 doi:10.20944/preprints202102.0327.v1

On the Recurrence Properties of Generalized Tribonacci Sequence

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Abstract. In this paper, we investigate the recurrence properties of the generalized Tribonacci sequence and present how the generalized Tribonacci sequence at negative indices can be expressed by the sequence itself at positive indices.

2020 Mathematics Subject Classification. 11B37, 11B39, 11B83.

Keywords. Tribonacci numbers, negatif indices, recurrence relations.

1. Introduction

In 2021, Lin [1] stated on page 4 of 12 that "Recently, Professor Tianxin Cai visited Northwest University and gave a talk about a series of linear recurrence sequences and their properties, which incited our interest in this field. There are many recursive identities concerning the Fibonacci, Tribonacci, and Lucas sequences, etc. However, few studies have been conducted regarding the Narayana sequence. Professor Cai proposed an open problem: Whether and how can the Narayana sequence at negative indices be expressed by the sequence itself at positive indices?" Then Lin presented the following theorem as a main result and proved it:

Theorem 1. *For $n \in \mathbb{Z}$, we have*

$$N_{-n} = 2N_n^2 + N_{2n} - 3N_{n+1}N_n.$$

Here, Narayana's cows sequence $\{N_n\}$ satisfies a third-order recurrence relation:

$$N_n = N_{n-1} + N_{n-3}, \text{ for } n \geq 3$$

with the initial values $N_0 = 0, N_2 = 1, N_2 = 1$. It can be extended to negative indices by defining

$$N_{-n} = -N_{-(n-2)} + N_{-(n-3)}, \text{ for } n = 1, 2, 3, ...$$

Lin also stated on the same page that "Theorem 1 solves Professor Cai's problem completely. It illustrates the connection between the Narayana sequence at the positive index and the negative index. By Theorem

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1, we obtain the recurrence property of the sequence at the negative index, which deepens our knowledge of the nature of the sequence."

Now, we can propose an open problem as follows: Whether and how can the generalized Tribonacci sequence W_n at negative indices be expressed by the sequence itself at positive indices?

We present our main result as follows which completely solves the above problem for the generalized Tribonacci sequence W_n.

THEOREM 2. *For $n \in \mathbb{Z}$, we have*

$$W_{-n} = t^{-n}(W_{2n} - H_n W_n + \frac{1}{2}(H_n^2 - H_{2n})W_0).$$

Note that H_n can be written in terms of W_n using Lemma 4 below. Next, we recall the definitions of generalized Tribonacci sequence W_n and its two special cases, namely (r,s,t) sequence G_n and (r,s,t) Lucas sequence H_n. The generalized (r,s,t) sequence (or generalized Tribonacci sequence or generalized 3-step Fibonacci sequence) $\{W_n(W_0,W_1,W_2;r,s,t)\}_{n\geq 0}$ (or shortly $\{W_n\}_{n\geq 0}$) is defined as follows:

$$W_n = rW_{n-1} + sW_{n-2} + tW_{n-3}, \qquad W_0 = a, W_1 = b, W_2 = c, \quad n \geq 3 \tag{1.1}$$

where W_0, W_1, W_2 are arbitrary complex (or real) numbers and r,s,t are real numbers. This sequence has been studied by many authors, see for example [3] and references therein. The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = -\frac{s}{t}W_{-(n-1)} - \frac{r}{t}W_{-(n-2)} + \frac{1}{t}W_{-(n-3)}$$

for $n = 1, 2, 3, ...$ when $t \neq 0$. Therefore, recurrence (1.1) holds for all integer n.

In the following Table 1 we present some special cases of generalized Tribonacci sequence.

Table 1 A few special case of generalized Tribonacci sequences.

