frac{\lambda_1}{c_1^2}\right)\right)\mathbf{e}_0+\right.$$

$$\left.+\left(\mu\left(1+\frac{\lambda_1\lambda_2}{c_1^2}\right)-t\left(\lambda_2+\lambda_1\right)\right)\mathbf{n}\right)=$$

$$= \frac{1}{\widetilde{\gamma}}\left((\alpha t+\beta\mu)\,\mathbf{e}_0+(\gamma t+\delta\mu)\,\mathbf{n}\right), \quad \text{where} \tag{23}$$

$$\left.\begin{array}{ll} \widetilde{\gamma}=\sqrt{\left(1-\frac{\lambda_1^2}{c_1^2}\right)\left(1-\frac{\lambda_2^2}{c_2^2}\right)}, & \\ \alpha=1+\dfrac{\lambda_1\lambda_2}{c_2^2}, & \beta=-\left(\dfrac{\lambda_2}{c_2^2}+\dfrac{\lambda_1}{c_1^2}\right), \\ \gamma=-\left(\lambda_2+\lambda_1\right), & \delta=1+\dfrac{\lambda_1\lambda_2}{c_1^2}, \end{array}\right\} \tag{24}$$

$$0<\lambda_i<c_i \ \left(i\in\overline{1,2}\right), \alpha,\delta,\widetilde{\gamma}>0. \tag{25}$$

Since $\mathbf{L}_{(\lambda_1,\lambda_2)}^{(c_1,c_2)} \in \mathcal{Q}$ then, according to inclusions (18), (19), we get $\mathbf{L}_{(\lambda_1,\lambda_2)}^{(c_1,c_2)} \in \mathfrak{P}(\mathfrak{H},[\vartheta])$. So, according to (16), (14) the elements $c\in(0,\infty)$, $\lambda\in[0,c)$, $s\in\{-1,1\}$, $\mathbf{n}\in\mathbf{B}_1(\mathfrak{H}_1)$, $J\in\mathfrak{U}(\mathfrak{H}_1)$ and $\mathbf{a}\in\mathcal{M}(\mathfrak{H})$ must exist such that $\mathbf{L}_{(\lambda_1,\lambda_2)}^{(c_1,c_2)}=\mathbf{W}_{\lambda,c}[s,\mathbf{n},J;\mathbf{a}]$. But by (8), $\mathbf{L}_{(\lambda_1,\lambda_2)}^{(c_1,c_2)}\mathbf{0}=\mathbf{W}_{\lambda_2,c_2}[1,\mathbf{n},\mathbb{I}_{-1,1}[\mathbf{n}]]\,\mathbf{W}_{\lambda_1,c_1}[1,\mathbf{n},\mathbb{I}_{-1,1}[\mathbf{n}]]\,\mathbf{0}=\mathbf{0}$. Hence $\mathbf{a}=\mathbf{0}$ and we have $\mathbf{L}_{(\lambda_1,\lambda_2)}^{(c_1,c_2)}=\mathbf{W}_{\lambda,c}[s,\mathbf{n},J;\mathbf{0}]=\mathbf{W}_{\lambda,c}[s,\mathbf{n},J]\in\mathfrak{O}(\mathfrak{H},c)$. So, according to Remark 1, for the operator $\mathbf{L}_{(\lambda_1,\lambda_2)}^{(c_1,c_2)}$ the condition (12) must be fulfilled. Hence, taking into account (11), we deliver:

$$\mathsf{M}_c\left(\mathbf{L}_{(\lambda_1,\lambda_2)}^{(c_1,c_2)}\mathbf{n}_{t,\mu}\right)=\mathsf{M}_c\left(\mathbf{n}_{t,\mu}\right)=\mu^2-c^2t^2 \qquad (\forall t,\mu\in\mathbb{R}). \tag{26}$$

