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Impulsive control and global stabilization of reaction-diffusion epidemic model

Ruofeng Rao[a,]

[a] *Department of Mathematics, Chengdu Normal University, Chengdu , 611130, China*

Abstract

In this paper, by using the variational method, a sufficient condition for the unique existence of the stationary solution of the reaction-diffusion ecosystem is obtained, which directly leads to the global asymptotic stability of the unique equilibrium point. Moreover, delayed feedback ecosystem with reaction-diffusion item is considered, and utilizing impulse control results in the globally exponential stability criterion of the delayed ecosystem. It is worth mentioning that the Neumann zero-boundary value that the infected and the susceptible people or animals should be controlled in the epidemic prevention area and not allowed to cross the border, which is a good simulation of the actual situation of epidemic prevention. And numerical examples illuminate the effectiveness of impulse control, which has a certain enlightening effect on the actual epidemic prevention work . That is, in the face of the epidemic situation, taking a certain frequency of positive and effective epidemic prevention measures is conducive to the stability and control of the epidemic situation. Particularly, the newly-obtained theorems quantifies this feasible step. Besides, utilizing Laplacian semigroup derives the pth moment stability criterion for the impulsive ecosystem.

Keywords: Neumann boundary value, Laplacian semigroup, Poincare inequality lemma, impulse control, Lyapunov-Razumikhin method

1. Introduction

Infectious diseases have existed for thousands of years in the history of human development and brought great harm to human beings. In the 6th century, the prevalence of smallpox in the Middle East reduced the population by nearly one tenth. From 1347 to 1352, the spread of plague in Europe killed nearly one third of the European population. AIDS is due to the infection of human immunodeficiency virus and make the human immune system destroyed, loss of immune capacity. In 1927, Kermack and McKendrick established the famous Sir chamber model in order to analyze the spreading law of London black death from 1665 to 1666 and Bombay plague in 1906 ([3]). In 1932, Kermack and McKendrick established the famous SIS model ([4,5]). The idea of compartment model is to divide the population into susceptible individuals, infected individuals and recovering individuals, and ignore the birth and death of the population, that is, the total population remains unchanged. Such infectious diseases as measles and chickenpox, patients with lifelong immunity after recovery, recovery individuals will not enter the susceptible

Email address: ruofengrao@163.com; ruofengrao@cdnu.edu.cn (Ruofeng Rao)

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individuals. On this basis, numerous scholars have done a lot of work ([6-10]). Stability analysis is an important part of infectious disease dynamics. The stability of equilibrium point is studied by qualitative theory of delay differential equations and uniform persistence theory. The stability of equilibrium includes local stability, uniform persistence and global stability. Local stability means that the initial value is near the equilibrium point, and the value at any time is near the equilibrium point [6]. Uniform persistence means that the system eventually has a positive lower bound, which reveals that infectious diseases eventually spread in the population [7]. Global stability means that the equilibrium point is stable. For any initial value, when the time is sufficiently large, the solution of the system will converge to the equilibrium point [7,8], The disease is extinct or epidemic. The research on the stability and consistent persistence of the dynamic model of infectious diseases is helpful for people to find the epidemic law of infectious diseases, and is of great significance for the prevention and control of infectious diseases. Time delay effect exists widely in the objective world. The epidemic law of infectious diseases not only depends on the current state, but also related to the previous state. In the process of infectious disease transmission, for some infectious diseases, such as tuberculosis, AIDS, etc., the susceptible become infected after infection, but they do not necessarily have the ability to infect immediately. It takes a period of time for the susceptible to become infected to have the ability to infect, so people need to take the infection delay into account in the infectious disease model. For influenza and other diseases, since the infected person can still reinfect the disease after recovery, it is necessary to consider the recovery delay. The recovery delay is the time from the recovery stage of the convalescent person to becoming a susceptible person again. In order to study gonorrhea, Cooke and Yorke established an epidemic dynamics model with time delay ([9]). There are a lot of classical works on the time-delay epidemic model (see [10 - 13]). In the virus dynamics model, since it takes a period of time from the virus infecting the host cell to the host cell producing virus particles, it is necessary to take the infection delay into account in the virus dynamics model. Because there is a period of time between the immune system receiving antigen stimulation and producing immune cells, it is necessary to take the immune delay into account in the virus dynamics model. In recent years, many scholars have obtained many interesting results by analyzing the stability of various infectious disease models ([14-23]). For example, in 2021, Xiaodan Chen and Renhao Cu investigated a diffusive cholera epidemic model with nonlinear incidence rate. By constructing suitable Lyapunov functionals, they derived the global stability of the disease free equilibrium and the endemic equilibrium ([23]). Besides, there are a lot of literature involved in impulsive epidemic model ([24-27]). In [24], Jianjun Jiao, Shaohong Cai and Limei Li established an SIR model with impulsive vaccination, impulsive dispersal and restricting infected individuals boarding transports, and proved that there exists globally asymptotically stable infection-free boundary periodic solution. Recently, reaction-diffusion epidemic model has been widely studied for its good simulation of epidemic infection (see, e.g. [31-36] and the related literature therein), for instance, Yantao Luo, Sitian Tang, Zhidong Teng and Long Zhang in [31] studied a reaction-diffusion multi-group SIR epidemic model with nonlinear incidence in spatially heterogeneous and homogeneous environment, and the global asymptotic stability for the disease-free and endemic steady states was obtained by way of Lyapunov functions method. In addition, Lyapunov-Razumikhin method of [28-30] inspires the author of this paper. For example, in [30], Dan Yang, Xiaodi Li and Shiji Song employed Lyapunov-Razumikhin method to derive the global stability criterion. In this paper, a dynamic model of infectious disease patients and susceptible patients with diffusion is studied , and a global stability result is obtained. Furthermore, since the artificial epidemic prevention can be regarded as an impulsive effect. So the impulsive delay dynamical system is also considered, and the global exponential asymptotic stability criterion is obtained by way of Lyapunov-Razumikhin method and impulse control technique. Besides, Laplacian semigroup, fixed point theory and inequality technique are applied to derive pth moment stability of the impulsive ecosystem by the inspiration of [46,47, 50].

In this paper, the innovative points are listed as follows,

⋄ By using the variational method, a sufficient condition for the unique existence of the stationary solution of the reaction-diffusion ecosystem is obtained, which directly leads to the global asymptotic stability of the unique equilibrium point.

⋄ Utilizing impulse control technique derives the globally exponential stability criterion of delayed feedback ecosystem. And numerical examples illuminate the effectiveness of impulse control, which has a certain enlightening effect on the actual epidemic prevention work .

⋄ The Neumann zero-boundary value that the infected and the susceptible people or animals should be controlled in the epidemic prevention area and not allowed to cross the border, which is a good simulation of the actual situation of epidemic prevention.

For convenience, throughout of this paper, the author denotes by λ_1 the smallest positive eigenvalue of the Neumann boundary problem in [1, Lemma 4]. Symmetric matrix $A > 0$ represents A is a positive definite symmetric matrix, and $A > B$ or $B < A$ represents $A - B$ is a positive definite symmetric matrix. $\lambda_{\max}A$ represents the maximum eigenvalue of the symmetric matrix A. For a matrix $A = (a_{ij})_{n\times n}$, denote $|A| = (|a_{ij}|)_{n\times n}$. For a vector $U = (U_1, U_2)^T$, denote $|U| = (|U_1|, |U_2|)^T$.

2. System descriptions

Consider the following ecosystem under Neumann boundary value:

$$\begin{cases} \dfrac{\partial u}{\partial t} =q_1\Delta u + a_1uv - b_1uv^2 - c_1u, \quad t \geqslant 0,\ x \in \Omega, \\ \dfrac{\partial v}{\partial t} =q_2\Delta v - a_2uv + b_2u^2v - c_2u + d, \quad t \geqslant 0,\ x \in \Omega, \\ \dfrac{\partial u(t,x)}{\partial \nu} =0 = \dfrac{\partial v(t,x)}{\partial \nu}, \quad x \in \partial\Omega,\ t \geqslant 0. \end{cases} \tag{2.1}$$

where u, v represent population density of people infected with virus and that of virus susceptible people or animal, respectively. $d > 0$ represents the birth rate of the virus susceptible, a_1uv represents the increasing degree on the infected with virus due to the cross infection, and $-a_2uv$ represents the decreasing degree on the susceptible due to the cross infection. b_1uv^2 represents decreasing degree on the infected due to epidemic preventions while b_2u^2v is the increasing degree on the susceptible after the patients are cured and became susceptible to infection. c_1u represents the decreasing degree on the patients due to the death of some patients. c_2u represents the decreasing degree on the virus susceptible due to increasing patients. If the diffusion phenomenon is ignored, the ecosystem (2.1) without reaction-diffusion items was investigated in many literature (see [45] and its references therein). However, it is well known that the phenomenon of diffusion can not be avoided in the ecosystem, and has a profound impact on the stability of the ecosystem. So, in this paper, the reaction-diffusion ecosystem with the infected and susceptible is considered for the first time.

