GENERALIZED GEOMETRY FOR SECOND-ORDER TANGENT GROUPS & AFFINE CONFIGURATION COMPLEXES

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ABSTRACT. In this work, generalized geometry of second-order tangent groups and affine configuration complexes is proposed. Initially, geometry for higher weights $n = 4$ and weights $n = 5$ is presented through some interesting and suitable homomorphisms, finally, this geometry is extended and generalized for any weight $n \in \mathbb{N}$.

1. INTRODUCTION

Suslin [12] connected free abelian groups by differential boundary maps to define his famous chain complex called Grassmannian configuration chain complex. Configuration chain complex has many applications in algebraic K-theory and geometry. Leibniz introduced Polylog function $Li_p(z) = \sum_{n=0}^{\infty} \frac{z^n}{n^p}$, $z \leq 1$, with relation $\log(uv) - \log(u) - \log(v) = 0$. For weight 1, Bloch defined polylog groups $\mathcal{B}_1(F)$, it is a quotient of $\mathbb{Z}$ -module $Z[\mathbb{P}_F^1/\{0, 1, \infty\}]$ by its subgroup, generated by the relation $[x] + [y] - [xy]$ [15]. Bloch also introduced dilogarithmic group $\mathcal{B}_2(F)$. It is the quotient of $Z[\mathbb{P}_F^1/\{0, 1, \infty\}]$ by the relation $\sum_{i=0}^4 (-1)^i r(v_0, \dots, \hat{v}_i, \dots, v_4)$. This relation is also called the five term relation of cross ratio of four points in two dimensional vector space. Bloch also introduced chain complex for dilogarithmic group called Bloch-Suslin chain complex. Goncharov [5, 6, 7] generalized Bloch group for any weight $n \in \mathbb{N}$ to define the generalized polylogarithmic chain complex called Goncharov complex.

Goncharov [5] connected Bloch Suslin and configuration chain complexes by morphisms for the geometry of weight 2 and 3. By making use of derivation, Cathelineau [4, 2, 3] introduced variant of Goncharov chain complex in two setting, one was infinitesimal while the other was tangential. Siddiqui [11] introduced both cross ratio and the Siegel cross ratio properties in tangential form. Siddiqui [11] also defined geometry of configuration and first order tangential chain complexes for weight 2 and weight 3. Khalid et. al [10] extended the work of [11] up to weight 5. Hussain [14] introduced second order tangent group and its chain complexes for weight 2 and 3. Hussain [14] also defined geometry of second order tangent group and configuration chain complexes both for weight 2 and 3.

The present work extends the work of [14], to introduce generalized geometry for affine configuration and 2nd order tangent groups chain complexes through generalized homomorphism. Section 2 provides basic ideas of the affine configuration chain complexes, truncated polynomial and its rings, cross ratio and dual numbers cross ratio, classical polylogarithmic groups and its complexes, first and second order Tangent groups and its generalized chain complexes, geometry of second order tangential and affine configuration chain complexes up to weight 3. Section 3 extends the work of [14] to define new geometry of

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2nd order tangent groups and affine configuration complexes for weight 4 and weight 5. For weight 4 initially the subcomplex of Grassmannian affine configuration chain complex and subcomplex of second order tangent group complex are connected through some interesting homomorphism. After connecting these two chain complexes we will show that the resultant diagram is commutative. Finally work done for weight 4 is extended for the geometry of weight 5 also and the resultant diagram is shown to be commutative. Section 4 describes generalization of the previous work. It propose generalized geometry for second order tangent groups and affine configuration complexes for any weight $n \in \mathbb{N}$. Last section is the conclusion of the whole research work.

2. BACKGROUND & LITERATURE REVIEW

2.1. Grassmannian Configuration Chain Complex. Let $GL_n(F)$ be a general linear group of order n . The group action of $GL_n(F) * V^n = V^n$, where the set V^n is an n -dimensional vector space with elements $(v_1, \dots, v_n)$ called configurations of V^n . Following morphisms are two types of differential maps defined as:

$$d(v_0, \dots, v_n) \rightarrow \sum_i^n (-1)^i (v_0, \dots, \hat{v}_i, \dots, v_n), \quad (1)$$

and

$$p(v_0, \dots, v_n) \rightarrow \sum_i^n (-1)^i (v_i | v_0, \dots, \hat{v}_i, \dots, v_n). \quad (2)$$

Let $G_m(n)$ be a free abelian group generated by m elements in n -dimensional vector space then following is the Grassmannian configuration chain complex:

$$\begin{array}{ccccc} G_{m+3}(n+3) & \xrightarrow{d} & G_{m+2}(n+3) & \xrightarrow{d} & G_{m+1}(n+3) \\ \downarrow p & & \downarrow p & & \downarrow p \\ G_{m+2}(n+2) & \xrightarrow{d} & G_{m+1}(n+2) & \xrightarrow{d} & G_m(n+2) \end{array} \quad (A)$$

Each square in diagram A is commutative (see [12]).

2.2. Affine Configuration Chain Complex. Let F be a field with characteristic 0. Let the k^{th} truncated polynomial ring be denoted by $F[\varepsilon]_k := F[\varepsilon]/\varepsilon^k, k \geq 1$. Let $\mathbb{A}_{F[\varepsilon]_n}^n$ be the affine space defined over the truncated polynomial $F[\varepsilon]_n$.

Let us denote the element $v = (a_1, a_2, a_3, \dots, a_n)^t \in \mathbb{A}_F^n \setminus (0, 0, 0, \dots, 0)^t$, $v_\varepsilon = (a_{1,\varepsilon}, \dots, a_{n,\varepsilon})^t \in \mathbb{A}_F^n$ also $v_{\varepsilon^n} = (a_{1,\varepsilon^{k-1}}, a_{2,\varepsilon^{k-1}}, a_{3,\varepsilon^{k-1}}, \dots, a_{n,\varepsilon^{k-1}})^t \in \mathbb{A}_F^n$ [14]. Let $G_m(\mathbb{A}_{F[\varepsilon]_k}^n)$ be the free abelian group, generated by $(v_1^*, \dots, v_m^*)$ vectors in the n -dimensional affine space $\mathbb{A}_{F[\varepsilon]_k}^n$, where the vector is defined as $v^* = v + v_\varepsilon \varepsilon + \dots + v_{\varepsilon^{k-1}} \varepsilon^{k-1}$ [14]. Now, let us introduced the following differential morphisms:

$$d : G_{m+1}(\mathbb{A}_{F[\varepsilon]_k}^n) \rightarrow G_m(\mathbb{A}_{F[\varepsilon]_k}^n)$$

and

$$p : G_{m+1}(\mathbb{A}_{F[\varepsilon]_k}^n) \rightarrow G_m(\mathbb{A}_{F[\varepsilon]_k}^{n-1}).$$

Following is the generalized affine configuration chain complex:

$$\begin{array}{ccccc} G_{m+1}(\mathbb{A}_{F[\varepsilon]_k}^n) & \xrightarrow{d} & G_m(\mathbb{A}_{F[\varepsilon]_k}^n) & \xrightarrow{d} & G_{m-1}(\mathbb{A}_{F[\varepsilon]_k}^n) \\ \downarrow p & & \downarrow p & & \downarrow p \\ G_m(\mathbb{A}_{F[\varepsilon]_k}^{n-1}) & \xrightarrow{d} & G_{m-1}(\mathbb{A}_{F[\varepsilon]_k}^{n-1}) & \xrightarrow{d} & G_{m-2}(\mathbb{A}_{F[\varepsilon]_k}^{n-1}) \end{array} \quad (\text{B})$$

2.3. Cross Ratio. Cross ratio of four points $(v_0, \dots, v_3)$ is defined as:

$$r(v_0, v_1, v_2, v_3) = \frac{\Delta(v_0, v_3)\Delta(v_1, v_2)}{\Delta(v_0, v_2)\Delta(v_1, v_3)},$$

where $(v_0, \dots, v_3) \in A_F^2$.

Following important property is called the Siegel [13] cross ratio property

$$1 = \frac{\Delta(v_0, v_3)\Delta(v_1, v_2)}{\Delta(v_0, v_2)\Delta(v_1, v_3)} + \frac{\Delta(v_0, v_1)\Delta(v_2, v_3)}{\Delta(v_0, v_2)\Delta(v_1, v_3)}. \quad (3)$$

2.4. Siegel cross ratio property in Polynomial Ring $F[\varepsilon]_k$.

Case 1 : For $k=1$ and $n=2$,

$$\Delta(v_1^*, v_2^*) = \Delta(v_1^*, v_2^*)_{\varepsilon^0} = \Delta(v_1, v_2). \quad (4)$$

Case 2 : For $k=2$ and $n=2$,

$$\Delta(v_1^*, v_2^*) = \Delta(v_1^*, v_2^*)_{\varepsilon^0} + \Delta(v_1^*, v_2^*)_{\varepsilon^1} \varepsilon, \quad (5)$$

$$\Delta(v_1^*, v_2^*)_{\varepsilon^1} = \Delta(v_1, v_{2,\varepsilon}) + \Delta(v_{1,\varepsilon}, v_2).$$

Case 3 : For $k=3$ and $n=2$,

$$\Delta(v_1^*, v_2^*) = \Delta(v_1^*, v_2^*)_{\varepsilon^0} + \Delta(v_1^*, v_2^*)_{\varepsilon^1} \varepsilon + \Delta(v_1^*, v_2^*)_{\varepsilon^2} \varepsilon^2, \quad (6)$$

$$\Delta(v_1^*, v_2^*)_{\varepsilon^2} = \Delta(v_1, v_{2,\varepsilon^2}) + \Delta(v_{1,\varepsilon}, v_{2,\varepsilon}) + \Delta(v_{1,\varepsilon^2}, v_2) \quad [11, 14].$$

Following are cross ratio properties and relation in $F[\varepsilon]_k$

$$\mathbf{r}(v_0^*, \dots, v_3^*) = (r_{\varepsilon^0} + r_{\varepsilon^1} \varepsilon + \dots + r_{\varepsilon^{k-1}} \varepsilon^{k-1})(v_0^*, \dots, v_3^*). \quad (7)$$