From the other hand, using (23) and (11), we obtain:

$$\mathsf{M}_c\left(\mathbf{L}_{(\lambda_1,\lambda_2)}^{(c_1,c_2)}\mathbf{n}_{t,\mu}\right)=\frac{1}{\widetilde{\gamma}^2}\left((\gamma t+\delta\mu)^2-c^2(\alpha t+\beta\mu)^2\right) \qquad (\forall t,\mu\in\mathbb{R}). \tag{27}$$

From equalities (26), (27) we deduce the equality:

$$\frac{1}{\widetilde{\gamma}^2}\left((\gamma t+\delta\mu)^2-c^2(\alpha t+\beta\mu)^2\right)=\mu^2-c^2t^2 \quad (\forall t,\mu\in\mathbb{R}). \tag{28}$$

Substituting the values $t:=1$, $\mu:=c$ and $t:=1$, $\mu:=-c$ as well as $t:=1$, $\mu:=0$ into (28) we conclude that the following equalities must hold:

$$\left\{\begin{array}{lcl} (\gamma+c\delta)^2 & = & c^2(\alpha+c\beta)^2; \\ (\gamma-c\delta)^2 & = & c^2(\alpha-c\beta)^2; \\ \gamma^2-c^2\alpha^2 & = & -c^2\,\widetilde{\gamma}^2. \end{array}\right. \tag{29}$$

System (29) may be performed only in the following four cases, which we consider below.

$$\textbf{Case 1}: \quad \begin{cases} \gamma + c\delta & = -c\left(\alpha + c\beta\right); \\ \gamma - c\delta & = c\left(\alpha - c\beta\right); \\ \gamma^2 - c^2\alpha^2 & = -c^2\,\widetilde{\gamma}^2. \end{cases} \tag{30}$$

In this case from the first two equalities of (30) we obtain the equality $2c\delta = -2c\alpha$, that is $\delta = -\alpha$, which contradicts to (25), because, according to (25), the both numbers δ and α must be positive.

$$\textbf{Case 2}: \quad \begin{cases} \gamma + c\delta & = -c\left(\alpha + c\beta\right); \\ \gamma - c\delta & = -c\left(\alpha - c\beta\right); \\ \gamma^2 - c^2\alpha^2 & = -c^2\,\widetilde{\gamma}^2. \end{cases} \tag{31}$$

In this case from the first two equalities of (31) we obtain the equality $\gamma = -c\alpha$. And substituting this value of γ into the third equality of (31) we obtain the equality $\widetilde{\gamma} = 0$, which contradicts to (25), because, according to (25), the number $\widetilde{\gamma}$ must be positive.

$$\textbf{Case 3}: \quad \begin{cases} \gamma + c\delta & = c\left(\alpha + c\beta\right); \\ \gamma - c\delta & = c\left(\alpha - c\beta\right); \\ \gamma^2 - c^2\alpha^2 & = -c^2\,\widetilde{\gamma}^2. \end{cases} \tag{32}$$

In this case from the first two equalities of (32) we obtain the equality $\gamma = c\alpha$. And substituting this value of γ into the third equality of (32) we obtain the equality $\widetilde{\gamma} = 0$, which contradicts to (25), because, according to (25), this number must be positive.

$$\textbf{Case 4}: \quad \begin{cases} \gamma + c\delta & = c\left(\alpha + c\beta\right); \\ \gamma - c\delta & = -c\left(\alpha - c\beta\right); \\ \gamma^2 - c^2\alpha^2 & = -c^2\,\widetilde{\gamma}^2. \end{cases} \tag{33}$$

In this case from the first two equalities of (33) we obtain the equality $\delta = \alpha$, where, according to (24), $\alpha = 1 + \frac{\lambda_1\lambda_2}{c_2^2}$, $\delta = 1 + \frac{\lambda_1\lambda_2}{c_1^2}$. So, taking into account (25), we obtain, $c_1 = c_2$.