No	Sequences (Numbers)	Notation
1	Generalized Tribonacci	$\{V_n\} = \{W_n(W_0,W_1,W_2;1,1,1)\}$
2	Generalized Third Order Pell	$\{V_n\} = \{W_n(W_0,W_1,W_2;2,1,1)\}$
3	Generalized Padovan	$\{V_n\} = \{W_n(W_0,W_1,W_2;0,1,1)\}$
4	Generalized Pell-Padovan	$\{V_n\} = \{W_n(W_0,W_1,W_2;0,2,1)\}$
5	Generalized Jacobsthal-Padovan	$\{V_n\} = \{W_n(W_0,W_1,W_2;0,1,2)\}$
6	Generalized Narayana	$\{V_n\} = \{W_n(W_0,W_1,W_2;1,0,1)\}$
7	Generalized Third Order Jacobsthal	$\{V_n\} = \{W_n(W_0,W_1,W_2;1,1,2)\}$
8	Generalized 3-primes	$\{V_n\} = \{W_n(W_0,W_1,W_2;2,3,5)\}$
9	Generalized Reverse 3-primes	$\{V_n\} = \{W_n(W_0,W_1,W_2;5,3,2)\}$

In literature, for example, the following names and notations (see Table 2) are used for the special case of r, s, t and initial values.

Table 2 A few special case of generalized (r,s,t) (generalized Tribonacci) sequence

No	Sequences (Numbers)	Notation	OEIS [2]
1	Tribonacci	$\{T_n\} = \{W_n(0,1,1;1,1,1)\}$	A000073, A057597
2	Tribonacci-Lucas	$\{K_n\} = \{W_n(3,1,3;1,1,1)\}$	A001644, A073145
3	Tribonacci-Perrin	$\{M_n\} = \{W_n(3,0,2;1,1,1)\}$	
4	modified Tribonacci	$\{U_n\} = \{W_n(1,1,1;1,1,1)\}$	
5	modified Tribonacci-Lucas	$\{G_n\} = \{W_n(4,4,10;1,1,1)\}$	
6	adjusted Tribonacci-Lucas	$\{H_n\} = \{W_n(4,2,0;1,1,1)\}$	
7	third order Pell	$\{P_n^{(3)}\} = \{W_n(0,1,2;2,1,1)\}$	A077939, A077978
8	third order Pell-Lucas	$\{Q_n^{(3)}\} = \{W_n(3,2,6;2,1,1)\}$	A276225, A276228
9	third order modified Pell	$\{E_n^{(3)}\} = \{W_n(0,1,1;2,1,1)\}$	A077997, A078049
10	third order Pell-Perrin	$\{R_n^{(3)}\} = \{W_n(3,0,2;2,1,1)\}$	
11	Padovan (Cordonnier)	$\{P_n\} = \{W_n(1,1,1;0,1,1)\}$	A000931
12	Perrin (Padovan-Lucas)	$\{E_n\} = \{W_n(3,0,2;0,1,1)\}$	A001608, A078712
13	Padovan-Perrin	$\{S_n\} = \{W_n(0,0,1;0,1,1)\}$	A000931, A176971
14	modified Padovan	$\{A_n\} = \{W_n(3,1,3;0,1,1)\}$	
15	adjusted Padovan	$\{U_n\} = \{W_n(0,1,0;0,1,1)\}$	
16	Pell-Padovan	$\{R_n\} = \{W_n(1,1,1;0,2,1)\}$	A066983, A128587
17	Pell-Perrin	$\{C_n\} = \{W_n(3,0,2;0,2,1)\}$	
18	third order Fibonacci-Pell	$\{G_n\} = \{W_n(1,0,2;0,2,1)\}$	
19	third order Lucas-Pell	$\{B_n\} = \{W_n(3,0,4;0,2,1)\}$	
20	adjusted Pell-Padovan	$\{M_n\} = \{W_n(0,1,0;0,2,1)\}$	
21	Jacobsthal-Padovan	$\{Q_n\} = \{W_n(1,1,1;0,1,2)\}$	A159284
22	Jacobsthal-Perrin (-Lucas)	$\{L_n\} = \{W_n(3,0,2;0,1,2)\}$	A072328
23	adjusted Jacobsthal-Padovan	$\{K_n\} = \{W_n(0,1,0;0,1,2)\}$	
24	modified Jacobsthal-Padovan	$\{M_n\} = \{W_n(3,1,3;0,1,2)\}$	
25	Narayana	$\{N_n\} = \{W_n(0,1,1;1,0,1)\}$	A078012
26	Narayana-Lucas	$\{U_n\} = \{W_n(3,1,1;1,0,1)\}$	A001609
27	Narayana-Perrin	$\{H_n\} = \{W_n(3,0,2;1,0,1)\}$	
28	third order Jacobsthal	$\{J_n^{(3)}\} = \{W_n(0,1,1;1,1,2)\}$	A077947
29	third order Jacobsthal-Lucas	$\{j_n^{(3)}\} = \{W_n(2,1,5;1,1,2)\}$	A226308
30	modified third order Jacobsthal-Lucas	$\{K_n^{(3)}\} = \{W_n(3,1,3;1,1,2)\}$	
31	third order Jacobsthal-Perrin	$\{Q_n^{(3)}\} = \{W_n(3,0,2;1,1,2)\}$	
32	3-primes	$\{G_n\} = \{W_n(0,1,2;2,3,5)\}$	
33	Lucas 3-primes	$\{H_n\} = \{W_n(3,2,10;2,3,5)\}$	
34	modified 3-primes	$\{E_n\} = \{W_n(0,1,1;2,3,5)\}$	
35	reverse 3-primes	$\{N_n\} = \{W_n(0,1,5;5,3,2)\}$	
36	reverse Lucas 3-primes	$\{S_n\} = \{W_n(3,5,31;5,3,2)\}$	
37	reverse modified 3-primes	$\{U_n\} = \{W_n(0,1,4;5,3,2)\}$	