It is easily known via a simple computation that if (u, v) is a constant equilibrium point of the system (2.1), then

$$\begin{cases} a_1uv - b_1uv^2 - c_1u = 0, \\ -a_2uv + b_2u^2v - c_2u + d = 0, \end{cases} \tag{2.2}$$

Throughout this paper, the following assumption is considered,

(H1) There are two positive constants $M_i (i = 1, 2)$ such that

$$0 \leqslant u \leqslant M_1, \quad 0 \leqslant v \leqslant M_2; \tag{2.3}$$

(H2) Assume that the positive numbers a_i, b_i, c_i, d satisfy

$$\begin{cases} a_1^2 = 4b_1c_1 \\ (a_1a_2 + 2b_1c_2)^2 = 8a_1b_1b_2d \end{cases} \tag{2.4}$$

Remark 1. Due to the limited natural resources, the boundedness hypothesis on the population density of the susceptible people or animals is reasonable. Especially due to the reliable guarantee of modern medicine and epidemic prevention science, the population density of the infected people or animals is also controllable and will not exceed a certain threshold. In sum, the boundedness assumption (H1) is practicable and workable.

Remark 2. For any given positive numbers a_i, b_i, c_i, d satisfying (2.4), the condition (H2) can guarantee that the positive solution $(u_*, v_*)^T$ of the equations (2.2) is unique, where

$$u_* = \frac{a_1a_2 + 2b_1c_2}{2a_1b_2}, \qquad v_* = \frac{a_1}{2b_1}. \tag{2.5}$$

Every regular or irregular killing and feeding of diseased species can be regarded as a kind of pulse, and hence, the impulsive model should be investigated. So the author may consider the following delayed feedback ecosystem:

$$\begin{cases} \frac{\partial u}{\partial t} = q_1\Delta u + a_1uv - b_1uv^2 - c_1u + p_1(u(t,x) - u(t-\tau(t),x)), \quad t \geqslant 0,\ t \neq t_k\ x \in \Omega, \\ \frac{\partial v}{\partial t} = q_2\Delta v - a_2uv + b_2u^2v - c_2u + d + p_2(v(t,x) - v(t-\tau(t),x)), \quad t \geqslant 0,\ t \neq t_k\ x \in \Omega, \\ (u(t_k^+,x) - u_*, v(t_k^+,x) - v_*)^T = B_k(u(t_k^-,x) - u_*, v(t_k^-,x) - v_*)^T, \quad k = 1,2,\cdots \\ \frac{\partial u}{\partial \nu} = 0 = \frac{\partial v}{\partial \nu}, \quad x \in \partial\Omega,\ t \geqslant 0, \\ (u(s,x) - u_*, v(s,x) - v_*) = \xi(s,x), \quad s \in [-\tau, 0],\ x \in \Omega, \end{cases} \tag{2.6}$$

where each $B_k > 0$ is a symmetric matrix, dependent on k. Time delay $\tau(t) \in [-\tau, 0]$.

Remark 3. It is obvious that under the assumption (H2), the point $(u_*, v_*)^T$ must be the unique stationary solution of the impulsive system (2.6), too. But the condition (H2) can not warrant that $(u_*, v_*)^T$ must be the unique stationary solution of the system (2.1) or (2.6). In fact, all the vector function $(u(x), v(x))^T$ satisfying the following elliptic equations (2.7) must be a stationary solution of the ecosystem (2.1) and also the impulsive system (2.6), where

$$\begin{cases} q_1\Delta u + a_1 uv - b_1 uv^2 - c_1 u = 0, & x \in \Omega, \\ q_2\Delta v - a_2 uv + b_2 u^2 v - c_2 u + d = 0, & x \in \Omega, \\ \dfrac{\partial u}{\partial \nu} = 0 = \dfrac{\partial v}{\partial \nu}, & x \in \partial\Omega. \end{cases} \tag{2.7}$$

The following lemma may be necessary to some extent (see, e.g. [1, Lemma 4]).

Lemma 2.1.(Poincar Integral Inequality). Let Ω be a bounded domain of R^m with a smooth boundary $\partial\Omega$ of class C^2 by Ω. $v(x)$ is a real-valued function belonging to $H_0^1(\Omega)$ and $\frac{\partial v(x)}{\partial \nu}|_{\partial\Omega} = 0$. Then

$$\lambda_1 \int_\Omega |v(x)|^2 dx \leqslant \int_\Omega |\nabla v(x)|^2 dx,$$

which λ_1 is the smallest positive eigenvalue of the Neumann boundary problem

$$\begin{cases} -\Delta\varphi = \lambda\varphi, & x \in \Omega, \\ \dfrac{\partial \varphi(x)}{\partial \nu} = 0, & x \in \partial\Omega. \end{cases} \tag{2.8}$$

Lemma 2.2 (Banach contraction mapping principle [48]) Let Θ be a contraction operator on a complete metric space Γ, then there exists a unique point $u \in \Gamma$ for which $\Theta(u) = u$.

Lemma 2.3. For any given $p \in [1, +\infty)$ and $m \in \mathbb{N}$, $a_j \geqslant 0,\ j = 1, 2, \cdots, m$, then the following inequality holds,

$$\Big(\sum_{j=1}^m a_j\Big)^p \leqslant m^{p-1} \sum_{j=1}^m a_j^p.$$

3. Mean square stability

Due to the viewpoint of Remark 3, the author has to prove that $(u_*, v_*)^T$ defined in (2.5) is the unique equilibrium point of the system (2.1). Assume, in this section, $u(t_k^+, x) = u(t_k, x)$, $v(t_k^+, x) = v(t_k, x)$.

Theorem 3.1. Suppose the conditions (H1) and (H2) hold, and

$$\begin{pmatrix} \alpha_1 & \frac{1}{2}(\alpha_2 + \beta_1) \\ * & \beta_2 \end{pmatrix} < \lambda_1 Q, \tag{3.1}$$

then $(u_*, v_*)^T$ defined in (2.5) is the unique stationary solution of the ecosystem (2.1) and also the impulsive system (2.6), where λ_1 is the smallest positive eigenvalue of the Neumann boundary problem ([1, Lemma 4]), and $Q = diag(q_1, q_2)$,

$$\begin{aligned}\alpha_1 =&|a_1 v_* - b_1 v_*^2 - c_1| + |a_1 - 2b_1 v_*| \cdot M_2, \quad \beta_1 = |a_1 u_* - 2b_1 u_* v_*| + b_1 u_* M_2 + b_1 M_1 M_2,\\ \alpha_2 =&| - a_2 v_* + 2b_2 u_* v_* - c_2| + b_2 v_* M_1 + b_2 M_1 M_2, \quad \beta_2 = | - a_2 u_* + b_2 u_*^2| + | - a_2 + 2b_2 u_*| \cdot M_1.\end{aligned} \tag{3.2}$$

Proof. Firstly, the author only need to prove that $(u_*, v_*)^T$ defined in (2.5) is the unique stationary solution of the ecosystem (2.1). In fact, if $(u_*, v_*)^T$ defined in (2.5) is the unique stationary solution of the ecosystem (2.1), the viewpoint of Remark 3 tells that the point $(u_*, v_*)^T$ must be the unique stationary solution of the impulsive system (2.6), too. Below, the author will prove that $(u_*, v_*)^T$ defined in (2.5) is the unique stationary solution of the ecosystem (2.1).