Therefore, Case 4 is the only possible and in this case we have, $c_1 = c_2$. Hence the equality $c_1 = c_2$ must hold. And, taking into account (22) we deduce $\vartheta\left(\lambda_1\right) = \vartheta\left(\lambda_2\right)$ $(\forall \lambda_1\lambda_2 \in \mathfrak{D}_*\left[\vartheta\right] \setminus \{0\})$. So the correlation (21) is performed. And, in accordance with (21) the number $c \in (0, \infty)$ exists such, that:

$$\vartheta(\lambda) = c \quad \left(\forall \lambda \in \mathfrak{D}_*\left[\vartheta\right] \setminus \{0\}\right). \tag{34}$$

The next aim is to prove that: $\mathfrak{D}_*\left[\vartheta\right] = [0, c)$. Using correlations (4) and (34) for each $\lambda \in \mathfrak{D}_*\left[\vartheta\right] \setminus \{0\}$ we obtain, $0 \leq \lambda < \vartheta\left(\lambda\right) = c$. So, we have the inclusion:

$$\mathfrak{D}_*\left[\vartheta\right] \subseteq [0, c). \tag{35}$$

So it remains to prove the inverse inclusion. Chose any fixed $\mathbf{n} \in \mathbf{B}_1\left(\mathfrak{H}_1\right)$. Applying the formula (23) for the case $c_1 = c_2 = c$ and $\lambda_1 = \lambda_2 = \lambda \in \mathfrak{D}_*\left[\vartheta\right] \subseteq [0, c)$ we obtain:

$$\begin{aligned} &\mathbf{W}_{\lambda,c}\left[1, \mathbf{n}, \mathbb{I}_{-1,1}\left[\mathbf{n}\right]\right]^2 \mathbf{n}_{t,\mu} = \mathbf{L}^{(c,c)}_{(\lambda,\lambda)} \mathbf{n}_{t,\mu} = \\ &\quad = \frac{1}{\sqrt{\left(1 - \frac{\lambda^2}{c^2}\right)^2}} \left(\left(t\left(1 + \frac{\lambda^2}{c^2}\right) - \mu\left(\frac{2\lambda}{c^2}\right)\right) \mathbf{e}_0 + \left(\mu\left(1 + \frac{\lambda^2}{c^2}\right) - t\cdot 2\lambda\right) \mathbf{n}\right) = \\ &\qquad = \frac{1}{\sqrt{1 - \frac{\xi(\lambda)^2}{c^2}}} \left(\left(t - \mu\,\frac{\xi\left(\lambda\right)}{c^2}\right) \mathbf{e}_0 + \left(\mu - t\,\xi\left(\lambda\right)\right)\mathbf{n}\right) \quad \left(\forall t, \mu \in \mathbb{R}\right), \end{aligned} \tag{36}$$

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$$\text{where} \quad \xi(x) = \frac{2x}{1+\frac{x^2}{c^2}} \quad (x \in \mathbb{R}).$$