Here, OEIS stands for On-line Encyclopedia of Integer Sequences. For easy writing, from now on, we drop the superscripts from the sequences, for example we write J_n for $J_n^{(3)}$.

It is well known that the generalized (r,s,t) numbers (the generalized Tribonacci numbers) can be expressed, for all integers n, using Binet's formula

$$W_n = A_1\alpha^n + A_2\beta^n + A_3\gamma^n \tag{1.2}$$

where

$$A_1 = \frac{W_2 - (\beta+\gamma)W_1 + \beta\gamma W_0}{(\alpha-\beta)(\alpha-\gamma)}, A_2 = \frac{W_2 - (\alpha+\gamma)W_1 + \alpha\gamma W_0}{(\beta-\alpha)(\beta-\gamma)}, A_3 = \frac{W_2 - (\alpha+\beta)W_1 + \alpha\beta W_0}{(\gamma-\alpha)(\gamma-\beta)}.$$

and α, β, γ are the roots of characteristic equation of W_n which is given by

$$x^3 - rx^2 - sx - t = 0 \tag{1.3}$$

Note that we have the following identities

$$\begin{cases} \alpha+\beta+\gamma = r, \\ \alpha\beta+\alpha\gamma+\beta\gamma = -s, \\ \alpha\beta\gamma = t. \end{cases} \tag{1.4}$$

Note that the Binet form of a sequence satisfying (1.3) for non-negative integers is valid for all integers n. Now we define two special cases of the generalized (r, s, t) sequence $\{W_n\}$. (r, s, t) sequence $\{G_n\}_{n\geq 0}$ and Lucas (r, s, t) sequence $\{H_n\}_{n\geq 0}$ are defined, respectively, by the third-order recurrence relations

$$\begin{aligned} G_{n+3} &= rG_{n+2} + sG_{n+1} + tG_n, \qquad G_0 = 0, G_1 = 1, G_2 = r, \\ H_{n+3} &= rH_{n+2} + sH_{n+1} + tH_n, \qquad H_0 = 3, H_1 = r, H_2 = 2s + r^2, \end{aligned}$$

The sequences $\{G_n\}_{n\geq 0}$ and $\{H_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$\begin{aligned} G_{-n} &= -\frac{s}{t}G_{-(n-1)} - \frac{r}{t}G_{-(n-2)} + \frac{1}{t}G_{-(n-3)}, \\ H_{-n} &= -\frac{s}{t}H_{-(n-1)} - \frac{r}{t}H_{-(n-2)} + \frac{1}{t}H_{-(n-3)}, \end{aligned}$$

for $n = 1, 2, 3, ...$ respectively. Some special cases of (r, s, t) sequence $\{G_n(0, 1, r; r, s, t)\}_{n\geq 0}$ and Lucas (r, s, t) sequence $\{H_n(3, r, 2s + r^2; r, s, t)\}_{n\geq 0}$ are as follows:

(1) $G_n(0, 1, 1; 1, 1, 1) = T_n$, Tribonacci sequence,
(2) $H_n(3, 1, 3; 1, 1, 1) = K_n$, Tribonacci-Lucas sequence,
(3) $G_n(0, 1, 2; 2, 1, 1) = P_n$, third order Pell sequence,
(4) $H_n(3, 2, 6; 2, 1, 1) = Q_n$, third order Pell-Lucas sequence,
(5) $G_n(0, 1, 0; 0, 1, 1) = U_n$, adjusted Padovan sequence,
(6) $H_n(3, 0, 2; 0, 1, 1) = E_n$, Perrin (Padovan-Lucas) sequence,
(7) $G_n(0, 1, 0; 0, 2, 1) = M_n$, adjusted Pell-Padovan sequence
(8) $H_n(3, 0, 4; 0, 2, 1) = B_n$, third order Lucas-Pell sequence,
(9) $G_n(0, 1, 0; 0, 1, 2) = K_n$, adjusted Jacobsthal-Padovan sequence,
(10) $H_n(3, 0, 2; 0, 1, 2) = L_n$, Jacobsthal-Perrin (-Lucas) sequence,
(11) $G_n(0, 1, 1; 1, 0, 1) = N_n$, Narayana sequence,
(12) $H_n(3, 1, 1; 1, 0, 1) = U_n$, Narayana-Lucas sequence,
(13) $G_n(0, 1, 1; 1, 1, 2) = J_n$, third order Jacobsthal sequence,
(14) $H_n(3, 1, 3; 1, 1, 2) = j_n$, modified third order Jacobsthal-Lucas sequence,

(15) $G_n(0,1,2;2,3,5) = G_n$, 3-primes sequence,

(16) $H_n(3,2,10;2,3,5) = H_n$, Lucas 3-primes sequence.

(17) $G_n(0,1,5;5,3,2) = N_n$, reverse 3-primes sequence,

(18) $H_n(3,5,31;5,3,2) = S_n$, reverse Lucas 3-primes sequence.

For all integers n, (r,s,t) and Lucas (r,s,t) numbers can be expressed using Binet's formulas as

$$\begin{aligned} G_n &= \frac{\alpha^{n+1}}{(\alpha-\beta)(\alpha-\gamma)} + \frac{\beta^{n+1}}{(\beta-\alpha)(\beta-\gamma)} + \frac{\gamma^{n+1}}{(\gamma-\alpha)(\gamma-\beta)}, \\ H_n &= \alpha^n + \beta^n + \gamma^n, \end{aligned}$$

respectively.

2. The Proof of Theorem 2

To prove Theorem 2, we need following lemma.

LEMMA 3. *For $n \in \mathbb{Z}$, denote*

$$S_n = \alpha^n\beta^n + \alpha^n\gamma^n + \beta^n\gamma^n$$

where α,β and γ are as in defined in Formula (1.4). Then the followings hold:

- **(a):** *For $n \in \mathbb{Z}$, we have $S_n = t^n H_{-n}$ and $S_{-n} = t^{-n} H_n$.*
- **(b):** *S_n has the recurrence relation so that*

$$S_n = -sS_{n-1} - rtS_{n-2} + t^2 S_{n-3}$$

with the initial conditions $S_0 = 3$, $S_1 = -s$, $S_2 = s^2 - 2rt$. The sequence at negative indices is given by

$$S_{-n} = -\frac{-rt}{t^2} S_{-(n-1)} - \frac{-s}{t^2} S_{-(n-2)} + \frac{1}{t^2} S_{-(n-3)}, \text{ for } n = 1,2,3,....$$

- **(c):** *S_n has the identity so that*

$$S_n = \frac{1}{2}(H_n^2 - H_{2n}).$$

Proof.

(a): From the definition of S_n and H_n, we obtain

$$t^n H_{-n} = \alpha^{-n}t^n + \beta^{-n}t^n + \gamma^{-n}t^n = \alpha^n\beta^n + \alpha^n\gamma^n + \beta^n\gamma^n = S_n$$

i.e., $S_n = t^n H_{-n}$ and so $S_{-n} = t^{-n} H_n$.