It is easy to verify that (u_*, v_*) defined in (2.5) is a solution of the following equations

$$\begin{cases} a_1 uv - b_1 uv^2 - c_1 u = 0,\\ - a_2 uv + b_2 u^2 v - c_2 u + d = 0,\end{cases} \tag{3.3}$$

which implies that $(u_*, v_*)^T$ is a positive equilibrium point of the system (2.1).

Let $(u(x), v(x))^T$ be any stationary solution of the system (2.1). Set $z_1 = u(x) - u_*$, $z_2 = v(x) - v_*$, and $z = (z_1, z_2)^T$, then

$$\begin{cases} q_1 \Delta z_1 + \phi_1(z_1, z_2) = 0, & x \in \Omega,\\ q_2 \Delta z_2 + \phi_2(z_1, z_2) = 0, & x \in \Omega,\\ \dfrac{\partial z_1}{\partial \nu} = 0 = \dfrac{\partial z_2}{\partial \nu}, & x \in \partial\Omega,\end{cases}$$

where

$$\begin{cases} \phi_1(z) =\Big(a_1 v_* - b_1 v_*^2 - c_1\Big) z_1 + \Big(a_1 u_* - 2b_1 u_* v_*\Big) z_2 + (a_1 - 2b_1 v_*) z_1 z_2 - b_1 u_* z_2^2 - b_1 z_1 z_2^2,\\ \phi_2(z) =\Big(- a_2 v_* + 2b_2 u_* v_* - c_2\Big) z_1 + \Big(- a_2 u_* + b_2 u_*^2\Big) z_2 + (-a_2 + 2b_2 u_*) z_1 z_2 + b_2 v_* z_1^2 + b_2 z_1^2 z_2,\end{cases} \tag{3.4}$$

Based on the boundedness assumptions (H1) on u, v, one can get

$$|\phi_1(z_1, z_2)| \leqslant \alpha_1 |z_1| + \beta_1 |z_2| \tag{3.5}$$

and

$$|\phi_2(z_1, z_2)| \leqslant \alpha_2 |z_1| + \beta_2 |z_2|, \tag{3.6}$$

where α_i, β_i are defined in (3.2).

Similarly as that of [1], employing Poincare integral inequality ([1, Lemma 4]) results in

$$\lambda_1 \int_\Omega |z|^T Q |z| \leqslant \int_\Omega \Big(\alpha_1 |z_1|^2 + (\alpha_2 + \beta_1)|z_1 z_2| + \beta_2 |z_2|^2\Big)$$

or

$$\lambda_1 \int_\Omega |z|^T Q |z| \leqslant \int_\Omega |z|^T \begin{pmatrix} \alpha_1 & \frac{1}{2}(\alpha_2 + \beta_1)\\ * & \beta_2 \end{pmatrix} |z|,$$

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and hence $z = 0$, and $u(x) = u_*$, $v(x) = v_*$. The proof of Theorem 3.1 is completed.

Remark 4. Both u_* and v_* are only the positive numbers, dependent upon a_i, b_i, c_i and d. Particularly, $a_1v_* - b_1v_*^2 - c_1 = 0$ and so $\alpha_1 = |a_1v_* - b_1v_*^2 - c_1| + |a_1 - 2b_1v_*| \cdot M_2$. Similarly, other parameters are also simplified to some extent.

Set $U = U(t,x) = (U_1(t,x), U_2(t,x))^T$ with

$$\begin{cases} U_1(t,x) = u(t,x) - u_* \\ U_2(t,x) = v(t,x) - v_*, \end{cases} \tag{3.7}$$

where u_* and v_* are positive numbers, defined in (2.5). Then it follows from (2.1) that

$$\begin{cases} \dfrac{\partial(u-u_*)}{\partial t} = q_1\Delta(u-u_*) + a_1uv - a_1u_*v_* - b_1uv^2 + b_1u_*v_* - c_1(u-u_*), & t \geqslant 0,\ x \in \Omega, \\ \dfrac{\partial(v-v_*)}{\partial t} = q_2\Delta(v-v_*) - a_2uv + a_2u_*v_* + b_2u^2v - b_2u_*^2v_* - c_2(u-u_*), & t \geqslant 0,\ x \in \Omega, \\ \dfrac{\partial(u-u_*)}{\partial \nu} = 0 = \dfrac{\partial(v-v_*)}{\partial \nu}, \quad x \in \partial\Omega,\ t \geqslant 0, \end{cases}$$

which can be rewritten as follows,

$$\begin{cases} \dfrac{\partial U}{\partial t} = Q\Delta U + \phi(U), & t \geqslant 0,\ x \in \Omega, \\ \dfrac{\partial U}{\partial \nu} = 0, & x \in \partial\Omega,\ t \geqslant 0, \end{cases} \tag{3.8}$$

where $\phi(U) = (\phi_1(U), \phi_2(U))^T$, and (3.4) yields

$$\begin{cases} \phi_1(U) = \Big(a_1v_* - b_1v_*^2 - c_1\Big)U_1 + \Big(a_1u_* - 2b_1u_*v_*\Big)U_2 + (a_1 - 2b_1v_*)U_1U_2 - b_1u_*U_2^2 - b_1U_1U_2^2, \\ \phi_2(U) = \Big(-a_2v_* + 2b_2u_*v_* - c_2\Big)U_1 + \Big(-a_2u_* + b_2u_*^2\Big)U_2 + (-a_2 + 2b_2u_*)U_1U_2 + b_2v_*U_1^2 + b_2U_1^2U_2. \end{cases} \tag{3.9}$$

Obviously, the positive equilibrium point $(u_*, v_*)^T$ of the system (2.1) is corresponding to the null solution $(0,0)^T$ of the system (3.8), where u_* and v_* are positive numbers, defined in (2.5).

Theorem 3.2. If all the conditions of Theorem 3.1 are satisfied, then the unique equilibrium point $(u_*, v_*)^T$ of the system (2.1) is globally asymptotically stable.

Proof. Consider the Lyapunov functional as follows

$$V(t,U) = \int_\Omega U^T(t,x)U(t,x)dx. \tag{3.10}$$

It is follows from (3.5) and (3.6) that

$$|\phi_1(U)| \leqslant \alpha_1|U_1| + \beta_1|U_2| \tag{3.11}$$

and

$$|\phi_2(U)| \leqslant \alpha_2|U_1| + \beta_2|U_2|, \tag{3.12}$$

Combining (3.11) and (3.12) results in

$$\begin{aligned}|\phi(U)^T U| =&|U^T\phi(U)| = |U_1\phi_1(U) + U_2\phi_2(U)|\\ \leqslant&|U|^T\begin{pmatrix}\alpha_1 & \frac{1}{2}(\alpha_2+\beta_1)\\ * & \beta_2\end{pmatrix}|U|\end{aligned}\tag{3.13}$$

Then Poincare inequality Lemma ([1, Lemma 4]), (3.1) and (3.8) yield

$$\begin{aligned}\frac{dU}{dt} =&2\int_\Omega U^T\Big(Q\Delta U + \phi(U)\Big)dx\\ \leqslant&2\int_\Omega\Big[-\lambda_1|U|^TQ|U| + |U|^T\begin{pmatrix}\alpha_1 & \frac{1}{2}(\alpha_2+\beta_1)\\ * & \beta_2\end{pmatrix}|U|\Big] \leqslant 0.\end{aligned}\tag{3.14}$$

Besides,

$$\|U\|^2_{L^2(\Omega)} \leqslant V(t,u) \leqslant 2\|U\|^2_{L^2(\Omega)},$$

which together with (3.14) implies that the null solution of the system (3.8) is globally asymptotically stable, and hence, the unique equilibrium point $(u_*, v_*)^T$ of the system (2.1) is globally asymptotically stable.