In the above eq.(7)

$$r_{\varepsilon^0}(v_0^*, \dots, v_3^*) = r(v_0, \dots, v_3) = \frac{\Delta(v_0, v_3)\Delta(v_1, v_2)}{\Delta(v_0, v_2)\Delta(v_1, v_3)}, \quad (8)$$

$$r_{\varepsilon^1}(v_0^*, \dots, v_3^*) = \frac{\{\Delta(v_0^*, v_3^*)\Delta(v_1^*, v_2^*)\}_{\varepsilon}}{\Delta(v_0, v_2)\Delta(v_1, v_3)} - r(v_0, \dots, v_3) \frac{\{\Delta(v_0^*, v_2^*)\Delta(v_1^*, v_3^*)\}_{\varepsilon}}{\Delta(v_0, v_2)\Delta(v_1, v_3)}, \quad (9)$$

$$\begin{aligned} r_{\varepsilon^2}(v_0^*, \dots, v_3^*) &= \frac{\{\Delta(v_0^*, v_3^*)\Delta(v_1^*, v_2^*)\}_{\varepsilon^2}}{\Delta(v_0, v_2)\Delta(v_1, v_3)} - r(v_0^*, \dots, v_3^*) \frac{\{\Delta(v_0^*, v_2^*)\Delta(v_1^*, v_3^*)\}_{\varepsilon}}{\Delta(v_0, v_2)\Delta(v_1, v_3)} \\ &\quad - r(v_0, \dots, v_3) \frac{\{\Delta(v_0^*, v_2^*)\Delta(v_1^*, v_3^*)\}_{\varepsilon}}{\Delta(v_0, v_2)\Delta(v_1, v_3)}, \end{aligned} \quad (10)$$

up to $r_{\varepsilon^{k-1}}(v_0, \dots, v_3)$.

2.5. Polylogarithmic Groups Complexes. Leibniz introduced polylogarithmic function

by a series $Li_p(z) = \sum_{n=0}^{\infty} \frac{z^n}{n^p}$, $z \leq 1$. This infinite series is absolutely convergent in a unit disc. Let us assume a $\mathbb{Z}$ -module defined over doubly punctured set denoted by $\mathbb{Z}[\mathbf{P}_F^1/\{0, 1, \infty\}]$. It is also a free abelian group generated by an element $[x]$.

2.5.1. Polylogarithmic Group. The polylog Scissor congruence group $\mathcal{B}(F)$ is a factor group of $Z[\mathbf{P}_F^1/\{0, 1, \infty\}]$ and its subgroup generated by the relation $[v_1] - [v_2] + \left[\frac{v_2}{v_1}\right] - \left[\frac{1-v_2^{-1}}{1-v_1^{-1}}\right] + \left[\frac{1-v_2}{1-v_1}\right]$, $v_1 \neq v_2$ and $v_1, v_2 \neq 0, 1$, also called Abel five term relation [5].

2.5.2. Bloch polylogarithmic group for Weight-1. In [15] Bloch defined a subgroup $R_1(F)$ of $Z[\mathbf{P}_F^1/\{0, 1, \infty\}]$, generated by the three relation $\{v_1 v_2\} - \{v_1\} - \{v_2\}$, $(v_1, v_2 \in F^\times)$. The quotient group $\mathcal{B}_1(F) = Z[\mathbf{P}_F^1/\{0, 1, \infty\}]/\langle R_1(F) \rangle$ is called Bloch classical polylogarithmic group for weight $n = 1$. Bloch also introduced an isomorphism $\delta_1 : \mathcal{B}_1(F) \rightarrow F^\times$, where $\delta_1 : [v] \rightarrow v$.

2.5.3. Polylog Bloch Group for weight 2 and Bloch Suslin Chain Complex. For weight 2 group, let $R_2(F) = \sum_{i=0}^4 (-1)^i r(v_0, \dots, \hat{v}_i, \dots, v_4)$ be a subgroup of $Z[\mathbf{P}_F^1/\{0, 1, \infty\}]$, generated by cross ratio of four points relation.

$\mathcal{B}_2(F) = Z[\mathbf{P}_F^1/\{0, 1, \infty\}]/\langle R_2(F) \rangle$. Bloch also connected $\mathcal{B}_2(F)$ and $\wedge^2 F^\times$ to form the following chain complex called the Bloch Suslin chain complex [15].

$$\mathcal{B}_2(F) \xrightarrow{\delta} \wedge^2 F^\times,$$

Where morphism δ is defined as $\delta[v]_2 = v \wedge v$.

2.5.4. Goncharov Chain Complex for Weight-3. Goncharov [5] introduced $R_3(F)$ subgroup of $Z[\mathbf{P}_F^1/\{0, 1, \infty\}]$, generated by the triple cross ratio relation of six points given by,

$$R_3(F) = \sum_{i=0}^6 (-1)^i Alt_6 \left[\frac{(v_0, v_1, v_3)(v_1, v_2, v_4)(v_0, v_2, v_5)}{(v_0, v_1, v_4)(v_1, v_2, v_5)(v_0, v_2, v_3)} \right]. \quad (11)$$

Goncharov [5] defined a factor group $\mathcal{B}_3(F) = Z[\mathbf{P}_F^1/\{0, 1, \infty\}]/\langle R_3(F) \rangle$. Goncharov also introduced the following chain complex for Bloch group $\mathcal{B}_3(F)$:

$$\mathcal{B}_3(F) \xrightarrow{\delta} \mathcal{B}_2(F) \otimes F^\times \xrightarrow{\delta} \wedge^3 F^\times$$

Lemma 2.1. $\delta \circ \delta = 0$ [5].

2.5.5. Weight- n . Goncharov [5] generalized Bloch group $\mathcal{B}_n(F) = Z[\mathbf{P}_F^1/\{0, 1, \infty\}]/\langle R_n(F) \rangle$, $R_n(F)$ is a subgroup of Z -module $Z[\mathbf{P}_F^1]$ and it is also the kernel of the morphism $\delta_n : Z[\mathbf{P}_F^1/\{0, 1, \infty\}] \rightarrow \mathcal{B}_{n-1}(F) \otimes F^\times$. Following is the generalized classical polylog chain complex called Goncharov complex:

$$\mathcal{B}_n(F) \xrightarrow{\delta_n} \mathcal{B}_{n-1}(F) \otimes F^\times \xrightarrow{\delta_{n-1}} \mathcal{B}_{n-2}(F) \otimes \wedge^2(F) \xrightarrow{\delta_{n-2}} \dots \xrightarrow{\delta_2} \mathcal{B}_2(F) \otimes \wedge^{n-2}(F) \xrightarrow{\delta_1} \wedge^n(F^\times) \quad (12)$$

Lemma 2.2. $\delta_{n-1} \circ \delta_n = 0$ [5].

2.6. First Order Tangent Group. Let us assume v and v' are elements of field F and $\langle v; v' \rangle_2 = [v + v' \varepsilon] - [v] \in \mathbb{Z}[F[\varepsilon]_2]$. Cathelineau [3] defined first order $T\mathcal{B}_2(F)$. It is a quotient group of Z -module generated by the elements $\langle x; x' \rangle_2 \in \mathbb{Z}[F[\varepsilon]_2]$ and the following relation

$$\langle v; v' \rangle - \langle w; w' \rangle + \left\langle \frac{w}{v}; \left(\frac{w}{v}\right)' \right\rangle - \left\langle \frac{1-w}{1-v}; \left(\frac{1-w}{1-v}\right)' \right\rangle + \left\langle \frac{v(1-w)}{w(1-v)}; \left(\frac{v(1-w)}{w(1-v)}\right)' \right\rangle,$$

$v \neq w$ and $v, w \neq 0, 1$. This relation is also called five term relation [3, 14].

2.6.1. *Tangent Group Chain Complex for Weight 2.* Following is the tangential chain complex to Bloch Suslin complex for weight 2 [3, 11]:

$$T\mathcal{B}_2(F) \xrightarrow{\delta_\varepsilon} F \otimes F^\times \oplus \wedge^2 F^\times ,$$

where

$$\delta_\varepsilon : \langle v, w \rangle_2 = \left(\frac{w}{v} \otimes (1 - v) + \frac{w}{(1 - v)} \otimes v \right) + \left(\frac{w}{(1 - v)} \wedge \frac{w}{v} \right). \quad (13)$$

2.6.2. *Tangent Group Chain Complex for Weight 3.* Cathelineau [3] introduced tangent group of order 3 defined as $T\mathcal{B}_3(F) = \mathbb{Z}[F[\varepsilon]_2]/\ker_3$, where $\ker_3$ is the kernel of the morphism $\delta_\varepsilon = \mathbb{Z}[F[\varepsilon]_2] \rightarrow T\mathcal{B}_2(F) \otimes F^\times \oplus F \otimes \mathcal{B}_2(F)$. Following chain complex is the tangential chain complex for weight 3 to Goncharov complex:

$$T\mathcal{B}_3(F) \xrightarrow{\delta_\varepsilon} T\mathcal{B}_2(F) \otimes F^\times \oplus F \otimes \mathcal{B}_2(F) \xrightarrow{\delta_\varepsilon} F \otimes \wedge^2 F^\times \oplus \wedge^3 F^\times .$$