(b): With Formula (1.4) or using the formula $S_n = (-u)^n H_{-n}$, we obtain initial values of S_n as

$$\begin{aligned} S_0 &= t^0 H_0 = 3, \\ S_1 &= t^1 H_{-1} = t \times (-\frac{s}{t}) = -s \\ S_2 &= t^2 H_{-2} = t^2 \times \frac{1}{t^2}(s^2 - 2rt) = s^2 - 2rt, \end{aligned}$$

or

$$S_2 = \alpha^2\beta^2 + \alpha^2\gamma^2 + \beta^2\gamma^2 = (\alpha\beta + \alpha\gamma + \beta\gamma)^2 - 2\alpha\beta\gamma(\alpha + \beta + \gamma) = s^2 - 2rt.$$

For $n \geq 3$, we have

$$\begin{aligned} S_1 S_{n-1} &= (\alpha^n\beta^n + \alpha^n\gamma^n + \beta^n\gamma^n) + \alpha\beta\gamma(\alpha^{n-2}\beta^{n-2}(\alpha+\beta) \\ &\quad +\alpha^{n-2}\gamma^{n-2}(\alpha+\gamma) + \beta^{n-2}\gamma^{n-2}(\beta+\gamma)) \\ &= S_n + rtS_{n-2} - t^2 S_{n-3} = (-s)S_{n-1}. \end{aligned}$$

(c): From the definition of S_n we get

$$2S_n = (\alpha^n + \beta^n + \gamma^n)^2 - (\alpha^{2n} + \beta^{2n} + \gamma^{2n}) = H_n^2 - H_{2n}. \ \square$$

Now, we shall complete the proof of Theorem 2.

The Proof of Theorem 2: For $n \in \mathbb{Z}$, we have

$$\begin{aligned} W_n H_n &= (A_1\alpha^n + A_2\beta^n + A_3\gamma^n)(\alpha^n + \beta^n + \gamma^n) \\ &= W_{2n} + (A_1 + A_2 + A_3)\alpha^n\beta^n + (A_1 + A_3 + A_2)\alpha^n\gamma^n \\ &\quad +(A_2 + A_3 + A_1)\beta^n\gamma^n - (A_3\alpha^n\beta^n + A_2\alpha^n\gamma^n + A_1\beta^n\gamma^n) \\ &= W_{2n} + (A_1 + A_2 + A_3)(\alpha^n\beta^n + \alpha^n\gamma^n + \beta^n\gamma^n) - (A_3\alpha^n\beta^n + A_2\alpha^n\gamma^n + A_1\beta^n\gamma^n) \\ &= W_{2n} + W_0 S_n - t^n W_{-n}. \end{aligned}$$

By Lemma 3 (c), it follows that

$$W_n H_n = W_{2n} + \frac{1}{2}(H_n^2 - H_{2n})W_0 - t^n W_{-n}$$

and so

$$W_{-n} = t^{-n}(W_{2n} - H_n W_n + \frac{1}{2}(H_n^2 - H_{2n})W_0). \ \square$$

Now, we present a few basic relation between $\{H_n\}$ and $\{W_n\}$ which can be used to write H_n in terms of W_n.

Lemma 4. *The following equality is true:*

$(W_2^3+(t+rs)W_1^3+t^2W_0^3+(r^2-s)W_1^2W_2-2rW_1W_2^2-sW_0W_2^2+rtW_0^2W_2+(s^2+rt)W_0W_1^2+2stW_0^2W_1+(rs-3t)W_0W_1W_2)H_n = (3W_2^2+(r^2-s)W_1^2+rtW_0^2-4rW_1W_2-2sW_0W_2+(rs-3t)W_0W_1)W_{n+2}+(-2rW_2^2+3tW_1^2-2sW_1W_2-3tW_0W_2+3rsW_1^2+2stW_0^2+2r^2W_1W_2+2s^2W_0W_1+rsW_0W_2+2rtW_0W_1)W_{n+1}+(-sW_2^2+(s^2+rt)W_1^2+3t^2W_0^2+(rs-3t)W_1W_2+2rtW_0W_2+4stW_0W_1)W_n.$

Proof. It is given in Soykan [3]. $\square$

Next, we present a remark which presents how H_n can be written in terms of W_n.