Taking into account (34) and (15), we see that $\mathbf{W}_{\lambda,c}[1,\mathbf{n},\mathbb{I}_{-1,1}[\mathbf{n}]] = \mathbf{W}_{\lambda,\vartheta(\lambda)}[1,\mathbf{n},\mathbb{I}_{-1,1}[\mathbf{n}]] \in \mathfrak{O}_+(\mathfrak{H},[\vartheta])$. So, according to inclusions (18), (19), we have: $\mathbf{W}_{\lambda,c}[1,\mathbf{n},\mathbb{I}_{-1,1}[\mathbf{n}]] \in \mathcal{Q}$. Since, according to the above assumption, the class $\mathcal{Q}$ is a group, the operator $\mathbf{W}_{\lambda,c}[1,\mathbf{n},\mathbb{I}_{-1,1}[\mathbf{n}]]^2$ also must belong to $\mathcal{Q}$, and, according to inclusions (18), (19), we get $\mathbf{W}_{\lambda,c}[1,\mathbf{n},\mathbb{I}_{-1,1}[\mathbf{n}]]^2 \in \mathfrak{P}(\mathfrak{H},[\vartheta])$. So, according to (16), (14) the elements $\lambda_* \in \mathfrak{D}_*[\vartheta]$, $s \in \{-1,1\}$, $\mathbf{n}_* \in \mathbf{B}_1(\mathfrak{H}_1)$, $J \in \mathfrak{U}(\mathfrak{H}_1)$ and $\mathbf{a} \in \mathcal{M}(\mathfrak{H})$ must exist such that $\mathbf{W}_{\lambda,c}[1,\mathbf{n},\mathbb{I}_{-1,1}[\mathbf{n}]]^2 = \mathbf{W}_{\lambda_*,\vartheta(\lambda_*)}[s,\mathbf{n}_*,J;\mathbf{a}]$. But by (8), $\mathbf{W}_{\lambda,c}[1,\mathbf{n},\mathbb{I}_{-1,1}[\mathbf{n}]]^2\mathbf{0} = \mathbf{0}$. Hence $\mathbf{W}_{\lambda_*,\vartheta(\lambda_*)}[s,\mathbf{n}_*,J]\,\mathbf{a} = \mathbf{W}_{\lambda_*,\vartheta(\lambda_*)}[s,\mathbf{n}_*,J;\mathbf{a}]\,\mathbf{0} = \mathbf{0}$. And, since, according to Remark 1, $\mathbf{W}_{\lambda_*,\vartheta(\lambda_*)}[s,\mathbf{n}_*,J]$ is a bijective linear operator, we have, $\mathbf{a} = \mathbf{0}$. Thus, $\mathbf{W}_{\lambda,c}[1,\mathbf{n},\mathbb{I}_{-1,1}[\mathbf{n}]]^2 = \mathbf{W}_{\lambda_*,\vartheta(\lambda_*)}[s,\mathbf{n}_*,J]$. And, taking into account (34), we get $\mathbf{W}_{\lambda,c}[1,\mathbf{n},\mathbb{I}_{-1,1}[\mathbf{n}]]^2 = \mathbf{W}_{\lambda_*,c}[s,\mathbf{n}_*,J]$. Thence, using the formula (13), we deduce:

$$\mathcal{V}\left(\mathbf{W}_{\lambda,c}[1,\mathbf{n},\mathbb{I}_{-1,1}[\mathbf{n}]]^2\right) = \lambda_* s\mathbf{n}_*. \tag{37}$$

From the other hand, applying the equality (36) for the vector $\mathbf{n}_{t_0^*,\mu_0^*}$, where $t_0^* = \frac{1}{\sqrt{1-\frac{\xi(\lambda)^2}{c^2}}}$, $\mu_0^* = \frac{\xi(\lambda)}{\sqrt{1-\frac{\xi(\lambda)^2}{c^2}}}$, we obtain:

$$\mathbf{W}_{\lambda,c}[1,\mathbf{n},\mathbb{I}_{-1,1}[\mathbf{n}]]^2\,\mathbf{n}_{t_0^*,\mu_0^*} = \\ = \frac{1}{\sqrt{1-\frac{\xi(\lambda)^2}{c^2}}}\left(\left(t_0^* - \mu_0^*\frac{\xi(\lambda)}{c^2}\right)\mathbf{e}_0 + \left(\mu_0^* - t_0^*\xi(\lambda)\right)\mathbf{n}\right) = \mathbf{e}_0.$$

So, by Definition 1, we deduce:

$$\mathcal{V}\left(\mathbf{W}_{\lambda,c}[1,\mathbf{n},\mathbb{I}_{-1,1}[\mathbf{n}]]^2\right) = \frac{\mathbf{X}\left(\mathbf{W}_{\lambda,c}[1,\mathbf{n},\mathbb{I}_{-1,1}[\mathbf{n}]]^2\right)^{-1}\mathbf{e}_0}{\mathcal{T}\left(\left(\mathbf{W}_{\lambda,c}[1,\mathbf{n},\mathbb{I}_{-1,1}[\mathbf{n}]]^2\right)^{-1}\mathbf{e}_0\right)} = \\ = \frac{\mathbf{X}\mathbf{n}_{t_0^*,\mu_0^*}}{\mathcal{T}\left(\mathbf{n}_{t_0^*,\mu_0^*}\right)} = \frac{\mu_0^*\mathbf{n}}{t_0^*} = \xi(\lambda)\,\mathbf{n}. \tag{38}$$