2.6.3. *Generalized First Order Tangent Group Chain Complex.* Cathelineau [3] introduced generalized tangent group $T\mathcal{B}_n(F)$ and then constructed the following generalized first order tangent group complex for any weight n :

$$T\mathcal{B}_n(F) \xrightarrow{\delta_{n,\varepsilon}} \frac{T\mathcal{B}_{n-1}(F) \otimes F^\times}{F \otimes \mathcal{B}_{n-1}(F)} \xrightarrow{\delta_{(n-1),\varepsilon}} \dots \xrightarrow{\delta_{1,\varepsilon}} \frac{T\mathcal{B}_2(F) \otimes \wedge^{n-2} F^\times}{F \otimes \mathcal{B}_2(F) \otimes \wedge^{n-3} F^\times} \xrightarrow{\delta_\varepsilon} (F \otimes \wedge^{n-1} F^\times) \oplus (\wedge^n F)$$

$$\delta_{(n-1),\varepsilon} \circ \delta_{n,\varepsilon} = 0 \quad [3].$$

2.7. **Second Order Tangent Group.** In [14] Hussain defined the second order tangent group $T\mathcal{B}_2^2(F)$ as the quotient group of $\mathbb{Z}$ -module generated by elements $\langle v; v', v'' \rangle \in \mathbb{Z}[F[\varepsilon]_3]$, where $\langle v; v', v'' \rangle = [v + v'\varepsilon + v''\varepsilon^2] - [v]$ and $v, v', v'' \in F$ and the following relation:

$$\begin{aligned} \langle v; v', v'' \rangle - \langle w; w', w'' \rangle + \left\langle \left(\frac{w}{v} \right); \left(\frac{w'}{v} \right), \left(\frac{w''}{v} \right) \right\rangle - \left\langle \left(\frac{1-w}{1-v} \right); \left(\frac{1-w'}{1-v} \right), \left(\frac{1-w''}{1-v} \right) \right\rangle + \\ \left\langle \left(\frac{v(1-w)}{w(1-v)} \right); \left(\frac{v(1-w')}{w(1-v)} \right), \left(\frac{v(1-w'')}{w(1-v)} \right) \right\rangle, \quad v, w \neq 0, 1, \quad v \neq w. \quad (14) \end{aligned}$$

2.7.1. *Second Order Tangent Group Chain Complex for Weight 2.* Hussain [14] also defined the following chain for second order tangential complex to the Bloch Suslin complex for weight 2:

$$T\mathcal{B}_2^2(F) \xrightarrow{\delta_{\varepsilon^2}} F \otimes F^\times \oplus \wedge^2 F^\times ,$$

where morphism δ_{ε^2} is defined as

$$\begin{aligned} \delta_{\varepsilon^2} \langle v; w_1, w_2 \rangle = \left(\frac{2w_2}{v} - \frac{w_1^2}{v^2} \right) \otimes (1 - v) + \left(\frac{2w_2}{(1-v)} - \frac{w_1^2}{(1-v)^2} \right) \otimes v + \\ \left(\frac{2w_2}{v} - \frac{w_1^2}{v^2} \right) \wedge \left(\frac{2w_2}{(1-v)} - \frac{w_1^2}{(1-v)^2} \right); \quad v, w_1, w_2 \in F. \end{aligned}$$

2.7.2. *Second Order Tangent Group Chain Complex for Weight 3.*

$$T\mathcal{B}_3^2(F) \xrightarrow{\delta_\varepsilon} T\mathcal{B}_2^2(F) \otimes F^\times \oplus F \otimes \mathcal{B}_2(F) \xrightarrow{\delta_\varepsilon} F \otimes \wedge^2 F^\times \oplus \wedge^3 F^\times .$$

2.7.3. Generalized Second Order Tangent Group Chain Complex. Following is the generalized second order tangent group chain complex:

$$TB_n^2(F) \xrightarrow{\delta_{n,\varepsilon}} \frac{TB_{n-1}^2(F) \otimes F^\times}{F \otimes B_{n-1}(F)} \xrightarrow{\delta_{(n-1),\varepsilon}} \cdots \xrightarrow{\delta_{1,\varepsilon}} \frac{TB_2^2(F) \otimes \wedge^{n-2} F^\times}{F \otimes B_2(F) \otimes \wedge^{n-3} F^\times} \xrightarrow{\delta_\varepsilon} (F \otimes \wedge^{n-1} F^\times) \oplus (\wedge^n F),$$

where $\delta_{(n-1),\varepsilon} \circ \delta_{n,\varepsilon} = 0$ [14].

2.8. Second order Tangential and Grassmannian Affine Configuration Complexes Geometry up to Weight 3.

2.8.1. Weight 2 Geometry. Hussain in [14] introduced the following geometry for weight-2:

$$\begin{array}{ccccc} G_5(\mathbb{A}_{F[\varepsilon]_3}^2) & \xrightarrow{p} & G_4(\mathbb{A}_{F[\varepsilon]_3}^2) & \xrightarrow{g_{1,\varepsilon}^2} & TB_2^2(F) \\ \downarrow d & & \downarrow d & & \downarrow \delta_{\varepsilon^2} \\ G_4(\mathbb{A}_{F[\varepsilon]_3}^2) & \xrightarrow{p} & G_3(\mathbb{A}_{F[\varepsilon]_3}^2) & \xrightarrow{g_{0,\varepsilon}^2} & F \otimes F^\times \oplus \wedge^2 F^\times \end{array} \quad (C)$$

Theorem 2.3. $g_{0,\varepsilon^2}^2 \circ d = \delta_{\varepsilon^2} \circ g_{1,\varepsilon}^2$ [14].

2.8.2. Weight 3 Geometry. Following geometry is for weight 3 as defined in [14]:

$$\begin{array}{ccccc} G_7(\mathbb{A}_{F[\varepsilon]_3}^3) & \xrightarrow{d} & G_6(\mathbb{A}_{F[\varepsilon]_3}^3) & & \\ \downarrow p & & \downarrow p & & \\ G_6(\mathbb{A}_{F[\varepsilon]_3}^3) & \xrightarrow{d} & G_5(\mathbb{A}_{F[\varepsilon]_3}^3) & \xrightarrow{g_{1,\varepsilon}^3} & TB_2^2(F) \otimes F^\times \oplus F \otimes B_2(F) \\ \downarrow p & & \downarrow p & & \downarrow \delta_{\varepsilon^2} \\ G_5(\mathbb{A}_{F[\varepsilon]_3}^3) & \xrightarrow{d} & G_4(\mathbb{A}_{F[\varepsilon]_3}^3) & \xrightarrow{g_{0,\varepsilon}^3} & F \otimes \wedge^2 F^\times \oplus \wedge^3 F^\times \end{array} \quad (D)$$

Theorem 2.4. $g_{0,\varepsilon^2}^3 \circ d = \delta_{\varepsilon^2} \circ g_{1,\varepsilon}^3$ [14].

3. GEOMETRY FOR HIGHER WEIGHTS

3.1. Second order Tangential and Grassmannian Affine Configuration Complexes Geometry up to Weight 4. For this weight connect the subcomplex of second order tangent group and affine configuration chain complexes. For the geometry of weight 4, take subcomplex of affine configuration and second order tangent group chain complexes. Now connect these two chain complexes by introducing two suitable and interesting homomorphisms g_{0,ε^2}^4 and g_{1,ε^2}^4 . These homomorphisms will help to produce a commutative diagram. The following commutative diagram is produced by connecting these two complexes:

$$\begin{array}{ccccc} G_7(\mathcal{A}_{F[\varepsilon]_r}^5) & \xrightarrow{p} & G_6(\mathcal{A}_{F[\varepsilon]_2}^4) & \xrightarrow{g_{1,\varepsilon^2}^4} & TB_2^2(F) \otimes \wedge^2 F^\times \oplus F \otimes B_2(F) \otimes F^\times \\ \downarrow d & & \downarrow d & & \downarrow \delta_{\varepsilon^2} \\ G_6(\mathcal{A}_{F[\varepsilon]_r}^5) & \xrightarrow{p} & G_5(\mathcal{A}_{F[\varepsilon]_2}^4) & \xrightarrow{g_{0,\varepsilon^2}^4} & F \otimes \wedge^3 F^\times \oplus \wedge^4 F^\times \end{array} \quad (E)$$

Where,

$$g_{0_{\varepsilon^2}}^4(v_0^*, \dots, v_4^*) = \sum_{i=0}^4 (-1)^i \left(2 \frac{\Delta(v_0^*, \dots, \hat{v}_i^*, \dots, v_4^*) \varepsilon^2}{\Delta(v_0, \dots, \hat{v}_i, \dots, v_4)} - \frac{\Delta(v_0^*, \dots, \hat{v}_i^*, \dots, v_4^*)^2 \varepsilon}{\Delta(v_0, \dots, \hat{v}_i, \dots, v_4)} \right) \otimes$$

$$\frac{\Delta(v_0, \dots, \hat{v}_{i+1}, \dots, v_4)}{\Delta(v_0, \dots, \hat{v}_{i+2}, \dots, v_4)} \wedge \frac{\Delta(v_0, \dots, \hat{v}_{i+2}, \dots, v_4)}{\Delta(v_0, \dots, \hat{v}_{i+3}, \dots, v_4)} \wedge \frac{\Delta(v_0, \dots, \hat{v}_{i+3}, \dots, v_4)}{\Delta(v_0, \dots, \hat{v}_{i+4}, \dots, v_4)} +$$

$$\sum_{i=0}^4 (-1)^{i+1} \bigwedge_{i=0}^4 \left(2 \frac{\Delta(v_0^*, \dots, \hat{v}_i^*, \dots, v_4^*) \varepsilon^2}{\Delta(v_0, \dots, \hat{v}_i, \dots, v_4)} - \frac{\Delta(v_0^*, \dots, \hat{v}_i^*, \dots, v_4^*)^2 \varepsilon}{\Delta(v_0, \dots, \hat{v}_i, \dots, v_4)} \right) (i \bmod 5) \quad (15)$$

and

$$g_{1_{\varepsilon^2}}^4(v_0^*, \dots, v_5^*) = \frac{1}{6} \sum_{i \neq j}^5 (-1)^i \left(\left\langle r(v_i, v_j | v_0, \dots, \hat{v}_i, \hat{v}_j, \dots, v_5); r_{\varepsilon}(v_i^*, v_j^* | v_0^*, \dots, \hat{v}_i^*, \hat{v}_j^*, \dots, v_5^*), \right. \right.$$

$$\left. r_{\varepsilon^2}(v_i^*, v_j^* | v_0^*, \dots, \hat{v}_i^*, \hat{v}_j^*, \dots, v_5^*) \right]_2^2 \otimes \prod_{i \neq r}^5 \Delta(v_0, \dots, \hat{v}_i, \hat{v}_r, \dots, v_5) \wedge$$

$$\prod_{j \neq r}^5 \Delta(v_0, \dots, \hat{v}_j, \hat{v}_r, \dots, v_5) + \sum_{i \neq r}^5 \frac{\Delta(v_0^*, \dots, \hat{v}_i^*, \hat{v}_r^*, \dots, v_5^*) \varepsilon^2}{\Delta(v_0, \dots, \hat{v}_i, \hat{v}_r, \dots, v_5)} \otimes$$

$$[r(v_i, v_j | v_0, \dots, \hat{v}_i, \hat{v}_j, \dots, v_5)]_2 \otimes$$

$$\prod_{j \neq r}^5 \Delta(v_0, \dots, \hat{v}_j, \hat{v}_r, \dots, v_5) + \sum_{j \neq r}^5 \frac{\Delta(v_0^*, \dots, \hat{v}_j^*, \hat{v}_r^*, \dots, v_5^*)^2 \varepsilon}{\Delta(v_0, \dots, \hat{v}_j, \hat{v}_r, \dots, v_5)} \otimes$$

$$[r(v_i, v_j | v_0, \dots, \hat{v}_i, \hat{v}_j, \dots, v_5)]_2 \otimes \prod_{i \neq r}^5 \Delta(v_0, \dots, \hat{v}_i, \hat{v}_r, \dots, v_5) \pmod{6}. \quad (16)$$