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REMARK 5. *To express W_{-n} by the sequence itself at positive indices we need that H_n can be written in terms of W_n. For this, writing*

$$H_n = a \times W_{n+2} + b \times W_{n+1} + c \times W_n$$

and solving the system of equations

$$\begin{aligned} H_0 &= a \times W_2 + b \times W_1 + c \times W_0 \\ H_1 &= a \times W_3 + b \times W_2 + c \times W_1 \\ H_2 &= a \times W_4 + b \times W_3 + c \times W_2 \end{aligned}$$

or

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} W_2 & W_1 & W_0 \\ W_3 & W_2 & W_1 \\ W_4 & W_3 & W_2 \end{pmatrix}^{-1} \begin{pmatrix} H_0 \\ H_1 \\ H_2 \end{pmatrix}$$

we find that H_n can be written in terms of W_n and we can replace this H_n in Theorem 2.

Using Theorem 2 and Lemma 4 or Remark 5, we have the following corollary.

COROLLARY 6. *For $n \in \mathbb{Z}$, we have*

(a): $G_{-n} = \frac{1}{t^{n+1}}((2rt - s^2)G_n^2 + tG_{2n} + sG_{n+2}G_n - (3t + rs)G_{n+1}G_n)$.

(b): $H_{-n} = \frac{1}{2t^n}(H_n^2 - H_{2n})$.

Note that if we take $r = 1, s = 0, t = 1$ and $G_n = N_n$ in the above Corollary, we obtain Lin's Theorem 1. Using Theorem 2 and Lemma 4 or Remark 5 (or using the last corollary for special cases), we can give some formulas for the special cases of generalized Tribonacci sequence (generalized (r,s,t)-sequence) as follows.

We have the following corollary which gives the connection between the special cases of generalized Tribonacci sequence at the positive index and the negative index.

COROLLARY 7. *For $n \in \mathbb{Z}$, we have the following recurrence relations:*

(a): *Tribonacci sequence:*

$T_{-n} = T_n^2 + T_{2n} + T_{n+2}T_n - 4T_{n+1}T_n$.

(b): *Tribonacci-Lucas sequence:*

$K_{-n} = \frac{1}{2}(K_n^2 - K_{2n})$.

(c): *Tribonacci-Perrin sequence:*

$M_{-n} = \frac{1}{3362}(243M_{n+2}^2 + 12M_{n+1}^2 + 805M_n^2 - 1107M_{2n+2} + 246M_{2n+1} - 943M_{2n} - 108M_{n+2}M_{n+1} + 1152M_{n+2}M_n - 256M_{n+1}M_n)$.

(d): *modified Tribonacci sequence:*

$U_{-n} = \frac{1}{2}(U_{n+2}^2 + 4U_{n+1}^2 + U_{2n+2} - 2U_{2n+1} - 4U_{n+2}U_{n+1} - 2U_{n+2}U_n + 4U_{n+1}U_n)$.

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(e): *modified Tribonacci-Lucas sequence:*

$$G_{-n} = \frac{1}{2}(G_{n+2}^2 + 4G_{n+1}^2 - 2G_{2n+2} + 4G_{2n+1} - 4G_{n+2}G_{n+1} + G_{n+2}G_n - 2G_{n+1}G_n).$$

(f): *adjusted Tribonacci-Lucas sequence:*

$$H_{-n} = \frac{1}{2}(H_{n+1}^2 - 2H_{2n+1} + H_{n+1}H_n).$$

The following corollary illustrates the connection between the special cases of generalized third-order Pell sequence at the positive index and the negative index.

Corollary 8. *For $n \in \mathbb{Z}$, we have the following recurrence relations:*

(a): *third order Pell sequence:*

$$P_{-n} = 3P_n^2 + P_{2n} + P_{n+2}P_n - 5P_{n+1}P_n.$$

(b): *third order Pell-Lucas sequence:*

$$Q_{-n} = \frac{1}{2}(Q_n^2 - Q_{2n}).$$

(c): *third order modified Pell sequence:*

$$E_{-n} = \frac{1}{3}(-E_n^2 + 3E_{2n} - 11E_{n+1}E_n + 2E_{n+2}E_n).$$

(d): *third order Pell-Perrin sequence:*

$$R_{-n} = \frac{1}{6962}(972R_{n+2}^2 + 48R_{n+1}^2 + 1081R_n^2 - 3186R_{2n+2} + 708R_{2n+1} - 1357R_{2n} - 432R_{n+2}R_{n+1} + 2952R_{n+2}R_n - 656R_{n+1}R_n).$$

The following corollary presents the connection between the special cases of generalized Padovan sequence at the positive index and the negative index.