Finally the author consider the following delayed feedback system with impulse:

$$\begin{cases}\frac{\partial U}{\partial t} =&Q\Delta U + \phi(U) + P(U(t,x) - U(t-\tau(t),x)), \quad t\geqslant 0,\ t\neq t_k,\ x\in\Omega,\\ U(t_k^+,x) =&B_kU(t_k^-,x), \quad k=1,2,\cdots\\ \frac{\partial U}{\partial \nu} =&0, \quad x\in\partial\Omega,\ t\geqslant 0,\\ U(s,x) =&\xi(s,x), \quad s\in[-\tau,0],\ x\in\Omega,\end{cases}\tag{3.15}$$

where U is defined in (3.7), $\tau(t)\in[-\tau,0]$, $U(t_k^+,x) = U(t_k,x)$ and

$$P = \begin{pmatrix}p_1 & 0\\ * & p_2\end{pmatrix}, \quad B_k = \begin{pmatrix}b_{k1} & 0\\ * & b_{k2}\end{pmatrix}.\tag{3.16}$$

Obviously, the null solution of (3.15) is corresponding to the equilibrium point $(u_*, v_*)^T$ of the impulsive system (2.6).

Theorem 3.1 tells that under the assumptions of Theorem 3.1, $(u_*, v_*)^T$ is the unique equilibrium point of the impulsive system (2.6). And then the author is able to consider its global stability.

Theorem 3.3. Suppose all the conditions of Theorem 3.1 hold. If, in addition, there exist several positive constants c, λ, σ with $\sigma - \lambda \geqslant c$, and $\gamma \geqslant 1$ such that

$$0 < \sup_{k\in\mathbb{N}} \lambda_{\max} B_k^2 < 1\tag{3.17}$$

$$\gamma \geqslant \frac{1}{\sup_{k\in\mathbb{N}} \lambda_{\max} B_k^2}\tag{3.18}$$

$$\ln(\sup_{k\in\mathbb{N}} \lambda_{\max} B_k^2) < -(\sigma-\lambda)(t_k - t_{k-1}),\ k\in\mathbb{N},\tag{3.19}$$

then the unique equilibrium point $(u_*, v_*)^T$ of the system (2.6) is globally exponentially stable with convergence rate $\frac{\lambda}{2}$, where I represents identity matrix, and

$$c = \lambda_{\max}\Big[-2\lambda_1 Q + 3P + 2\begin{pmatrix} \alpha_1 & \frac{1}{2}(\alpha_2+\beta_1) \\ * & \beta_2 \end{pmatrix}\Big] + \gamma e^{\lambda\tau}(\lambda_{\max}P)I\Big] > 0. \tag{3.20}$$

Proof. Firstly, the conditions of Theorem 3.1 warrant that $(u_*, v_*)^T$ is the unique equilibrium point of the system (2.6).

Consider the Lyapunov functional as follows

$$V(t,U) = \int_\Omega U^T(t,x)U(t,x)dx = \int_\Omega |U(t,x)|^T|U(t,x)|dx.$$

Then Poincare inequality Lemma ([1, Lemma 4]) and (3.15) yield that for $t \in [t_{k-1}, t_k), t \in \mathbb{N}$,

$$\begin{aligned} D^+V(t,U(t)) =& 2\int_\Omega U^T\Big(Q\Delta U + \phi(U) + P(U(t,x) - U(t-\tau(t),x))\Big)dx \\ \leqslant& \int_\Omega |U|^T\Big[-2\lambda_1 Q + 3P + 2\begin{pmatrix} \alpha_1 & \frac{1}{2}(\alpha_2+\beta_1) \\ * & \beta_2 \end{pmatrix}\Big]|U|dx + \lambda_{\max}P\int_\Omega |U(t-\tau(t),x))|^T|U(t-\tau(t),x))|dx \end{aligned} \tag{3.21}$$

According to [2, Theorem 2.1], when

$$\int_\Omega |U(t-\tau(t),x))|^T|U(t-\tau(t),x))|dx \leqslant \gamma e^{\lambda\tau}\int_\Omega |U|^T|U|dx,$$

(3.21) yields

$$\begin{aligned} D^+V(t,U(t)) \leqslant& \int_\Omega |U|^T\Big[-2\lambda_1 Q + 3P + 2\begin{pmatrix} \alpha_1 & \frac{1}{2}(\alpha_2+\beta_1) \\ * & \beta_2 \end{pmatrix}\Big]|U|dx + \lambda_{\max}P\int_\Omega |U(t-\tau(t),x))|^T|U(t-\tau(t),x))|dx \\ \leqslant& cV(t,U(t)), \qquad t \in [t_{k-1}, t_k), t \in \mathbb{N}, \end{aligned} \tag{3.22}$$

where $c > 0$ is defined in (3.20).

On the other hand,

$$V(t_k, U(t_k)) = \int_\Omega U(t_k^-, x)^T B_k B_k U(t_k^-, x)dx \leqslant \lambda_{\max}B_k^2 V(t_k^-, U(t_k^-)),$$

which together with (3.22), and (3.17)-(3.19) means that all the conditions of [1, Theorem 2.1] are satisfied. According to [2, Theorem 2.1], the null solution of the system (3.15) is globally exponentially stable with convergence rate $\frac{\lambda}{2}$. And hence, the unique equilibrium point $(u_*, v_*)^T$ of the system (2.6) is globally exponentially stable with convergence rate $\frac{\lambda}{2}$.

4. Exponential stability in the pth moment ($p \geqslant 1$)

In this section, the system (3.15) is rewritten as follows,

$$\begin{cases} \dfrac{\partial U_1}{\partial t} = q_1\Delta U_1 + \phi_1(U) + p_1(U_1(t,x) - U_1(t-\tau(t),x)), & t \geqslant 0,\ t \neq t_k,\ x \in \Omega, \\ \dfrac{\partial U_2}{\partial t} = q_2\Delta U_2 + \phi_2(U) + p_2(U_2(t,x) - U_2(t-\tau(t),x)), & t \geqslant 0,\ t \neq t_k,\ x \in \Omega, \\ U(t_k^+,x) = B_k U(t_k^-,x), \quad k = 1,2,\cdots \\ \dfrac{\partial U}{\partial \nu} = 0, \quad x \in \partial\Omega,\ t \geqslant 0, \\ U(s,x) = \xi(s,x), \quad s \in [-\tau,0],\ x \in \Omega, \end{cases} \tag{4.1}$$

where $\phi_i(U)$ is defined in (3.9). Obviously, the null solution of (4.1) is corresponding to the equilibrium point $(u_*, v_*)^T$ of the impulsive ecosystem (2.6). In this section, the author assumes $u(t_k^-,x) = u(t_k,x)$ and $v(t_k^-,x) = v(t_k,x)$. Denote by $\mathcal{L}^2(\Omega)$ the space of all real-valued square integrable functions with the inner product $\langle\chi,\eta\rangle = \int_\Omega \chi(x)\eta(x)dx$, for $\chi,\eta \in \mathcal{L}^2(\Omega)$ which derives the norm $\|\chi\| = (\int_\Omega \chi^2(x)dx)^{\frac{1}{2}}$ for $\chi \in \mathcal{L}^2(\Omega)$. Denote by $\Delta = \sum_{j=1}^{m} \frac{\partial^2}{\partial x_j^2}$ the Laplace operator, with domain $\mathcal{D}(\Delta) = W_0^{1,2}(\Omega) \bigcap W_0^{2,2}(\Omega)$, which generates a strongly continuous semigroup $e^{t\Delta}$, where $W_0^{1,2}(\Omega)$ and $W_0^{2,2}(\Omega)$ are the Sobolev spaces with compactly supported sets. Next, the definition of the mild solution of the system (4.1) will be introduced. For convenience, the author takes $U(t) = (U_1(t,\cdot), U_2(t,\cdot))^T$ and $U = \{U(t,\cdot)\}_{[0,T]}$ for any given $T > 0$ such that each $U_i(t)$ is a $\mathcal{L}^2(\Omega)$-valued function.