Theorem 3.1. $g_{0_{\varepsilon^2}}^4 \circ d = \delta_{\varepsilon^2} \circ g_{1_{\varepsilon^2}}^4$

Proof: Let $(v_0, \dots, v_5) \in G_6(\mathcal{A}_{F[\varepsilon]_2}^4)$, apply morphism d

$$d(v_0, \dots, v_5) = \sum_{j=0}^5 (-1)^j (v_0, \dots, \hat{v}_j, \dots, v_5), \quad (17)$$

now apply $g_{0_{\varepsilon^2}}^4$

$$g_{0_{\varepsilon^2}}^4 \circ d = \sum_{j=0}^5 (-1)^j \sum_{i=0}^5 (-1)^i \left(2 \frac{\Delta(v_0^*, \dots, \hat{v}_i^*, \hat{v}_j^*, \dots, v_5^*) \varepsilon^2}{\Delta(v_0, \dots, \hat{v}_i, \hat{v}_j, \dots, v_5)} - \frac{\Delta(v_0^*, \dots, \hat{v}_i^*, \hat{v}_j^*, \dots, v_5^*)^2 \varepsilon}{\Delta(v_0, \dots, \hat{v}_i, \hat{v}_j, \dots, v_5)} \right) \otimes$$

$$\frac{\Delta(v_0, \dots, \hat{v}_{i+1}, \hat{v}_j, \dots, v_5)}{\Delta(v_0, \dots, \hat{v}_{i+2}, \hat{v}_j, \dots, v_5)} \wedge \frac{\Delta(v_0, \dots, \hat{v}_{i+2}, \hat{v}_j, \dots, v_5)}{\Delta(v_0, \dots, \hat{v}_{i+3}, \hat{v}_j, \dots, v_5)} \wedge \frac{\Delta(v_0, \dots, \hat{v}_{i+3}, \hat{v}_j, \dots, v_5)}{\Delta(v_0, \dots, \hat{v}_{i+4}, \hat{v}_j, \dots, v_5)} +$$

$$\sum_{j=0}^5 (-1)^j \sum_{i=0}^5 (-1)^{i+1} \bigwedge_{i \neq j}^5 \left(2 \frac{\Delta(v_0^*, \dots, \hat{v}_i^*, \hat{v}_j^*, \dots, v_5^*) \varepsilon^2}{\Delta(v_0, \dots, \hat{v}_i, \hat{v}_j, \dots, v_5)} - \frac{\Delta(v_0^*, \dots, \hat{v}_i^*, \hat{v}_j^*, \dots, v_5^*)^2 \varepsilon}{\Delta(v_0, \dots, \hat{v}_i, \hat{v}_j, \dots, v_5)} \right). \quad (18)$$

Consider again $(v_0, \dots, v_5) \in G_6(\mathcal{A}_{F[\varepsilon]_2}^4)$, apply morphism $g_{1\varepsilon^2}^4$

$$\begin{aligned}
 g_{1\varepsilon^2}^4(v_0^*, \dots, v_5^*) &= \frac{1}{6} \sum_{i \neq j}^5 (-1)^i \left(\left(r(v_i, v_j | v_0, \dots, \hat{v}_i, \hat{v}_j, \dots, v_5); r_\varepsilon(v_i^*, v_j^* | v_0^*, \dots, \hat{v}_i^*, \hat{v}_j^*, \dots, v_5^*), \right. \right. \\
 &\quad \left. \left. r_{\varepsilon^2}(v_i^*, v_j^* | v_0^*, \dots, \hat{v}_i^*, \hat{v}_j^*, \dots, v_5^*) \right)_2^2 \otimes \prod_{i \neq r}^5 \Delta(v_0, \dots, \hat{v}_i, \hat{v}_r, \dots, v_5) \wedge \right. \\
 &\quad \left. \prod_{j \neq r}^5 \Delta(v_0, \dots, \hat{v}_j, \hat{v}_r, \dots, v_5) + \sum_{\substack{i \neq r \\ i=0}}^5 \frac{\Delta(v_0^*, \dots, \hat{v}_i^*, \hat{v}_r^*, \dots, v_5^*)_{\varepsilon^2}}{\Delta(v_0, \dots, \hat{v}_i, \hat{v}_r, \dots, v_5)} \otimes \right. \\
 &\quad \left. [r(v_i, v_j | v_0, \dots, \hat{v}_i, \hat{v}_j, \dots, v_5)]_2 \otimes \right. \\
 &\quad \left. \prod_{j \neq r}^5 \Delta(v_0, \dots, \hat{v}_j, \hat{v}_r, \dots, v_5) + \sum_{\substack{j \neq r \\ j=0}}^5 \frac{\Delta(v_0^*, \dots, \hat{v}_j^*, \hat{v}_r^*, \dots, v_5^*)_{\varepsilon^2}}{\Delta(v_0, \dots, \hat{v}_j, \hat{v}_r, \dots, v_5)} \otimes \right. \\
 &\quad \left. [r(v_i, v_j | v_0, \dots, \hat{v}_i, \hat{v}_j, \dots, v_5)]_2 \otimes \prod_{i \neq r}^5 \Delta(v_0, \dots, \hat{v}_i, \hat{v}_r, \dots, v_5) \right), \quad (19)
 \end{aligned}$$

now applying morphism δ_{ε^2} and using wedge, tensor and Siegel cross ratio properties [13], we get

$$\begin{aligned}
 \delta_{\varepsilon^2} \circ g_{1\varepsilon^2}^4 &= \sum_{j=0}^5 (-1)^j \sum_{i=0}^5 (-1)^i \left(2 \frac{\Delta(v_0^*, \dots, \hat{v}_i^*, \hat{v}_j^*, \dots, v_5^*)_{\varepsilon^2}}{\Delta(v_0, \dots, \hat{v}_i, \hat{v}_j, \dots, v_5)} - \frac{\Delta(v_0^*, \dots, \hat{v}_i^*, \hat{v}_j^*, \dots, v_5^*)^2_{\varepsilon^2}}{\Delta(v_0, \dots, \hat{v}_i, \hat{v}_j, \dots, v_5)} \right) \otimes \\
 &\quad \frac{\Delta(v_0, \dots, \hat{v}_{i+1}, \hat{v}_j, \dots, v_5)}{\Delta(v_0, \dots, \hat{v}_{i+2}, \hat{v}_j, \dots, v_5)} \wedge \frac{\Delta(v_0, \dots, \hat{v}_{i+2}, \hat{v}_j, \dots, v_5)}{\Delta(v_0, \dots, \hat{v}_{i+3}, \hat{v}_j, \dots, v_5)} \wedge \frac{\Delta(v_0, \dots, \hat{v}_{i+3}, \hat{v}_j, \dots, v_5)}{\Delta(v_0, \dots, \hat{v}_{i+4}, \hat{v}_j, \dots, v_5)} + \\
 &\quad \sum_{j=0}^5 (-1)^j \sum_{i=0}^5 (-1)^{i+1} \bigwedge_{i \neq j}^5 \left(2 \frac{\Delta(v_0^*, \dots, \hat{v}_i^*, \hat{v}_j^*, \dots, v_5^*)_{\varepsilon^2}}{\Delta(v_0, \dots, \hat{v}_i, \hat{v}_j, \dots, v_5)} - \frac{\Delta(v_0^*, \dots, \hat{v}_i^*, \hat{v}_j^*, \dots, v_5^*)^2_{\varepsilon^2}}{\Delta(v_0, \dots, \hat{v}_i, \hat{v}_j, \dots, v_5)} \right). \quad (20)
 \end{aligned}$$

So from Eq.(18) and Eq.(20), $g_{0\varepsilon^2}^4 \circ d = \delta_{\varepsilon^2} \circ g_{1\varepsilon^2}^4$.