From equalities (37) and (38) it follows that $\lambda_* s\mathbf{n}_* = \xi(\lambda)\,\mathbf{n}$. And taking into account that $\lambda, \lambda_* \geq 0$, $\xi(\lambda) = \frac{2\lambda}{1+\frac{\lambda^2}{c^2}} \geq 0$, $|s| = 1$, $\|\mathbf{n}\| = \|\mathbf{n}_*\| = 1$, we obtain the equality $\xi(\lambda) = \lambda_*$. So, since $\lambda_* \in \mathfrak{D}_*[\vartheta]$, we have, $\xi(\lambda) \in \mathfrak{D}_*[\vartheta]$. Thus, we have proven the following statement:

$$\forall \lambda \in \mathfrak{D}_*[\vartheta] \quad (\xi(\lambda) \in \mathfrak{D}_*[\vartheta]). \tag{39}$$

The function $\xi(x)$ obeys the following three properties:

1^0. *$\xi(x)$ is continuous and increasing on $[0,c]$.*

Indeed, it is apparently that $\xi(x)$ is differentiable and thus continuous on $[0,c]$. Moreover, simple calculation shows that $\frac{d}{dx}\xi(x) = 2\frac{1-\frac{x^2}{c^2}}{\left(1+\frac{x^2}{c^2}\right)^2} > 0$ for $x \in [0,c)$.

2^0. *$\xi(\lambda) > \lambda$ for $\lambda \in (0,c)$.*

Indeed, for $\lambda \in (0,c)$ we have, $\xi(\lambda) = \frac{2\lambda}{1+\frac{\lambda^2}{c^2}} > \frac{2\lambda}{1+\frac{c^2}{c^2}} = \lambda$.

3^0. *$\xi(\lambda) \in [0,c)$ for $\lambda \in [0,c)$.*

Indeed, using Property 1^0, for $\lambda \in [0,c)$ we obtain, $0 \leq \xi(\lambda) < \xi(c) = \frac{2c}{1+\frac{c^2}{c^2}} = c$.

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According to (5), the number $\eta_0 \in (0,c)$ exists such that:

$$[0,\eta_0) \subseteq \mathfrak{D}_*[\vartheta] \tag{40}$$

Taking into account that $\xi(0)=0$ and using Property 1^0 as well as statement (39), from inclusion (40) we deduce, that $[0,\eta_1) \subseteq \mathfrak{D}_*[\vartheta]$, where $\eta_1 = \xi(\eta_0)$. Thus, recursively we obtain the inclusions:

$$\begin{gathered}[0,\eta_n) \subseteq \mathfrak{D}_*[\vartheta] \qquad (\forall n \in \mathbb{N}), \\ \text{where} \quad \eta_{k+1} = \xi(\eta_k) \quad (k \in \mathbb{N}).\end{gathered} \tag{41}$$

From Property 2^0 it follows that the sequence $(\eta_n)_{n=0}^{\infty}$ is increasing. From Property 3^0, taking into account that $\eta_0 \in (0,c)$, we obtain:

$$\eta_n \in (0,c) \quad (n \in \mathbb{N}). \tag{42}$$

Thus, the sequence $(\eta_n)_{n=0}^{\infty}$ is monotonous and bounded. And, according to Weierstrass theorem, there exists the number $\eta_\infty \in \mathbb{R}$ such, that $\eta_\infty = \lim\limits_{n\to\infty} \eta_n$. Since the sequence $(\eta_n)_{n=0}^{\infty}$ is increasing, from the correlation (42) we obtain the inequality:

$$0 < \eta_\infty \le c \tag{43}$$

and from the correlation (41) we obtain the inclusion:

$$[0,\eta_\infty) = \bigcup_{n=1}^{\infty} [0,\eta_n) \subseteq \mathfrak{D}_*[\vartheta]. \tag{44}$$

Since $\eta_{k+1} = \xi(\eta_k)$ $(\forall k \in \mathbb{N})$, we obtain the following equality $\eta_\infty = \xi(\eta_\infty)$. So, according to inequality (43), the number η_∞ is the positive solution of the equation $x = \frac{2x}{1+\frac{x^2}{c^2}}$. Simple calculation shows that the last equation has only one positive solution $x = c$. Therefore, $\eta_\infty = c$. And taking into account the inclusion (44), we deliver, $[0,c) \subseteq \mathfrak{D}_*[\vartheta]$. The last inclusion together with the inclusion (35) proves the equality

$$\mathfrak{D}_*[\vartheta] = [0,c), \tag{45}$$

which had to be proven.