Corollary 9. *For $n \in \mathbb{Z}$, we have the following recurrence relations:*

(a): *Padovan (Cordonnier) sequence:*

$$P_{-n} = \frac{1}{2}(9P_{n+2}^2 + 4P_{n+1}^2 + 8P_n^2 + 3P_{2n+2} - 2P_{2n+1} - 2P_{2n} - 12P_{n+2}P_{n+1} - 18P_{n+2}P_n + 12P_{n+1}P_n).$$

(b): *Perrin (Padovan-Lucas) sequence:*

$$E_{-n} = \frac{1}{2}(E_n^2 - E_{2n}).$$

(c): *Padovan-Perrin sequence:*

$$S_{-n} = S_n^2 + S_{2n} - 3S_{n+2}S_n.$$

(d): *modified Padovan sequence:*

$$A_{-n} = \frac{1}{722}(3A_{n+2}^2 + 108A_{n+1}^2 + 616A_n^2 + 57A_{2n+2} + 342A_{2n+1} - 532A_{2n} + 36A_{n+2}A_{n+1} - 94A_{n+2}A_n - 564A_{n+1}A_n).$$

(e): *adjusted Padovan sequence:*

$$U_{-n} = -U_n^2 + U_{2n} + U_{n+2}U_n - 3U_{n+1}U_n.$$

We have the following corollary which gives the connection between the special cases of generalized Pell-Padovan sequence at the positive index and the negative index.

Corollary 10. *For $n \in \mathbb{Z}$, we have the following recurrence relations:*

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(a): *Pell-Padovan sequence:*

$$R_{-n}=\frac{1}{8}(16R_{n+1}^2+9R_{n+2}^2+5R_n^2+6R_{2n+2}-8R_{2n+1}-2R_{2n}-24R_{n+2}R_{n+1}-18R_{n+2}R_n+24R_{n+1}R_n).$$

(b): *Pell-Perrin sequence:*

$$C_{-n}=\frac{1}{242}(432C_{n+2}^2+972C_{n+1}^2+665C_n^2+396C_{2n+2}-594C_{2n+1}-385C_{2n}-1296C_{n+2}C_{n+1}-1104C_{n+2}C_n+1656C_{n+1}C_n).$$

(c): *third order Fibonacci-Pell sequence:*

$$G_{-n}=\frac{1}{2}(16G_{n+2}^2+4G_{n+1}^2+35G_n^2-4G_{2n+2}+2G_{2n+1}+7G_{2n}-16G_{n+2}G_{n+1}-48G_{n+2}G_n+24G_{n+1}G_n).$$

(d): *third order Lucas-Pell sequence:*

$$B_{-n}=\frac{1}{2}(B_n^2-B_{2n}).$$

(e): *adjusted Pell-Padovan sequence:*

$$M_{-n}=-4M_n^2+M_{2n}+2M_{n+2}M_n-3M_{n+1}M_n.$$

The following corollary illustrates the connection between the special cases of generalized generalized Jacobsthal-Padovan sequence at the positive index and the negative index.

Corollary 11. *For $n\in\mathbb{Z}$, we have the following recurrence relations:*

(a): *Jacobsthal-Padovan sequence:*

$$Q_{-n}=\frac{1}{2^{n+3}}(9Q_{n+2}^2+4Q_{n+1}^2+21Q_n^2+6Q_{2n+2}-4Q_{2n+1}-6Q_{2n}-12Q_{n+2}Q_{n+1}-30Q_{n+2}Q_n+20Q_{n+1}Q_n).$$

(b): *Jacobsthal-Perrin (-Lucas) sequence:*

$$L_{-n}=\frac{1}{2^{n+1}}(L_n^2-L_{2n}).$$

(c): *adjusted Jacobsthal-Padovan sequence:*

$$K_{-n}=\frac{1}{2^{n+1}}(-K_n^2+2K_{2n}+K_{n+2}K_n-6K_{n+1}K_n).$$

(d): *modified Jacobsthal-Padovan sequence:*

$$M_{-n}=\frac{1}{529\times 2^{n+3}}(108M_{n+1}^2+75M_{n+2}^2+3551M_n^2+690M_{2n+2}+828M_{2n+1}-3082M_{2n}+180M_{n+2}M_{n+1}-1130M_{n+2}M_n-1356M_{n+1}M_n).$$

The following corollary presents the connection between the special cases of generalized Narayana sequence at the positive index and the negative index.