3.1.1. Weight 5 Geometry. For the geometry of weight 5, we extend the geometry of weight 4. Connect the subcomplex of second order tangent complex for weight 5 with subcomplex of Grassmannian affine configuration chain complex by two new homomorphisms g_{0,ε^2}^5 and g_{1,ε^2}^5 . Following is the resultant commutative diagram for weight 5 after connecting these chain complexes:

$$\begin{array}{ccccc}
 G_8(\mathcal{A}_{F[\varepsilon]_2}^6) & \xrightarrow{p} & G_7(\mathcal{A}_{F[\varepsilon]_2}^5) & \xrightarrow{g_{1,\varepsilon^2}^5} & T\mathcal{B}_2^2(F) \otimes \wedge^3 F^\times \oplus F \otimes \mathcal{B}_2(F) \otimes \wedge^2 F^\times \\
 \downarrow d & & \downarrow d & & \downarrow \delta_{\varepsilon^2} \\
 G_7(\mathcal{A}_{F[\varepsilon]_2}^6) & \xrightarrow{p} & G_6(\mathcal{A}_{F[\varepsilon]_2}^5) & \xrightarrow{g_{0,\varepsilon^2}^5} & F \otimes \wedge^4 F^\times \oplus \wedge^5 F^\times
 \end{array} \quad (F)$$

where,

$$g_{0,\varepsilon^2}^5(v_0^*, \dots, v_5^*) = \sum_{i=j+1}^5 (-1)^{i+1} \left(2 \frac{\Delta(v_0^*, \dots, \hat{v}_i^*, \dots, v_5^*)_{\varepsilon^2}}{\Delta(v_0, \dots, \hat{v}_i, \dots, v_5)} - \frac{\Delta(v_0^*, \dots, \hat{v}_i^*, \dots, v_5^*)^2_{\varepsilon^2}}{\Delta(v_0, \dots, \hat{v}_i, \dots, v_5)} \right) \otimes$$

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$$\begin{aligned}
& \frac{\Delta(v_0, \dots, \hat{v}_{i+1}, \dots, v_5)}{\Delta(v_0, \dots, \hat{v}_{i+2}, \dots, v_5)} \wedge \frac{\Delta(v_0, \dots, \hat{v}_{i+2}, \dots, v_5)}{\Delta(v_0, \dots, \hat{v}_{i+3}, \dots, v_5)} \wedge \frac{\Delta(v_0, \dots, \hat{v}_{i+3}, \dots, v_5)}{\Delta(v_0, \dots, \hat{v}_{i+4}, \dots, v_5)} \wedge \\
& \frac{\Delta(v_0, \dots, \hat{v}_{i+4}, \dots, v_5)}{\Delta(v_0, \dots, \hat{v}_{i+5}, \dots, v_5)} + \\
& \sum_{j=0}^5 (-1)^{j+1} \bigwedge_{\substack{j \neq i \\ j=0}}^5 \left(2 \frac{\Delta(v_0^*, \dots, \hat{v}_j^*, \dots, v_5^*) \mathcal{E}^2}{\Delta(v_0, \dots, \hat{v}_j, \dots, v_5)} - \frac{\Delta(v_0^*, \dots, \hat{v}_j^*, \dots, v_5^*) \mathcal{E}}{\Delta(v_0, \dots, \hat{v}_j, \dots, v_5)} \right) (i \bmod 6)
\end{aligned} \tag{21}$$

and

$$\begin{aligned}
g_{1\mathcal{E}^2}^5(v_0^*, \dots, v_6^*) &= \frac{1}{10} \sum_{i \neq j}^6 (-1)^i \left(\left\langle r(v_i, v_j, v_j | v_0, \dots, \hat{v}_i, \hat{v}_j, \hat{v}_k, \dots, v_6); r_{\mathcal{E}}(v_i^*, v_j^*, v_k^* | v_0^*, \dots, \right. \right. \\
& \left. \left. \hat{v}_i^*, \hat{v}_j^*, \hat{v}_k^*, \dots, v_6^*) \right\rangle_2^2 \otimes \prod_{i \neq r \neq s}^6 \Delta(v_0, \dots, \hat{v}_i, \hat{v}_r, \hat{v}_s, \dots, v_6) \wedge \right. \\
& \prod_{j \neq r \neq s}^6 \Delta(v_0, \dots, \hat{v}_j, \hat{v}_r, \hat{v}_s, \dots, v_6) \wedge \prod_{k \neq r \neq s}^6 \Delta(v_0, \dots, \hat{v}_k, \hat{v}_r, \hat{v}_s, \dots, v_6) \\
& \left. + \sum_{\substack{i \neq r \neq s \\ i=0}}^6 \left(\frac{\Delta(v_0^*, \dots, \hat{v}_i^*, \hat{v}_r^*, \hat{v}_s^*, \dots, v_6^*) \mathcal{E}^2}{\Delta(v_0, \dots, \hat{v}_i, \hat{v}_r, \hat{v}_s, \dots, v_6)} - \frac{\Delta(v_0^*, \dots, \hat{v}_i^*, \hat{v}_r^*, \hat{v}_s^*, \dots, v_6^*)^2 \mathcal{E}}{\Delta(v_0, \dots, \hat{v}_i, \hat{v}_r, \hat{v}_s, \dots, v_6)} \right) \otimes \right. \\
& \left. [r(v_i, v_j, v_k | v_0, \dots, \hat{v}_i, \hat{v}_j, \hat{v}_k, \dots, v_6)]_2 \otimes \prod_{j \neq r \neq s}^6 \Delta(v_0, \dots, \hat{v}_j, \hat{v}_r, \hat{v}_s, \dots, v_6) \wedge \right. \\
& \prod_{k \neq r \neq s}^6 \Delta(v_0, \dots, \hat{v}_k, \hat{v}_r, \hat{v}_s, \dots, v_6) \\
& \left. + \sum_{\substack{j \neq r \neq s \\ j=0}}^6 \left(\frac{\Delta(v_0^*, \dots, \hat{v}_j^*, \hat{v}_r^*, \hat{v}_s^*, \dots, v_6^*) \mathcal{E}^2}{\Delta(v_0, \dots, \hat{v}_j, \hat{v}_r, \hat{v}_s, \dots, v_6)} - \frac{\Delta(v_0^*, \dots, \hat{v}_j^*, \hat{v}_r^*, \hat{v}_s^*, \dots, v_6^*)^2 \mathcal{E}}{\Delta(v_0, \dots, \hat{v}_j, \hat{v}_r, \hat{v}_s, \dots, v_6)} \right) \otimes \right. \\
& \left. [r(v_i, v_j, v_k | v_0, \dots, \hat{v}_i, \right. \\
& \left. \hat{v}_j, \hat{v}_k, \dots, v_6)]_2 \otimes \prod_{k \neq r \neq s}^6 \Delta(v_0, \dots, \hat{v}_k, \hat{v}_r, \hat{v}_s, \dots, v_6) \wedge \prod_{i \neq r \neq s}^6 \Delta(v_0, \dots, \hat{v}_i, \hat{v}_r, \hat{v}_s, \dots, v_6) \right. \\
& \left. + \sum_{\substack{k \neq r \neq s \\ k=0}}^6 \left(\frac{\Delta(v_0^*, \dots, \hat{v}_k^*, \hat{v}_r^*, \hat{v}_s^*, \dots, v_6^*) \mathcal{E}^2}{\Delta(v_0, \dots, \hat{v}_k, \hat{v}_r, \hat{v}_s, \dots, v_6)} - \frac{\Delta(v_0^*, \dots, \hat{v}_k^*, \hat{v}_r^*, \hat{v}_s^*, \dots, v_6^*)^2 \mathcal{E}}{\Delta(v_0, \dots, \hat{v}_k, \hat{v}_r, \hat{v}_s, \dots, v_6)} \right) \otimes \right. \\
& \left. [r(v_i, v_j, v_k | v_0, \dots, \hat{v}_i, \hat{v}_j, \hat{v}_k, \dots, v_6)]_2 \otimes \prod_{i \neq r \neq s}^6 \Delta(v_0, \dots, \hat{v}_i, \hat{v}_r, \hat{v}_s, \dots, v_6) \wedge \right. \\
& \left. \prod_{j \neq r \neq s}^6 \Delta(v_0, \dots, \hat{v}_j, \hat{v}_r, \hat{v}_s, \dots, v_6) \right) \pmod{6}.
\end{aligned} \tag{22}$$

Theorem 3.2. $g_{0,\varepsilon^2}^5 \circ d = \delta_{\varepsilon^2} \circ g_{1,\varepsilon^2}^5$

Proof: Let $(v_0, \dots, v_6)$ be points of $G_7(\mathcal{A}_{F[\varepsilon]_2}^5)$, applying morphism d

$$d(v_0, \dots, v_6) = \sum_{j=0}^6 (-1)^j (v_0, \dots, \hat{v}_j, \dots, v_6), \quad (23)$$

now applying g_{0,ε^2}^5 , we get

$$\begin{aligned} g_{0,\varepsilon^2}^5 \circ d = & \sum_{j=0}^6 (-1)^j \sum_{i=j+1}^6 (-1)^i \left(2 \frac{\Delta(v_0^*, \dots, \hat{v}_i^*, \hat{v}_j^*, \dots, v_6^*) \varepsilon^2}{\Delta(v_0, \dots, \hat{v}_i, \hat{v}_j, \dots, v_6)} - \frac{\Delta(v_0^*, \dots, \hat{v}_i, \hat{v}_j, \dots, v_6^*)^2 \varepsilon}{\Delta(v_0, \dots, \hat{v}_i, \hat{v}_j, \dots, v_6)} \right) \otimes \\ & \frac{\Delta(v_0, \dots, \hat{v}_{i+1}, \hat{v}_j, \dots, v_6)}{\Delta(v_0, \dots, \hat{v}_{i+2}, \hat{v}_j, \dots, v_6)} \wedge \frac{\Delta(v_0, \dots, \hat{v}_{i+2}, \hat{v}_j, \dots, v_6)}{\Delta(v_0, \dots, \hat{v}_{i+3}, \hat{v}_j, \dots, v_6)} \wedge \frac{\Delta(v_0, \dots, \hat{v}_{i+3}, \hat{v}_j, \dots, v_6)}{\Delta(v_0, \dots, \hat{v}_{i+4}, \hat{v}_j, \dots, v_6)} \wedge \\ & \frac{\Delta(v_0, \dots, \hat{v}_{i+4}, \hat{v}_j, \dots, v_6)}{\Delta(v_0, \dots, \hat{v}_{i+5}, \hat{v}_j, \dots, v_6)} + \\ & \sum_{j=0}^6 (-1)^j \sum_{i=j+1}^6 (-1)^i \bigwedge_{j \neq i}^6 \left(2 \frac{\Delta(v_0^*, \dots, \hat{v}_i^*, \hat{v}_j^*, \dots, v_6^*) \varepsilon^2}{\Delta(v_0, \dots, \hat{v}_i, \hat{v}_j, \dots, v_6)} - \frac{\Delta(v_0^*, \dots, \hat{v}_i^*, \hat{v}_j^*, \dots, v_6^*) \varepsilon}{\Delta(v_0, \dots, \hat{v}_i, \hat{v}_j, \dots, v_6)} \right). \end{aligned} \quad (24)$$