Definition 1. A $\mathcal{L}^2(\Omega)$-valued function $U = \{U(t)\}_{[0,T]}$ is called a mild solution of (4.1) if for any $i \in \{1,2\}$, $U_i(t,x) \in C([0,T];\mathcal{L}^2(\Omega))$ satisfies $\int_0^t \|U_i(s)\|^p ds < \infty$, $i = 1,2$, and the following integral equations hold for any $t \in [0,T]$ and $x \in \Omega$,

$$\begin{cases} U_1(t,x) = e^{q_1 t\Delta}\xi_1(0,x) + \displaystyle\int_0^t e^{q_1(t-s)\Delta}[\phi_1(U(s,x)) + p_1(U_1(s,x) - U_1(s-\tau(s),x))]ds + e^{q_1 t\Delta}\sum_{0<t_k<t} e^{-q_1 t_k\Delta}(b_{k1}-1)U_1(t_k,x), & t \geqslant 0, \\ U_2(t,x) = e^{q_2 t\Delta}\xi_2(0,x) + \displaystyle\int_0^t e^{q_2(t-s)\Delta}[\phi_2(U(s,x)) + p_2(U_2(s,x) - U_2(s-\tau(s),x))]ds + e^{q_2 t\Delta}\sum_{0<t_k<t} e^{-q_2 t_k\Delta}(b_{k2}-1)U_2(t_k,x), & t \geqslant 0, \end{cases} \tag{4.2}$$

and

$$\frac{\partial U}{\partial \nu} = 0, \quad x \in \partial\Omega,\ t \geqslant 0,$$
$$U(s,x) = \xi(s,x), \quad s \in [-\tau,0],\ x \in \Omega.$$

Assume, in addition,

(H3) $\|e^{t\Delta}\| \leqslant Me^{-\gamma t}$, where $M > 0$ and $\gamma > 0$ are constants.

Remark 5. $\|e^{t\Delta}\| \triangleq \|e^{t\Delta}\|_2 = \sup\limits_{\|w\|_{L^2(\Omega)}=1} \|e^{t\Delta}w\|_{L^2(\Omega)}$ (see [46]). And Definition 1 is well defined in view of [46] and [49, Lemma 3.1].

Theorem 4.1. Assume that the condition (H3) and all the conditions of Theorem 3.1 hold, and if, in addition,

$$0 < \beta < 1, \tag{4.3}$$

then the unique equilibrium point $(u_*, v_*)^T$ of the system (2.6) is globally exponential stability in the pth moment ($p \geqslant 1$), where $\mu = \inf_{k\in\mathbb{N}}(t_{k+1} - t_k) > 0$, $\beta = \max\{\beta_1, \beta_2\}$,

$$\begin{aligned}\beta_1 =& 4^{p-1}\Big[\frac{M^p 2^{p-1}}{q_1^p\gamma^p}\Big((|a_1 v_* - b_1 v_*^2 - c_1| + M_2|a_1 - 2b_1 v_*| + b_1 M_2^2)^p \\ &+ (|a_1 u_* - 2b_1 u_* v_*| + M_1|a_1 - 2b_1 v_*| + 2M_2 b_1 u_* + 2M_1 M_2 b_1)^p\Big) + \frac{2M^p p_1^p}{q_1^p\gamma^p} + M^{2p}(\max_k |b_{k1} - 1|)^p\Big(1 + \frac{1}{q_1\gamma\mu}\Big)^p\Big]\end{aligned} \tag{4.4}$$

and

$$\begin{aligned}\beta_2 =& 4^{p-1}\Big[\frac{M^p 2^{p-1}}{q_2^p\gamma^p}\Big((| - a_2 v_* + 2b_2 u_* v_* - c_2| + M_2|a_2 - 2b_2 u_*| + 2M_1 b_2 v_* + 2M_1 M_2 b_2)^p \\ &+ (|a_2 u_* - b_2 u_*^2| + M_1|a_2 - 2b_2 u_*| + b_2 M_1^2)^p\Big) + \frac{2p_2^p M^p}{q_2^p\gamma^p} + M^{2p}(\max_k |b_{k2} - 1|)^p\Big(1 + \frac{1}{q_2\gamma\mu}\Big)^p\Big]\end{aligned} \tag{4.5}$$

Proof. Under all the conditions of Theorem 3.1, obviously $(u_*, v_*)^T$ is the unique equilibrium point of the system (2.6). Below, it only need to prove that the unique equilibrium point is globally exponential stability in the pth moment, or the null solution of the system (4.1) is globally exponential stability in the pth moment, which can be derived by the Banach contraction mapping principle. To utilize the principle, the author need the following five main steps to achieve the goal.

Step 1. Formulating a contraction mapping on a suitable complete metric space.

Let Γ be the normed space of all pth moment continuous processes consisting of functions $\begin{pmatrix} U_1(t,x) \\ U_2(t,x)\end{pmatrix}$ at $t \geqslant 0$ with $t \neq t_k$ such that $e^{\alpha t}\|U_i(t)\|^p \to 0$ for each $i = 1, 2$ if $t \to +\infty$. Moreover, the author equips the space Γ with the following distance

$$dist\Big(\begin{pmatrix} U_1 \\ U_2\end{pmatrix}, \begin{pmatrix} V_1 \\ V_2\end{pmatrix}\Big) = \Big[\max\Big\{\sup_{t\geqslant-\tau}\|U_1(t,x) - V_1(t,x)\|^p, \sup_{t\geqslant-\tau}\|U_2(t,x) - V_2(t,x)\|^p\Big\}\Big]^{\frac{1}{p}}, \quad \forall \begin{pmatrix} U_1 \\ U_2\end{pmatrix}, \begin{pmatrix} V_1 \\ V_2\end{pmatrix} \in \Gamma, \tag{4.6}$$

then Γ is a complete metric space under the distance (4.6), where α is a positive scalar with $0 < \alpha < \min\{q_1\gamma, q_2\gamma\}$.

Construct an operator Θ with $\Theta = (\Theta_1, \Theta_2)^T$ such that for any given $U = (U_1, U_2)^T \in \Gamma$,

$$\begin{cases} \Theta_1(U_1)(t,x) = e^{q_1 t\Delta}\xi_1(0,x) + \int_0^t e^{q_1(t-s)\Delta}[\phi_1(U(s,x)) + p_1(U_1(s,x) - U_1(s-\tau(s),x))]ds + e^{q_1 t\Delta}\sum_{0<t_k<t} e^{-q_1 t_k\Delta}(b_{k1} - 1)U_1(t_k,x), & t \geqslant 0, \\ \Theta_2(U_2)(t,x) = e^{q_2 t\Delta}\xi_2(0,x) + \int_0^t e^{q_2(t-s)\Delta}[\phi_2(U(s,x)) + p_2(U_2(s,x) - U_2(s-\tau(s),x))]ds + e^{q_2 t\Delta}\sum_{0<t_k<t} e^{-q_2 t_k\Delta}(b_{k2} - 1)U_2(t_k,x), & t \geqslant 0, \\ \frac{\partial\Theta(U)}{\partial\nu} = 0, \quad x \in \partial\Omega,\ t \geqslant 0, \\ \Theta(U)(s,x) = \xi(s,x), \quad s \in [-\tau, 0],\ x \in \Omega. \end{cases} \tag{4.7}$$

Below, the author has to prove that the above-defined Θ is really a contraction mapping $\Theta : \Gamma \to \Gamma$.

Step 2. For $t \in [0, +\infty) \setminus \{t_k\}_{k=1}^{\infty}$, the author claims that both $\Theta_1(U_1)(t,x)$ and $\Theta_2(U_2)(t,x)$ are pth moment continuous.

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Indeed, let δ be small enough scalar, a routine proof yields that if $\delta \to 0$,

$$\begin{aligned}
&\|\Theta_1(U_1)(t+\delta,x)-\Theta_1(U_1)(t,x)\|^p \leqslant 4^{p-1}\|e^{q_1(t+\delta)\Delta}\xi_1(0,x)-e^{q_1t\Delta}\xi_1(0,x)\|^p\\
&+4^{p-1}\|\int_0^{t+\delta} e^{q_1(t+\delta-s)\Delta}\phi_1(U(s,x))ds-\int_0^t e^{q_1(t-s)\Delta}\phi_1(U(s,x))ds\|^p\\
&+4^{p-1}\|\int_0^{t+\delta} e^{q_1(t+\delta-s)\Delta}[p_1(U_1(s,x)-U_1(s-\tau(s),x))]ds-\int_0^t e^{q_1(t-s)\Delta}[p_1(U_1(s,x)-U_1(s-\tau(s),x))]ds\|^p\\
&+4^{p-1}\|e^{q_1(t+\delta)\Delta}\sum_{0<t_k<t+\delta} e^{-q_1t_k\Delta}(b_{k1}-1)U_1(t_k,x)-e^{q_1t\Delta}\sum_{0<t_k<t} e^{-q_1t_k\Delta}(b_{k1}-1)U_1(t_k,x)\|^p\to 0,\ \ \forall\, t\in[0,+\infty)\setminus\{t_k\}_{k=1}^{\infty}.
\end{aligned}\tag{4.8}$$

Similarly, if $\delta \to 0$,

$$\|\Theta_2(U_2)(t+\delta,x)-\Theta_2(U_2)(t,x)\|^p\to 0,\ \ \forall\, t\in[0,+\infty)\setminus\{t_k\}_{k=1}^{\infty}.\tag{4.9}$$

Hence, (4.8) and (4.9) prove the claim.