Corollary 12. *For $n\in\mathbb{Z}$, we have the following recurrence relations:*

(a): *Narayana sequence:*

$$N_{-n}=2N_n^2+N_{2n}-3N_{n+1}N_n.$$

(b): *Narayana-Lucas sequence:*

$$U_{-n}=\frac{1}{2}(U_n^2-U_{2n}).$$

(c): *Narayana-Perrin sequence:*

$H_{-n} = \frac{1}{5618}(2028H_{n+1}^2+1323H_{n+2}^2+429H_n^2-3339H_{2n+2}+4134H_{2n+1}-583H_{2n}-3276H_{n+2}H_{n+1}+2688H_{n+2}H_n-3328H_{n+1}H_n)$.

We have the following corollary which gives the connection between the special cases of generalized third-order Jacobsthal sequence at the positive index and the negative index.

Corollary 13. *For $n \in \mathbb{Z}$, we have the following recurrence relations:*

(a): *third order Jacobsthal sequence:*

$J_{-n} = \frac{1}{2^{n+1}}(3J_n^2 + 2J_{2n} + J_{n+2}J_n - 7J_{n+1}J_n)$.

(b): *third order Jacobsthal-Lucas sequence:*

$j_{-n} = \frac{1}{9\times 2^{n+6}}(121j_{n+2}^2 + 441j_{n+1}^2 - 95j_n^2 - 264j_{2n+2} + 504j_{2n+1} + 120j_{2n} - 462j_{n+2}j_{n+1} + 154j_{n+2}j_n - 294j_{n+1}j_n)$.

(c): *modified third order Jacobsthal-Lucas sequence:*

$K_{-n} = \frac{1}{2^{n+1}}(K_n^2 - K_{2n})$.

(d): *third order Jacobsthal-Perrin sequence:*

$Q_{-n} = \frac{1}{1225\times 2^{n+3}}(243Q_{n+2}^2+3Q_{n+1}^2+3328Q_n^2-1890Q_{2n+2}+210Q_{2n+1}-3640Q_{2n}-54Q_{n+2}Q_{n+1}+2196Q_{n+2}Q_n - 244Q_{n+1}Q_n)$.

The following corollary illustrates the connection between the special cases of generalized generalized 3-primes sequence at the positive index and the negative index.

Corollary 14. *For $n \in \mathbb{Z}$, we have the following recurrence relations:*

(a): *3-primes sequence:*

$G_{-n} = \frac{1}{5^{n+1}}(11G_n^2 + 5G_{2n} + 3G_{n+2}G_n - 21G_{n+1}G_n)$.

(b): *Lucas 3-primes sequence:*

$H_{-n} = \frac{1}{2\times 5^n}(H_n^2 - H_{2n})$.

(c): *modified 3-primes sequence:*

$E_{-n} = \frac{1}{9\times 5^n}(-7E_n^2 + 9E_{2n} + 4E_{n+2}E_n - 31E_{n+1}E_n)$.

The following corollary presents the connection between the special cases of generalized reverse 3-primes sequence at the positive index and the negative index.

Corollary 15. *For $n \in \mathbb{Z}$, we have the following recurrence relations:*

(a): *reverse 3-primes sequence:*

$N_{-n} = \frac{1}{2^{n+1}}(11N_n^2 + 2N_{2n} + 3N_{n+2}N_n - 21N_{n+1}N_n)$.

(b): *reverse Lucas 3-primes sequence:*

$S_{-n} = \frac{1}{2^{n+1}}(S_n^2 - S_{2n})$.

(c): *reverse modified 3-primes sequence:*

$U_{-n} = \frac{1}{9\times 2^n}(-7U_n^2 + 9U_{2n} + 10U_{n+2}U_n - 67U_{n+1}U_n)$.

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