Consider $(v_0, \dots, v_6) \in G_7(\mathcal{A}_{F[\varepsilon]_2}^5)$, we apply map g_{1,ε^2}^5

$$\begin{aligned} g_{1,\varepsilon^2}^5(v_0^*, \dots, v_6^*) = & \frac{1}{10} \sum_{i \neq j}^6 (-1)^i \left(\langle r(v_i, v_j, v_k | v_0, \dots, \hat{v}_i, \hat{v}_j, \hat{v}_k, \dots, v_6); r_\varepsilon(v_i^*, v_j^*, v_k^* | v_0^*, \dots, \right. \\ & \left. \hat{v}_i^*, \hat{v}_j^*, \hat{v}_k^*, \dots, v_6^*) \rangle_2 \otimes \prod_{i \neq r \neq s}^6 \Delta(v_0, \dots, \hat{v}_i, \hat{v}_r, \hat{v}_s, \dots, v_6) \wedge \right. \\ & \left. \prod_{j \neq r \neq s}^6 \Delta(v_0, \dots, \hat{v}_j, \hat{v}_r, \hat{v}_s, \dots, v_6) \wedge \prod_{k \neq r \neq s}^6 \Delta(v_0, \dots, \hat{v}_k, \hat{v}_r, \hat{v}_s, \dots, v_6) \right. \\ & \left. + \sum_{\substack{i \neq r \neq s \\ i=0}}^6 \left(\frac{\Delta(v_0^*, \dots, \hat{v}_i^*, \hat{v}_r^*, \hat{v}_s^*, \dots, v_6^*) \varepsilon^2}{\Delta(v_0, \dots, \hat{v}_i, \hat{v}_r, \hat{v}_s, \dots, v_6)} - \frac{\Delta(v_0^*, \dots, \hat{v}_i^*, \hat{v}_r^*, \hat{v}_s^*, \dots, v_6^*)^2 \varepsilon}{\Delta(v_0, \dots, \hat{v}_i, \hat{v}_r, \hat{v}_s, \dots, v_6)} \right) \otimes \right. \\ & \left. [r(v_i, v_j, v_k | v_0, \dots, \hat{v}_i, \hat{v}_j, \hat{v}_k, \dots, v_6)]_2 \otimes \prod_{j \neq r \neq s}^6 \Delta(v_0, \dots, \hat{v}_j, \hat{v}_r, \hat{v}_s, \dots, v_6) \wedge \right. \\ & \left. \prod_{k \neq r \neq s}^6 \Delta(v_0, \dots, \hat{v}_k, \hat{v}_r, \hat{v}_s, \dots, v_6) \right. \\ & \left. + \sum_{\substack{j \neq r \neq s \\ j=0}}^6 \left(\frac{\Delta(v_0^*, \dots, \hat{v}_j^*, \hat{v}_r^*, \hat{v}_s^*, \dots, v_6^*) \varepsilon^2}{\Delta(v_0, \dots, \hat{v}_j, \hat{v}_r, \hat{v}_s, \dots, v_6)} - \frac{\Delta(v_0^*, \dots, \hat{v}_j^*, \hat{v}_r^*, \hat{v}_s^*, \dots, v_6^*)^2 \varepsilon}{\Delta(v_0, \dots, \hat{v}_j, \hat{v}_r, \hat{v}_s, \dots, v_6)} \right) \otimes \right. \\ & \left. [r(v_i, v_j, v_k | v_0, \dots, \hat{v}_i, \right. \\ & \left. \hat{v}_j, \hat{v}_k, \dots, v_6)]_2 \otimes \prod_{k \neq r \neq s}^6 \Delta(v_0, \dots, \hat{v}_k, \hat{v}_r, \hat{v}_s, \dots, v_6) \wedge \prod_{i \neq r \neq s}^6 \Delta(v_0, \dots, \hat{v}_i, \hat{v}_r, \hat{v}_s, \dots, v_6) \right) \end{aligned}$$

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$$\begin{aligned}
& + \sum_{\substack{k \neq r \neq s \\ k=0}}^6 \left(\frac{\Delta(v_0^*, \dots, \hat{v}_k^*, \hat{v}_r^*, \hat{v}_s^*, \dots, v_6^*) \varepsilon^2}{\Delta(v_0, \dots, \hat{v}_k, \hat{v}_r, \hat{v}_s, \dots, v_6)} - \frac{\Delta(v_0^*, \dots, \hat{v}_k^*, \hat{v}_r^*, \hat{v}_s^*, \dots, v_6^*)^2 \varepsilon}{\Delta(v_0, \dots, \hat{v}_k, \hat{v}_r, \hat{v}_s, \dots, v_6)} \right) \otimes \\
& [r(v_i, v_j, v_k | v_0, \dots, \hat{v}_i, \hat{v}_j, \hat{v}_k, \dots, v_6)]_2 \otimes \prod_{i \neq r \neq s}^6 \Delta(v_0, \dots, \hat{v}_i, \hat{v}_r, \hat{v}_s, \dots, v_6) \wedge \\
& \prod_{j \neq r \neq s}^6 \Delta(v_0, \dots, \hat{v}_j, \hat{v}_r, \hat{v}_s, \dots, v_6). \quad (25)
\end{aligned}$$

Now apply morphism δ_{ε^2} then using wedge, tensor and Siegel cross ratio properties [13], we get

$$\begin{aligned}
\delta_{\varepsilon^2} \circ g_{1\varepsilon^2}^5 &= \sum_{j=0}^6 (-1)^j \sum_{i=j+1}^6 (-1)^i \left(2 \frac{\Delta(v_0^*, \dots, \hat{v}_i^*, \hat{v}_j^*, \dots, v_6^*) \varepsilon^2}{\Delta(v_0, \dots, \hat{v}_i, \hat{v}_j, \dots, v_6)} - \frac{\Delta(v_0^*, \dots, \hat{v}_i^*, \hat{v}_j^*, \dots, v_6^*)^2 \varepsilon}{\Delta(v_0, \dots, \hat{v}_i, \hat{v}_j, \dots, v_6)} \right) \otimes \\
& \frac{\Delta(v_0, \dots, \hat{v}_{i+1}, \hat{v}_j, \dots, v_6)}{\Delta(v_0, \dots, \hat{v}_{i+2}, \hat{v}_j, \dots, v_6)} \wedge \frac{\Delta(v_0, \dots, \hat{v}_{i+2}, \hat{v}_j, \dots, v_6)}{\Delta(v_0, \dots, \hat{v}_{i+3}, \hat{v}_j, \dots, v_6)} \wedge \frac{\Delta(v_0, \dots, \hat{v}_{i+3}, \hat{v}_j, \dots, v_6)}{\Delta(v_0, \dots, \hat{v}_{i+4}, \hat{v}_j, \dots, v_6)} \wedge \\
& \frac{\Delta(v_0, \dots, \hat{v}_{i+4}, \hat{v}_j, \dots, v_6)}{\Delta(v_0, \dots, \hat{v}_{i+5}, \hat{v}_j, \dots, v_6)} + \\
& \sum_{j=0}^6 (-1)^j \sum_{i=j+1}^6 (-1)^i \bigwedge_{j \neq i}^6 \left(2 \frac{\Delta(v_0^*, \dots, \hat{v}_i^*, \hat{v}_j^*, \dots, v_6^*) \varepsilon^2}{\Delta(v_0, \dots, \hat{v}_i, \hat{v}_j, \dots, v_6)} - \frac{\Delta(v_0^*, \dots, \hat{v}_i^*, \hat{v}_j^*, \dots, v_6^*) \varepsilon}{\Delta(v_0, \dots, \hat{v}_i, \hat{v}_j, \dots, v_6)} \right). \quad (26)
\end{aligned}$$

From Eq.(24) and Eq.(26), $g_{0\varepsilon^2}^5 \circ d = \delta_{\varepsilon^2} \circ g_{1\varepsilon^2}^5$.

4. GENERALIZED GEOMETRY FOR WEIGHT $n \in \mathbb{N}$

Following is the generalized geometry for any weight n:

$$\begin{array}{ccccc}
C_{n+3}(\mathcal{A}_{F[\varepsilon]_2}^{n+1}) & \xrightarrow{p} & C_{n+2}(\mathcal{A}_{F[\varepsilon]_2}^n) & \xrightarrow{g_{1\varepsilon^2}^n} & T\mathcal{B}_2^2(F) \otimes \wedge^{n-2} F^\times \oplus F \otimes \mathcal{B}_2(F) \otimes \wedge^{n-3} F^\times \quad (H) \\
\downarrow d & & \downarrow d & & \downarrow \delta_{\varepsilon^2} \\
C_{n+2}(\mathcal{A}_{F[\varepsilon]_2}^{n+1}) & \xrightarrow{p} & C_{n+1}(\mathcal{A}_{F[\varepsilon]_2}^n) & \xrightarrow{g_{0\varepsilon^2}^n} & F \otimes \wedge^{n-1} F^\times \oplus \wedge^n F^\times
\end{array}$$

where,

$$\begin{aligned}
g_{0,\varepsilon^2}^n(v_0^*, \dots, v_n^*) &= \sum_{i=0}^n (-1)^{i+1} \left(2 \frac{\Delta(v_0^*, \dots, \hat{v}_i^*, \dots, v_n^*) \varepsilon^2}{\Delta(v_0, \dots, \hat{v}_i, \dots, v_n)} - \frac{\Delta(v_0^*, \dots, \hat{v}_i^*, \dots, v_n^*)^2 \varepsilon}{\Delta(v_0, \dots, \hat{v}_i, \dots, v_n)} \right) \otimes \\
& \frac{\Delta(v_0, \dots, \hat{v}_{i+1}, \dots, v_n)}{\Delta(v_0, \dots, \hat{v}_{i+2}, \dots, v_n)} \wedge \frac{\Delta(v_0, \dots, \hat{v}_{i+2}, \dots, v_n)}{\Delta(v_0, \dots, \hat{v}_{i+3}, \dots, v_n)} \wedge \frac{\Delta(v_0, \dots, \hat{v}_{i+3}, \dots, v_n)}{\Delta(v_0, \dots, \hat{v}_{i+4}, \dots, v_n)} \wedge \\
& \frac{\Delta(v_0, \dots, \hat{v}_{i+4}, \dots, v_n)}{\Delta(v_0, \dots, \hat{v}_{i+5}, \dots, v_n)} \wedge \dots \wedge \frac{\Delta(v_0, \dots, \hat{v}_{i+(n-1)}, \dots, v_n)}{\Delta(v_0, \dots, \hat{v}_{i+n}, \dots, v_n)} \bigg) (i \bmod n), \quad (27)
\end{aligned}$$