Step 3. It follows by a routine proof that

$$\begin{aligned}
&\lim_{\delta\to0^+}\Theta_1(U_1)(t_k+\delta,x)=b_{k1}\Theta_1(U_1)(t_k,x)\text{ and }\lim_{\delta\to0^-}\Theta_1(U_1)(t_k+\delta,x)=\Theta_1(U_1)(t_k,x)\,;\\
&\lim_{\delta\to0^+}\Theta_2(U_2)(t_k+\delta,x)=b_{k2}\Theta_2(U_2)(t_k,x)\text{ and }\lim_{\delta\to0^-}\Theta_2(U_2)(t_k+\delta,x)=\Theta_2(U_2)(t_k,x).
\end{aligned}\tag{4.10}$$

Step 4. The author claims that

$$e^{\alpha t}\left\|\Theta\begin{pmatrix}U_1\\U_2\end{pmatrix}\right\|^p\to 0,\ \ \text{if } t\to+\infty,\tag{4.11}$$

i.e.,

$$\begin{cases} e^{\alpha t}\|\Theta_1(U_1(t,x))\|^p\to 0,\\ e^{\alpha t}\|\Theta_2(U_2(t,x))\|^p\to 0,\end{cases}\ \ \text{if } t\to+\infty.\tag{4.12}$$

Indeed,

$$\begin{aligned}
&e^{\alpha t}\|\Theta_1(U_1)(t,x)\|^p=e^{\alpha t}\|e^{q_1t\Delta}\xi_1(0,x)+\int_0^t e^{q_1(t-s)\Delta}[\phi_1(U(s,x))+p_1(U_1(s,x)-U_1(s-\tau(s),x))]ds+e^{q_1t\Delta}\sum_{0<t_k<t}e^{-q_1t_k\Delta}(b_{k1}-1)U_1(t_k,x)\|^p\\
&\leqslant 5^{p-1}e^{\alpha t}\|e^{q_1t\Delta}\xi_1(0,x)\|^p+5^{p-1}e^{\alpha t}\|\int_0^t e^{q_1(t-s)\Delta}\phi_1(U(s,x))ds\|^p+5^{p-1}e^{\alpha t}\|\int_0^t e^{q_1(t-s)\Delta}p_1U_1(s,x)ds\|^p\\
&+5^{p-1}e^{\alpha t}\|\int_0^t e^{q_1(t-s)\Delta}p_1U_1(s-\tau(s),x)ds\|^p+5^{p-1}e^{\alpha t}\|e^{q_1t\Delta}\sum_{0<t_k<t}e^{-q_1t_k\Delta}(b_{k1}-1)U_1(t_k,x)\|^p,\ t\geqslant 0,
\end{aligned}\tag{4.13}$$

Moreover,

$$e^{\alpha t}\|e^{q_1t\Delta}\xi_1(0,x)\|^p\leqslant M^pe^{\alpha t}e^{-\gamma q_1t}\|\xi_1(0,x)\|^p\to 0,\text{ if } t\to+\infty.\tag{4.14}$$

The condition (H3), (3.11) and Holder inequality yield

$$\begin{aligned}
&e^{\alpha t}\|\int_0^t e^{q_1(t-s)\Delta}\phi_1(U(s,x))ds\|^p\\
\leqslant&2^{p-1}M^pe^{\alpha t}\Big[\Big(\alpha_1\int_0^t e^{-q_1\gamma(t-s)}\|U_1\|ds\Big)^p+\Big(\beta_1\int_0^t e^{-q_1\gamma(t-s)}\|U_2\|ds\Big)^p\Big]\\
\leqslant&2^{p-1}M^pe^{\alpha t}\Big[\alpha_1^p(\frac{1}{q_1\gamma})^{p-1}\int_0^t e^{-q_1\gamma(t-s)}\|U_1\|^pds+\beta_1^p(\frac{1}{q_1\gamma})^{p-1}\int_0^t e^{-q_1\gamma(t-s)}\|U_2\|^pds\Big]\\
\leqslant&2^{p-1}M^p\Big[\alpha_1^p(\frac{1}{q_1\gamma})^{p-1}\int_0^t e^{-(q_1\gamma-\alpha)(t-s)}e^{\alpha s}\|U_1(s,x)\|^pds+\beta_1^p(\frac{1}{q_1\gamma})^{p-1}\int_0^t e^{-q_1\gamma(t-s)}\|U_2\|^pds\Big].
\end{aligned}\tag{4.15}$$

On the other hard, $e^{\alpha t}\|U_i(t)\|^p\to 0$ means that for any $\varepsilon>0$, there exists $t^*>0$ such that all $e^{\alpha t}\|U_i(t)\|^p<\varepsilon$. And so,

$$\begin{aligned}
&\int_0^t e^{-(q_1\gamma-\alpha)(t-s)}e^{\alpha s}\|U_1(s,x)\|^pds\leqslant\int_0^{t^*}e^{-(q_1\gamma-\alpha)(t-s)}\max_{s\in[0,t^*]}(e^{\alpha s}\|U_1(s,x)\|^p)ds+\int_{t^*}^t e^{-(q_1\gamma-\alpha)(t-s)}\varepsilon ds\\
\leqslant&\max_{s\in[0,t^*]}(e^{\alpha s}\|U_1(s,x)\|^p)e^{-(q_1\gamma-\alpha)t}\frac{1}{q_1\gamma-\alpha}e^{(q_1\gamma-\alpha)t^*}+\varepsilon\frac{1}{q_1\gamma-\alpha},
\end{aligned}$$

which together with the arbitrariness of ε implies that $\int_0^t e^{-(q_1\gamma-\alpha)(t-s)}e^{\alpha s}\|U_1(s,x)\|^pds\to 0$ if $t\to+\infty$. Similarly, $\int_0^t e^{-q_1\gamma(t-s)}\|U_2\|^pds\to 0$ if $t\to+\infty$. And hence, if $t\to+\infty$,

$$e^{\alpha t}\|\int_0^t e^{q_1(t-s)\Delta}\phi_1(U(s,x))ds\|^p\to 0.\tag{4.16}$$

Similarly, it is not difficult to prove that if $t\to+\infty$,

$$5^{p-1}e^{\alpha t}\|\int_0^t e^{q_1(t-s)\Delta}p_1U_1(s,x)ds\|^p+5^{p-1}e^{\alpha t}\|\int_0^t e^{q_1(t-s)\Delta}p_1U_1(s-\tau(s),x)ds\|^p\to 0.\tag{4.17}$$