Using (45), for $\lambda \in (0,c) = \mathfrak{D}_*[\vartheta] \setminus \{0\}$ we obtain, $\vartheta(\lambda) = c$ (by (34)). For $\lambda \in [c,\infty)$, according to (45), we have, $\lambda \notin \mathfrak{D}_*[\vartheta]$. Therefore in this case by (4) and (3), we obtain $\vartheta(\lambda) = \lambda$. Thus, by (6), we obtain $\vartheta(\lambda) = \vartheta_c(\lambda)$ $(\forall\, \lambda \in (0,\infty))$, which had to be proven.

Conversely, if $\vartheta(\lambda) = \vartheta_c(\lambda)$ $(\forall\, \lambda \in (0,\infty))$ then, according to equalities (20) and Remark 1, the class $\mathcal{Q} \in \{\mathfrak{O}(\mathfrak{H},[\vartheta]),\ \mathfrak{O}_+(\mathfrak{H},[\vartheta]),\ \mathfrak{P}(\mathfrak{H},[\vartheta]),\ \mathfrak{P}_+(\mathfrak{H},[\vartheta])\}$ is a group of operators in the space $\mathcal{M}(\mathfrak{H})$. □

Remark 4. Thus, Theorem 1 asserts that in the case of non-constant velocity of light (where there do not exist the number $c \in (0,\infty)$ such that $\vartheta(\lambda) = \vartheta_c(\lambda)$ $(\forall\, \lambda \in (0,\infty))$), the each of the classes of operators $\mathfrak{O}(\mathfrak{H},[\vartheta])$, $\mathfrak{O}_+(\mathfrak{H},[\vartheta])$, $\mathfrak{P}(\mathfrak{H},[\vartheta])$, $\mathfrak{P}_+(\mathfrak{H},[\vartheta])$ is not a group of operators over the space $\mathcal{M}(\mathfrak{H})$, because the composition of two operators from the class $\mathcal{Q} \in \{\mathfrak{O}(\mathfrak{H},[\vartheta]),\ \mathfrak{O}_+(\mathfrak{H},[\vartheta]),\ \mathfrak{P}(\mathfrak{H},[\vartheta]),\ \mathfrak{P}_+(\mathfrak{H},[\vartheta])\}$ does not belong to $\mathcal{Q}$ in the general case.

From the other hand these classes of operators have the following properties.

Properties 1. *Let* $\vartheta \in \Upsilon$ *and* $\mathcal{Q} \in \{\mathfrak{O}(\mathfrak{H},[\vartheta]),\ \mathfrak{O}_+(\mathfrak{H},[\vartheta]),\ \mathfrak{P}(\mathfrak{H},[\vartheta]),\ \mathfrak{P}_+(\mathfrak{H},[\vartheta])\}$. *Then:*

1. $\mathbb{I} \in \mathcal{Q}$, *where* $\mathbb{I} = \mathbb{I}_{\mathcal{M}(\mathfrak{H})}$ *is the identity operator over the space* $\mathcal{M}(\mathfrak{H})$.
2. *If* $U \in \mathcal{Q}$ *then* $U^{-1} \in \mathcal{Q}$.