and

$$\begin{aligned}
g_{1\varepsilon^2}^n(v_0^*, \dots, v_{n+1}^*) &= \sum_{i=0}^7 (-1)^i \left(\left\langle r(v_{i_1}, v_{i_2}, \dots, v_{i_{n-2}} | v_0, \dots, \hat{v}_{i_1}, \hat{v}_{i_2}, \dots, \hat{v}_{i_{n-2}}, \dots, v_{n+1}) \right\rangle; r_\varepsilon(v_{i_1}^*, v_{i_2}^*, \dots, \right. \\
& \left. v_{i_{n-2}}^* | v_0^*, \dots, v_{i_1}^*, v_{i_2}^*, \dots, v_{i_{n-2}}^*, \dots, v_{n+1}^*) \right]_2^2 \otimes \prod_{i_1 \neq j_1 \neq j_2 \neq \dots \neq j_{n-2}}^{n+2} \Delta(v_0, \dots, \hat{v}_{i_1}, \hat{v}_{j_1}, \hat{v}_{j_2}, \dots, \hat{v}_{j_{n-2}},
\end{aligned}$$

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$$\begin{aligned}
& \dots, v_{n+2}) \wedge \prod_{i_2 \neq j_1 \neq j_2 \neq \dots \neq j_{n-2}}^{n+2} \Delta(v_0, \dots, \hat{v}_{i_2}, \hat{v}_{j_1}, \hat{v}_{j_2}, \dots, \hat{v}_{j_{n-2}}, \dots, v_{n+2}) \wedge \dots \\
& \wedge \prod_{i_{n-2} \neq j_1 \neq j_2 \neq \dots \neq j_{n-2}}^{n+2} \Delta(v_0, \dots, \hat{v}_{i_{n-2}}, \hat{v}_{j_1}, \hat{v}_{j_2}, \dots, \hat{v}_{j_{n-2}}, \dots, v_{n+2}) \\
& + \sum_{\substack{i_1 \neq j_1 \neq j_2 \neq \dots \neq j_{n-2} \\ i_1=0}}^{n+2} \left(\frac{\Delta(v_0^*, \dots, \hat{v}_{i_1}^*, \hat{v}_{j_1}^*, \hat{v}_{j_2}^*, \dots, \hat{v}_{j_{n-2}}^*, \dots, v_{n+2}^*) \varepsilon^2}{\Delta(v_0, \dots, \hat{v}_{i_1}, \hat{v}_{j_1}, \hat{v}_{j_2}, \dots, \hat{v}_{j_{n-2}}, \dots, v_{n+2})} \right. \\
& \left. \frac{\Delta(v_0^*, \dots, \hat{v}_{i_1}^*, \hat{v}_{j_1}^*, \hat{v}_{j_2}^*, \dots, \hat{v}_{j_{n-2}}^*, \dots, v_{n+2}^*)^2 \varepsilon}{\Delta(v_0, \dots, \hat{v}_{i_1}, \hat{v}_{j_1}, \hat{v}_{j_2}, \dots, \hat{v}_{j_{n-2}}, \dots, v_{n+2})} \right) \otimes [v_{i_1}, v_{i_2}, \dots, v_{i_{n-2}} | v_0, \dots, \\
& \hat{v}_{i_1}, \hat{v}_{i_2}, \hat{v}_{i_{n-2}}, \dots, v_{n+2})]_2 \otimes \prod_{i_2 \neq j_1 \neq j_2 \neq \dots \neq j_{n-2}t}^{n+2} \Delta(v_0, \dots, \hat{v}_{i_2}, \hat{v}_{j_1}, \hat{v}_{j_2}, \dots, \hat{v}_{j_{n-2}}, \dots, v_{n+2}) \wedge \\
& \prod_{i_3 \neq j_1 \neq j_2 \neq \dots \neq j_{n-2}}^{n+2} \Delta(v_0, \dots, \hat{v}_{i_3}, \hat{v}_{j_1}, \hat{v}_{j_2}, \dots, \hat{v}_{j_{n-2}}, \dots, v_{n+2}) \dots \\
& \prod_{i_{n-2} \neq j_1 \neq j_2 \neq \dots \neq j_{n-2}}^{n+2} \Delta(v_0, \dots, \hat{v}_{i_{n-2}}, \hat{v}_{j_1}, \hat{v}_{j_2}, \dots, \hat{v}_{j_{n-2}}, \dots, v_{n+2}) + \\
& \dots \\
& + \sum_{\substack{i_{n-2} \neq j_1 \neq j_2 \neq \dots \neq j_{n-2} \\ i_{n-2}=0}}^{n+2} \left(\frac{\Delta(v_0^*, \dots, \hat{v}_{i_{n-2}}^*, \hat{v}_{j_1}^*, \hat{v}_{j_2}^*, \dots, \hat{v}_{j_{n-2}}^*, \dots, v_{n+2}^*) \varepsilon^2}{\Delta(v_0, \dots, \hat{v}_{i_{n-2}}, \hat{v}_{j_1}, \hat{v}_{j_2}, \dots, \hat{v}_{j_{n-2}}, \dots, v_{n+2})} \right. \\
& \left. \frac{\Delta(v_0^*, \dots, \hat{v}_{i_{n-2}}^*, \hat{v}_{j_1}^*, \hat{v}_{j_2}^*, \dots, \hat{v}_{j_{n-2}}^*, \dots, v_{n+2}^*)^2 \varepsilon}{\Delta(v_0, \dots, \hat{v}_{i_{n-2}}, \hat{v}_{j_1}, \hat{v}_{j_2}, \dots, \hat{v}_{j_{n-2}}, \dots, v_{n+2})} \right) \otimes [v_{i_1}, v_{i_2}, \dots, v_{i_{n-2}} | v_0, \dots, \\
& \hat{v}_{i_1}, \hat{v}_{i_2}, \hat{v}_{i_{n-2}}, \dots, v_{n+2})]_2 \otimes \prod_{i_1 \neq j_1 \neq j_2 \neq \dots \neq j_{n-2}t}^{n+2} \Delta(v_0, \dots, \hat{v}_{i_1}, \hat{v}_{j_1}, \hat{v}_{j_2}, \dots, \hat{v}_{j_{n-2}}, \dots, v_{n+2}) \wedge \\
& \prod_{i_2 \neq j_1 \neq j_2 \neq \dots \neq j_{n-2}}^{n+2} \Delta(v_0, \dots, \hat{v}_{i_2}, \hat{v}_{j_1}, \hat{v}_{j_2}, \dots, \hat{v}_{j_{n-2}}, \dots, v_{n+2}) \dots \prod_{i_{n-3} \neq j_1 \neq j_2 \neq \dots \neq j_{n-2}}^{n+2} \Delta(v_0, \dots, \\
& \hat{v}_{i_{n-3}}, \hat{v}_{j_1}, \hat{v}_{j_2}, \dots, \hat{v}_{j_{n-2}}, \dots, v_{n+2}). \tag{28}
\end{aligned}$$

Theorem 4.1. $g_{0,\varepsilon^2}^n \circ d = \delta_{\varepsilon^2} \circ g_{1,\varepsilon^2}^n$

Proof: Let $(v_0, \dots, v_{n+1})$ be $(n+2)$ points of the group $C_{n+2}(\mathcal{A}_{F[\varepsilon]_2}^n)$. We apply morphism d

$$d(v_0, \dots, v_{n+1}) = \sum_{k=0}^{n+1} (-1)^k (v_0, \dots, \hat{v}_k, \dots, v_{n+1}), \tag{29}$$

then apply homomorphism g_{0,ε^2}^n

$$g_{0,\varepsilon^2}^n \circ d = \sum_{i=0}^n (-1)^{i+1} \sum_{k=0}^{n+1} (-1)^k \left(2 \frac{\Delta(v_0^*, \dots, \hat{v}_i^*, \hat{v}_k, \dots, v_{n+1}^*) \varepsilon^2}{\Delta(v_0, \dots, \hat{v}_i, \hat{v}_k, \dots, v_{n+1})} - \frac{\Delta(v_0^*, \dots, \hat{v}_i^*, \hat{v}_k, \dots, v_{n+1}^*)^2 \varepsilon}{\Delta(v_0, \dots, \hat{v}_i, \hat{v}_k, \dots, v_{n+1})} \right) \otimes$$

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$$\frac{\Delta(v_0, \dots, \hat{v}_{i+1}, \hat{v}_k, \dots, v_{n+1})}{\Delta(v_0, \dots, \hat{v}_{i+2}, \hat{v}_k, \dots, v_{n+1})} \wedge \frac{\Delta(v_0, \dots, \hat{v}_{i+2}, \hat{v}_k, \dots, v_{n+1})}{\Delta(v_0, \dots, \hat{v}_{i+3}, \hat{v}_k, \dots, v_{n+1})} \wedge \frac{\Delta(v_0, \dots, \hat{v}_{i+3}, \hat{v}_k, \dots, v_{n+1})}{\Delta(v_0, \dots, \hat{v}_{i+4}, \hat{v}_k, \dots, v_{n+1})} \\ \wedge \frac{\Delta(v_0, \dots, \hat{v}_{i+4}, \hat{v}_k, \dots, v_{n+1})}{\Delta(v_0, \dots, \hat{v}_{i+5}, \hat{v}_k, \dots, v_{n+1})} \wedge \dots \wedge \frac{\Delta(v_0, \dots, \hat{v}_{i+(n-1)}, \hat{v}_k, \dots, v_{n+1})}{\Delta(v_0, \dots, \hat{v}_{i+n}, \hat{v}_k, \dots, v_{n+1})}. \quad (30)$$