Next, using the definition of Riemann integral $\int_a^b e^sds$ results in

$$\begin{aligned}
&e^{\alpha t}\|e^{q_1t\Delta}\sum_{0<t_k<t}e^{-q_1t_k\Delta}(b_{k1}-1)U_1(t_k,x)\|^p\\
\leqslant&2^{p-1}\max_k|b_{k1}-1|\Big[e^{-(pq_1\gamma-\alpha)t}\Big(\sum_{0<t_k\leqslant t^*}e^{q_1\gamma t_k}\|U_1(t_k,x)\|\Big)^p+\Big(e^{-q_1\gamma t}\sum_{t^*<t_k<t}e^{q_1\gamma t_k}e^{\frac{1}{p}\alpha(t-t_k)}[e^{\alpha t_k}\|U_1(t_k,x)\|^p]^{\frac{1}{p}}\Big)^p\Big]\\
\leqslant&2^{p-1}\max_k|b_{k1}-1|\Big[e^{-(pq_1\gamma-\alpha)t}\Big(\sum_{0<t_k\leqslant t^*}e^{q_1\gamma t_k}\|U_1(t_k,x)\|\Big)^p+\Big(\varepsilon^{\frac{1}{p}}e^{-q_1\gamma t}\sum_{t^*<t_k<t}e^{q_1\gamma t_k}e^{\frac{1}{p}\alpha(t-t_k)}\Big)^p\Big]\\
\leqslant&2^{p-1}\max_k|b_{k1}-1|\Big[e^{-(pq_1\gamma-\alpha)t}\Big(\sum_{0<t_k\leqslant t^*}e^{q_1\gamma t_k}\|U_1(t_k,x)\|\Big)^p+\varepsilon\Big(e^{-(q_1\gamma-\frac{\alpha}{p})t}\frac{1}{\mu}\sum_{t^*<t_k<t}e^{(q_1\gamma-\frac{\alpha}{p})t_k}(t-t_k)\Big)^p\Big]\\
\leqslant&2^{p-1}\max_k|b_{k1}-1|\Big[e^{-(pq_1\gamma-\alpha)t}\Big(\sum_{0<t_k\leqslant t^*}e^{q_1\gamma t_k}\|U_1(t_k,x)\|\Big)^p+\varepsilon\frac{1}{(q_1\gamma-\frac{\alpha}{p})^p}\Big]\to 0.
\end{aligned}\tag{4.18}$$

Combining (4.13)-(4.18) yields

$$e^{\alpha t}\|\Theta_1(U_1)(t,x)\|^p\to 0,\quad t\to+\infty,\tag{4.19}$$

and similarly

$$e^{\alpha t}\|\Theta_2(U_2)(t,x)\|^p\to 0,\quad t\to+\infty.$$

It follows from the above three steps that

$$\Theta(\Gamma) \subset \Gamma. \tag{4.20}$$

Step 5. Finally, the author will prove that Θ is a contractive mapping on Γ.

Indeed, for any $\begin{pmatrix} U_1 \\ U_2 \end{pmatrix}, \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} \in \Gamma$,

Obviously, $|U_i| \leqslant M_i$. And then it follows from (3.9) that

$$\begin{aligned}
&|\phi_1(U) - \phi_1(V)| \leqslant |a_1 v_* - b_1 v_*^2 - c_1| \cdot |U_1 - V_1| + |a_1 u_* - 2b_1 u_* v_*| \cdot |U_2 - V_2| + |a_1 - 2b_1 v_*| \cdot |U_1 U_2 - V_1 V_2| \\
&+ b_1 u_* |U_2^2 - V_2^2| + b_1 |U_1 U_2^2 - V_1 V_2^2| \\
&\leqslant \Big(|a_1 v_* - b_1 v_*^2 - c_1| + M_2 |a_1 - 2b_1 v_*| + b_1 M_2^2\Big)|U_1 - V_1| + \Big(|a_1 u_* - 2b_1 u_* v_*| + M_1 |a_1 - 2b_1 v_*| + 2M_2 b_1 u_* + 2M_1 M_2 b_1\Big)|U_2 - V_2|
\end{aligned} \tag{4.21}$$

$$\begin{aligned}
&|\phi_2(U) - \phi_2(V)| \leqslant |-a_2 v_* + 2b_2 u_* v_* - c_2| \cdot |U_1 - V_1| + |-a_2 u_* + b_2 u_*^2| \cdot |U_2 - V_2| + |-a_2 + 2b_2 u_*| \cdot |U_1 U_2 - V_1 V_2| \\
&+ b_2 v_* |U_1^2 - V_1^2| + b_2 |U_1^2 U_2 - V_1^2 V_2| \\
&\leqslant \Big(|-a_2 v_* + 2b_2 u_* v_* - c_2| + M_2 |a_2 - 2b_2 u_*| + 2M_1 b_2 v_* + 2M_1 M_2 b_2\Big)|U_1 - V_1| + \Big(|a_2 u_* - b_2 u_*^2| + M_1 |a_2 - 2b_2 u_*| + b_2 M_1^2\Big)|U_2 - V_2|
\end{aligned} \tag{4.22}$$

Combining (4.21), Lemma 2.3, Holder inequality, the condition (H3) and (4.6) results in

$$\begin{aligned}
&\| \int_0^t e^{q_1(t-s)\Delta}[\phi_1(U(s,x)) - \phi_1(V(s,x))]ds\|^p \\
&\leqslant M^p \Big[2^{p-1}(|a_1 v_* - b_1 v_*^2 - c_1| + M_2|a_1 - 2b_1 v_*| + b_1 M_2^2)^p \frac{1}{(q_1\gamma)^{p-1}} \int_0^t e^{-q_1\gamma(t-s)} \sup_{t \geqslant -\tau} \|U_1 - V_1\|^p ds \\
&+ 2^{p-1}(|a_1 u_* - 2b_1 u_* v_*| + M_1|a_1 - 2b_1 v_*| + 2M_2 b_1 u_* + 2M_1 M_2 b_1)^p \frac{1}{(q_1\gamma)^{p-1}} \int_0^t e^{-q_1\gamma(t-s)} \sup_{t \geqslant -\tau} \|U_2 - V_2\|^p ds\Big] \\
&\leqslant \frac{M^p 2^{p-1}}{(q_1\gamma)^p} \Big[(|a_1 v_* - b_1 v_*^2 - c_1| + M_2|a_1 - 2b_1 v_*| + b_1 M_2^2)^p \\
&+ (|a_1 u_* - 2b_1 u_* v_*| + M_1|a_1 - 2b_1 v_*| + 2M_2 b_1 u_* + 2M_1 M_2 b_1)^p\Big] \cdot [dist(U,V)]^p,
\end{aligned}$$

and so

$$\begin{aligned}
&\sup_{t \geqslant -\tau} \| \int_0^t e^{q_1(t-s)\Delta}[\phi_1(U(s,x)) - \phi_1(V(s,x))]ds\|^p \\
&\leqslant \frac{M^p 2^{p-1}}{q_1^p \gamma^p} \Big[(|a_1 v_* - b_1 v_*^2 - c_1| + M_2|a_1 - 2b_1 v_*| + b_1 M_2^2)^p \\
&+ (|a_1 u_* - 2b_1 u_* v_*| + M_1|a_1 - 2b_1 v_*| + 2M_2 b_1 u_* + 2M_1 M_2 b_1)^p\Big] \cdot [dist(U,V)]^p
\end{aligned} \tag{4.23}$$

Similarly,

$$\begin{aligned}
&\| \int_0^t e^{q_1(t-s)\Delta} p_1 [U_1(s,x) - V_1(s,x)]ds\|^p \\
&\leqslant M^p p_1^p \frac{1}{(q_1\gamma)^{p-1}} \cdot \int_0^t e^{-q_1\gamma(t-s)} ds \cdot \sup_{t \geqslant -\tau} \|U_1(t,x) - V_1(t,x)\|^p \\
&\leqslant \frac{M^p p_1^p}{q_1^p \gamma^p} \sup_{t \geqslant -\tau} \|U_1(t,x) - V_1(t,x)\|^p,
\end{aligned}$$

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and then

$$\sup_{t\geqslant-\tau}\|\int_0^t e^{q_1(t-s)\Delta}p_1[U_1(s,x)-V_1(s,x)]ds\|^p\leqslant\frac{M^pp_1^p}{q_1^p\gamma^p}\sup_{t\geqslant-\tau}\|U_1(t,x)-V_1(t,x)\|^p\leqslant\frac{M^pp_1^p}{q_1^p\gamma^p}[dist(U,V)]^p, \quad (4.24)$$

$$\sup_{t\geqslant-\tau}\|\int_0^t e^{q_1(t-s)\Delta}p_1[U_1(s-\tau(s),x)-V_1(s-\tau(s),x)]ds\|^p\leqslant\frac{M^pp_1^p}{q_1^p\gamma^p}[dist(U,V)]^p, \quad (4.25)$$