Proof. **1**. Chose any vector $\mathbf{n} \in \mathbf{B}_1(\mathfrak{H}_1)$. Using formula (8) we readily obtain the equality:

$$\mathbf{W}_{0,\vartheta(0)}[1, \mathbf{n}, \mathbb{I}_{-1,1}[\mathbf{n}]] = \mathbb{I}.$$

So, according to inclusions, we have (18), (19) $\mathbb{I} \in \mathfrak{O}_+(\mathfrak{H}, [\vartheta]) \subseteq \mathcal{Q} \in \{\mathfrak{O}(\mathfrak{H}, [\vartheta]), \mathfrak{O}_+(\mathfrak{H}, [\vartheta]), \mathfrak{P}(\mathfrak{H}, [\vartheta]), \mathfrak{P}_+(\mathfrak{H}, [\vartheta])\}$.

2. Using [16, Corollary 4] for any $s \in \{-1, 1\}$, $\lambda \in \mathfrak{D}_*[\vartheta]$, $\mathbf{n} \in \mathbf{B}_1(\mathfrak{H}_1)$, $J \in \mathfrak{U}(\mathfrak{H}_1)$ and $\mathbf{a} \in \mathcal{M}(\mathfrak{H})$ we obtain:

$$\left(\mathbf{W}_{\lambda,\vartheta(\lambda)}[s, \mathbf{n}, J; \mathbf{a}]\right)^{-1} = \mathbf{W}_{\lambda,\vartheta(\lambda)}\left[s,\ J\mathbf{n},\ J^{-1}; \widetilde{\mathbf{a}}\right],$$

where $\widetilde{\mathbf{a}} = -\left(\mathbf{W}_{\lambda,\vartheta(\lambda)}[s,\ J\mathbf{n}, J^{-1}]\right)^{-1}\mathbf{a} \in \mathcal{M}(\mathfrak{H})$ (so if $\mathbf{a} = \mathbf{0}$ then $\widetilde{\mathbf{a}} = \mathbf{0}$). The last formula shows that the operation of taking the inverse operator does not take us beyond the class $\mathcal{Q} \in \{\mathfrak{O}(\mathfrak{H}, [\vartheta]), \mathfrak{O}_+(\mathfrak{H}, [\vartheta]), \mathfrak{P}(\mathfrak{H}, [\vartheta]), \mathfrak{P}_+(\mathfrak{H}, [\vartheta])\}$. □

4 Generalized Hassani Kinematics and Relativity Principle

In this section we are going to to construct universal kinematics, based on generalized Hassani transforms and demonstrate that this kinematics does not satisfy of the relativity principle in the general case. Further we use the system of notations and definitions from the theory of changeable sets and universal kinematics [14, 17–20, etc] (the most complete and detailed explanation of these theories can be found in [12]).

Let $(\mathfrak{H}, \|\cdot\|, \langle\cdot,\cdot\rangle)$ be a real Hilbert space with $\dim(\mathfrak{H}) \geq 1$, $\mathcal{B}$ be any base changeable set such, that $\mathfrak{Bs}(\mathcal{B}) \subseteq \mathfrak{H}$ and $\mathbb{T}\mathbf{m}(\mathcal{B}) = (\mathbb{R}, \leq)$, where $\leq$ is the standard order in the field of real numbers $\mathbb{R}$ and ϑ be a function from the class Υ, (see (3)). Applying the results of [12, 20], to the classes of operators $\mathfrak{P}(\mathfrak{H}, [\vartheta])$ and $\mathfrak{P}_+(\mathfrak{H}, [\vartheta])$ we can introduce the following universal kinematics:

$$\mathfrak{U}\mathrm{H}_0(\mathfrak{H}, \mathcal{B}, \vartheta) := \mathfrak{Ku}(\mathfrak{P}(\mathfrak{H}, [\vartheta]), \mathcal{B}; \mathfrak{H});$$
$$\mathfrak{U}\mathrm{H}(\mathfrak{H}, \mathcal{B}, \vartheta) := \mathfrak{Ku}(\mathfrak{P}_+(\mathfrak{H}, [\vartheta]), \mathcal{B}; \mathfrak{H}),$$