Let us again take again $(n+2)$ points $(v_0, \dots, v_{n+1}) \in C_{n+2}(\mathcal{A}_{F[\mathcal{E}]_2}^n)$ and apply morphism $g_{1\mathcal{E}^2}^n$

$$g_{1\mathcal{E}^2}^n(v_0^*, \dots, v_{n+1}^*) = \sum_{i=0}^7 (-1)^i \left(\left(r(v_{i_1}, v_{i_2}, \dots, v_{i_{n-2}} | v_0, \dots, \hat{v}_{i_1}, \hat{v}_{i_2}, \dots, \hat{v}_{i_{n-2}}, \dots, v_{n+1}) ; r_{\mathcal{E}}(v_{i_1}^*, v_{i_2}^*, \dots, \right. \right. \\ \left. \left. v_{i_{n-2}}^* | v_0^*, \dots, v_{i_1}^*, v_{i_2}^*, \dots, v_{i_{n-2}}^*, \dots, v_{n+1}^*) \right]_2^2 \otimes \prod_{i_1 \neq j_1 \neq j_2 \neq \dots \neq j_{n-2}}^{n+2} \Delta(v_0, \dots, \hat{v}_{i_1}, \hat{v}_{j_1}, \hat{v}_{j_2}, \dots, \hat{v}_{j_{n-2}}, \dots, v_{n+2}) \wedge \right. \\ \left. \dots, v_{n+2}) \wedge \prod_{i_2 \neq j_1 \neq j_2 \neq \dots \neq j_{n-2}}^{n+2} \Delta(v_0, \dots, \hat{v}_{i_2}, \hat{v}_{j_1}, \hat{v}_{j_2}, \dots, \hat{v}_{j_{n-2}}, \dots, v_{n+2}) \wedge \dots \right. \\ \left. \wedge \prod_{i_{n-2} \neq j_1 \neq j_2 \neq \dots \neq j_{n-2}}^{n+2} \Delta(v_0, \dots, \hat{v}_{i_{n-2}}, \hat{v}_{j_1}, \hat{v}_{j_2}, \dots, \hat{v}_{j_{n-2}}, \dots, v_{n+2}) \right. \\ \left. + \sum_{\substack{i_1 \neq j_1 \neq j_2 \neq \dots \neq j_{n-2} \\ i_1=0}}^{n+2} \left(\frac{\Delta(v_0^*, \dots, \hat{v}_{i_1}^*, \hat{v}_{j_1}^*, \hat{v}_{j_2}^*, \dots, \hat{v}_{j_{n-2}}^*, \dots, v_{n+2}^*) \mathcal{E}^2}{\Delta(v_0, \dots, \hat{v}_{i_1}, \hat{v}_{j_1}, \hat{v}_{j_2}, \dots, \hat{v}_{j_{n-2}}, \dots, v_{n+2})} - \right. \right. \\ \left. \frac{\Delta(v_0^*, \dots, \hat{v}_{i_1}^*, \hat{v}_{j_1}^*, \hat{v}_{j_2}^*, \dots, \hat{v}_{j_{n-2}}^*, \dots, v_{n+2}^*)^2 \mathcal{E}}{\Delta(v_0, \dots, \hat{v}_{i_1}, \hat{v}_{j_1}, \hat{v}_{j_2}, \dots, \hat{v}_{j_{n-2}}, \dots, v_{n+2})} \right) \otimes [v_{i_1}, v_{i_2}, \dots, v_{i_{n-2}} | v_0, \dots, \\ \hat{v}_{i_1}, \hat{v}_{i_2}, \hat{v}_{i_{n-2}}, \dots, v_{n+2})]_2 \otimes \prod_{i_2 \neq j_1 \neq j_2 \neq \dots \neq j_{n-2}}^{n+2} \Delta(v_0, \dots, \hat{v}_{i_2}, \hat{v}_{j_1}, \hat{v}_{j_2}, \dots, \hat{v}_{j_{n-2}}, \dots, v_{n+2}) \wedge \\ \prod_{i_3 \neq j_1 \neq j_2 \neq \dots \neq j_{n-2}}^{n+2} \Delta(v_0, \dots, \hat{v}_{i_3}, \hat{v}_{j_1}, \hat{v}_{j_2}, \dots, \hat{v}_{j_{n-2}}, \dots, v_{n+2}) \dots \\ \prod_{i_{n-2} \neq j_1 \neq j_2 \neq \dots \neq j_{n-2}}^{n+2} \Delta(v_0, \dots, \hat{v}_{i_{n-2}}, \hat{v}_{j_1}, \hat{v}_{j_2}, \dots, \hat{v}_{j_{n-2}}, \dots, v_{n+2}) + \\ \cdot \\ \cdot \\ \cdot \\ + \sum_{\substack{i_{n-2} \neq j_1 \neq j_2 \neq \dots \neq j_{n-2} \\ i_{n-2}=0}}^{n+2} \left(\frac{\Delta(v_0^*, \dots, \hat{v}_{i_{n-2}}^*, \hat{v}_{j_1}^*, \hat{v}_{j_2}^*, \dots, \hat{v}_{j_{n-2}}^*, \dots, v_{n+2}^*) \mathcal{E}^2}{\Delta(v_0, \dots, \hat{v}_{i_{n-2}}, \hat{v}_{j_1}, \hat{v}_{j_2}, \dots, \hat{v}_{j_{n-2}}, \dots, v_{n+2})} - \right. \\ \left. \frac{\Delta(v_0^*, \dots, \hat{v}_{i_{n-2}}^*, \hat{v}_{j_1}^*, \hat{v}_{j_2}^*, \dots, \hat{v}_{j_{n-2}}^*, \dots, v_{n+2}^*)^2 \mathcal{E}}{\Delta(v_0, \dots, \hat{v}_{i_{n-2}}, \hat{v}_{j_1}, \hat{v}_{j_2}, \dots, \hat{v}_{j_{n-2}}, \dots, v_{n+2})} \right) \otimes [v_{i_1}, v_{i_2}, \dots, v_{i_{n-2}} | v_0, \dots, \\ \hat{v}_{i_1}, \hat{v}_{i_2}, \hat{v}_{i_{n-2}}, \dots, v_{n+2})]_2 \otimes \prod_{i_1 \neq j_1 \neq j_2 \neq \dots \neq j_{n-2}}^{n+2} \Delta(v_0, \dots, \hat{v}_{i_1}, \hat{v}_{j_1}, \hat{v}_{j_2}, \dots, \hat{v}_{j_{n-2}}, \dots, v_{n+2}) \wedge \\ \prod_{i_2 \neq j_1 \neq j_2 \neq \dots \neq j_{n-2}}^{n+2} \Delta(v_0, \dots, \hat{v}_{i_2}, \hat{v}_{j_1}, \hat{v}_{j_2}, \dots, \hat{v}_{j_{n-2}}, \dots, v_{n+2}) \dots \prod_{i_{n-3} \neq j_1 \neq j_2 \neq \dots \neq j_{n-2}}^{n+2} \Delta(v_0, \dots, \end{math>$$

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$$\hat{v}_{i-3}, \hat{v}_{j_1}, \hat{v}_{j_2}, \dots, \hat{v}_{j_{n-2}}, \dots, v_{n+2}). \quad (31)$$

Now apply map δ_{ε^2} and simplify using wedge, tensor and Siegel cross ratio properties [13], we get

$$\begin{aligned} \delta_{\varepsilon^2} \circ g_{1_{\varepsilon^2}}^n &= \sum_{i=0}^n (-1)^{i+1} \sum_{k=0}^{n+1} (-1)^k \left(2 \frac{\Delta(v_0^*, \dots, \hat{v}_i^*, \hat{v}_k, \dots, v_{n+1}^*) \varepsilon^2}{\Delta(v_0, \dots, \hat{v}_i, \hat{v}_k, \dots, v_{n+1})} - \frac{\Delta(v_0^*, \dots, \hat{v}_i^*, \hat{v}_k, \dots, v_{n+1}^*)^2 \varepsilon}{\Delta(v_0, \dots, \hat{v}_i, \hat{v}_k, \dots, v_{n+1})} \right) \otimes \\ &\quad \frac{\Delta(v_0, \dots, \hat{v}_{i+1}, \hat{v}_k, \dots, v_{n+1})}{\Delta(v_0, \dots, \hat{v}_{i+2}, \hat{v}_k, \dots, v_{n+1})} \wedge \frac{\Delta(v_0, \dots, \hat{v}_{i+2}, \hat{v}_k, \dots, v_{n+1})}{\Delta(v_0, \dots, \hat{v}_{i+3}, \hat{v}_k, \dots, v_{n+1})} \wedge \frac{\Delta(v_0, \dots, \hat{v}_{i+3}, \hat{v}_k, \dots, v_{n+1})}{\Delta(v_0, \dots, \hat{v}_{i+4}, \hat{v}_k, \dots, v_{n+1})} \\ &\quad \wedge \frac{\Delta(v_0, \dots, \hat{v}_{i+4}, \hat{v}_k, \dots, v_{n+1})}{\Delta(v_0, \dots, \hat{v}_{i+5}, \hat{v}_k, \dots, v_{n+1})} \wedge \dots \wedge \frac{\Delta(v_0, \dots, \hat{v}_{i+(n-1)}, \hat{v}_k, \dots, v_{n+1})}{\Delta(v_0, \dots, \hat{v}_{i+n}, \hat{v}_k, \dots, v_{n+1})}. \end{aligned} \quad (32)$$

From Eq.(30) and Eq.(32), $g_{0_{\varepsilon^2}}^6 \circ d = \delta_{\varepsilon^2} \circ g_{1_{\varepsilon^2}}^6$.

5. CONCLUSION

In this work generalized commutative diagrams is presented for the geometry of second order tangent groups and affine configuration chain complexes. Previous works defined geometry only up to weight $n = 3$. Here geometry is generalized for any weight $n \in \mathbb{N}$. Tangent groups have other form and order so generalized homomorphism $g_{0_{\varepsilon^2}}^n$ and $g_{1_{\varepsilon^2}}^n$ will help us define homology for affine configuration and other form of tangent groups complexes.

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