where the notation $\mathfrak{Ku}(\cdot, \cdot; \cdot)$ was introduced in [20, pg. 112], [12, pg. 166]. In Theorem 1 it was proven, that the classes of operators $\mathfrak{P}(\mathfrak{H}, [\vartheta])$ and $\mathfrak{P}_+(\mathfrak{H}, [\vartheta])$ do not form a group over $\mathcal{M}(\mathfrak{H})$. This means, that the kinematics $\mathfrak{U}\mathrm{H}_0(\mathfrak{H}, \mathcal{B}, \vartheta)$ and $\mathfrak{U}\mathrm{H}(\mathfrak{H}, \mathcal{B}, \vartheta)$, constructed on the basis of these classes, do not satisfy the relativity principle. Indeed, according to [12, Property 3.23.1(1)] for universal kinematics $\mathcal{F} \in \{\mathfrak{U}\mathrm{H}_0(\mathfrak{H}, \mathcal{B}, \vartheta), \mathfrak{U}\mathrm{H}(\mathfrak{H}, \mathcal{B}, \vartheta)\}$ any reference frame $\mathfrak{l} \in \mathcal{L}k(\mathcal{F})$ can be represented in the form $\mathfrak{l} = (U, U[\mathcal{B}])$, where

$$U \in \mathcal{Q} = \begin{cases} \mathfrak{P}(\mathfrak{H}, [\vartheta]), & \mathcal{F} = \mathfrak{U}\mathrm{H}_0(\mathfrak{H}, \mathcal{B}, \vartheta) \\ \mathfrak{P}_+(\mathfrak{H}, [\vartheta]), & \mathcal{F} = \mathfrak{U}\mathrm{H}(\mathfrak{H}, \mathcal{B}, \vartheta). \end{cases}$$

So, according to [12, Property 3.23.1(7)], the subset of universal coordinate transforms:

$$\mathbb{UP}(\mathfrak{l}) = \{[\mathfrak{m} \leftarrow \mathfrak{l}, \mathcal{F}] \mid \mathfrak{m} \in \mathcal{L}k(\mathcal{F})\} = \left\{VU^{-1} \mid V \in \mathcal{Q}\right\},$$

providing transition from some reference frame $\mathfrak{l} = (U, U[\mathcal{B}]) \in \mathcal{L}k(\mathcal{F})$ to all other frames $\mathfrak{m} \in \mathcal{L}k(\mathcal{F})$, is different for different frames $\mathfrak{l}$. Moreover, assume that there does not exist the number $c \in (0, \infty)$ such that $\vartheta(\lambda) = \vartheta_c(\lambda)$ $(\forall \lambda \in (0, \infty))$. Then, taking into account Remark 4 and Properties 1, we can prove that the set $\mathbb{UP}(\mathfrak{l})$ coincides with the starting class of transforms $\mathcal{Q}$ only for some (but not all) reference frames, for example for the frame $\mathfrak{l}_{0,\mathcal{B}} = (\mathbb{I}, \mathbb{I}[\mathcal{B}]) = (\mathbb{I}, \mathcal{B}) \in \mathcal{L}k(\mathcal{F})$.

But, the principle of relativity is only one of the experimentally established facts, which must not be satisfied when we exit out of the light barrier or may be satisfied only approximatively with the great accuracy even in subluminal case. In this regard it is useful to consider the kinematics $\mathfrak{U}\mathrm{H}(\mathfrak{H}, \mathcal{B}, \vartheta)$ in the case, where the continuous function ϑ satisfies the following conditions:

1) $\vartheta(\lambda) > \lambda$ $(\forall \lambda \in [0, \infty))$;

2) $c - \varepsilon_1 < \vartheta(\lambda) < c$ for $0 \leq \lambda < c - \varepsilon$, where c is the speed of light in vacuum and $\varepsilon, \varepsilon_1 \in (0, c)$ are some small positive numbers.

In this case we obtain the kinematics, which may be arbitrarily close to classical special relativity one and in which the hard-transversal light barrier may be overcome by means of the continuous change of the velocity of particle. And in this kinematics the principle of relativity is satisfied only approximatively. The possibility of violation the relativity principle is discussed in the physical literature (see for example [21–28]).

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