Suppose $t_{j-1}<t\leqslant t_j$, then the definition of Riemann integral $\int_a^b e^s ds$ yields

$$\begin{aligned}
&\|e^{q_1t\Delta}\sum_{0<t_k<t}e^{-q_1t_k\Delta}(b_{k1}-1)[U_1(t_k,x)-V_1(t_k,x)]\|^p\\
\leqslant&M^{2p}(\max_k|b_{k1}-1|)^p\Big[e^{-q_1\gamma t}\Big(e^{q_1\gamma t_{j-1}}\|U_1(t_{j-1},x)-V_1(t_{j-1},x)\|+\frac{1}{\mu}\sum_{0<t_k\leqslant t_{j-2}}e^{q_1\gamma t_k}\|U_1(t_k,x)-V_1(t_k,x)\|(t_{k+1}-t_k)\Big)\Big]^p\\
\leqslant&M^{2p}(\max_k|b_{k1}-1|)^p\Big[e^{-q_1\gamma t}\Big(e^{q_1\gamma t_{j-1}}+\frac{1}{\mu}\sum_{0<t_k\leqslant t_{j-2}}e^{q_1\gamma t_k}(t_{k+1}-t_k)\Big)\cdot dist(U,V)\Big]^p\\
\leqslant&M^{2p}(\max_k|b_{k1}-1|)^p\Big(1+\frac{1}{q_1\gamma\mu}\Big)^p\cdot[dist(U,V)]^p.
\end{aligned} \quad (4.26)$$

It follows by (4.23)-(4.26) that

$$\begin{aligned}
&\sup_{t\geqslant-\tau}\|\Theta_1(U_1)(t,x)-\Theta_1(V_1)(t,x)\|^p\\
\leqslant&4^{p-1}\sup_{t\geqslant-\tau}\|\int_0^t e^{q_1(t-s)\Delta}[\phi_1(U(s,x))-\phi_1(V(s,x))]ds\|^p+4^{p-1}\sup_{t\geqslant-\tau}\|\int_0^t e^{q_1(t-s)\Delta}p_1[U_1(s,x)-V_1(s,x)]ds\|^p\\
&+4^{p-1}\sup_{t\geqslant-\tau}\|\int_0^t e^{q_1(t-s)\Delta}p_1[U_1(s-\tau(s),x)-V_1(s-\tau(s),x)]ds\|^p+4^{p-1}\sup_{t\geqslant-\tau}\|e^{q_1t\Delta}\sum_{0<t_k<t}e^{-q_1t_k\Delta}(b_{k1}-1)[U_1(t_k,x)-V_1(t_k,x)]\|^p\\
\leqslant&4^{p-1}\Bigg[\frac{M^p2^{p-1}}{q_1^p\gamma^p}\Big((|a_1v_*-b_1v_*^2-c_1|+M_2|a_1-2b_1v_*|+b_1M_2^2)^p+(|a_1u_*-2b_1u_*v_*|+M_1|a_1-2b_1v_*|+2M_2b_1u_*+2M_1M_2b_1)^p\Big)\\
&+\frac{2M^pp_1^p}{q_1^p\gamma^p}+M^{2p}(\max_k|b_{k1}-1|)^p\Big(1+\frac{1}{q_1\gamma\mu}\Big)^p\Bigg][dist(U,V)]^p\\
=&\beta_1[dist(U,V)]^p\leqslant\beta[dist(U,V)]^p
\end{aligned} \quad (4.27)$$

(4.22), Holder inequality, Lemma 2.3 and the condition (H3) yield

$$\begin{aligned}
&\|\int_0^t e^{q_2(t-s)\Delta}[\phi_2(U(s,x))-\phi_2(V(s,x))]ds\|^p\\
\leqslant&M^p2^{p-1}\Big(|-a_2v_*+2b_2u_*v_*-c_2|+M_2|a_2-2b_2u_*|+2M_1b_2v_*+2M_1M_2b_2\Big)^p\Big(\int_0^t e^{-q_2\gamma(t-s)}\|U_1-V_1\|ds\Big)^p\\
&+M^p2^{p-1}\Big(|a_2u_*-b_2u_*^2|+M_1|a_2-2b_2u_*|+b_2M_1^2\Big)^p\Big(\int_0^t e^{-q_2\gamma(t-s)}\|U_2-V_2\|ds\Big)^p\\
\leqslant&M^p2^{p-1}\Big(|-a_2v_*+2b_2u_*v_*-c_2|+M_2|a_2-2b_2u_*|+2M_1b_2v_*+2M_1M_2b_2\Big)^p(\frac{1}{q_2\gamma})^{p-1}\int_0^t e^{-q_2\gamma(t-s)}ds\cdot[dist(U,V)]^p \quad (4.28)\\
&+M^p2^{p-1}\Big(|a_2u_*-b_2u_*^2|+M_1|a_2-2b_2u_*|+b_2M_1^2\Big)^p(\frac{1}{q_2\gamma})^{p-1}\int_0^t e^{-q_2\gamma(t-s)}ds\cdot[dist(U,V)]^p\\
\leqslant&\frac{M^p2^{p-1}}{q_2^p\gamma^p}\Big[\Big(|-a_2v_*+2b_2u_*v_*-c_2|+M_2|a_2-2b_2u_*|+2M_1b_2v_*+2M_1M_2b_2\Big)^p\\
&+\Big(|a_2u_*-b_2u_*^2|+M_1|a_2-2b_2u_*|+b_2M_1^2\Big)^p\Big][dist(U,V)]^p.
\end{aligned}$$

Similarly as (4.24) and (4.25), one can get

$$\begin{aligned}&\|\int_0^t e^{q_2(t-s)\Delta}p_2[U_2(s,x)-V_2(s,x)]ds\|^p\\ \leqslant & p_2^pM^p\Big(\int_0^t e^{-q_2\gamma(t-s)}\|U_2(s,x)-V_2(s,x)\|ds\Big)^p\\ \leqslant & \frac{p_2^pM^p}{q_2^p\gamma^p}[dist(U,V)]^p\end{aligned} \tag{4.29}$$

$$\|\int_0^t e^{q_2(t-s)\Delta}p_2[U_2(s-\tau(s),x)-V_2(s-\tau(s),x)]ds\|^p\leqslant\frac{p_2^pM^p}{q_2^p\gamma^p}[dist(U,V)]^p \tag{4.30}$$

$$\sup_{t\geqslant-\tau}\|e^{q_2t\Delta}\sum_{0<t_k<t}e^{-q_2t_k\Delta}(b_{k2}-1)[U_2(t_k,x)-V_2(t_k,x)]\|^p\leqslant M^{2p}(\max_k|b_{k2}-1|)^p\Big(1+\frac{1}{q_2\gamma\mu}\Big)^p\cdot[dist(U,V)]^p \tag{4.31}$$

Combining (4.28)-(4.31) results in

$$\begin{aligned}&\sup_{t\geqslant-\tau}\|\Theta_2(U_2)(t,x)-\Theta_2(V_2)(t,x)\|^p\\ \leqslant & 4^{p-1}\sup_{t\geqslant-\tau}\|\int_0^t e^{q_2(t-s)\Delta}[\phi_2(U(s,x))-\phi_2(V(s,x))]ds\|^p+4^{p-1}\sup_{t\geqslant-\tau}\|\int_0^t e^{q_2(t-s)\Delta}p_2[U_2(s,x)-V_2(s,x)]ds\|^p\\ &+4^{p-1}\sup_{t\geqslant-\tau}\|\int_0^t e^{q_2(t-s)\Delta}p_2[U_2(s-\tau(s),x)-V_2(s-\tau(s),x)]ds\|^p+4^{p-1}\sup_{t\geqslant-\tau}\|e^{q_2t\Delta}\sum_{0<t_k<t}e^{-q_2t_k\Delta}(b_{k2}-1)[U_2(t_k,x)-V_2(t_k,x)]\|^p\\ \leqslant & 4^{p-1}\Big[\frac{M^p2^{p-1}}{q_2^p\gamma^p}\Big((|-a_2v_*+2b_2u_*v_*-c_2|+M_2|a_2-2b_2u_*|+2M_1b_2v_*+2M_1M_2b_2)^p\\ &+(|a_2u_*-b_2u_*^2|+M_1|a_2-2b_2u_*|+b_2M_1^2)^p\Big)+\frac{2p_2^pM^p}{q_2^p\gamma^p}+M^{2p}(\max_k|b_{k2}-1|)^p\Big(1+\frac{1}{q_2\gamma\mu}\Big)^p\Big][dist(U,V)]^p\\ =&\beta_2[dist(U,V)]^p\leqslant\beta[dist(U,V)]^p\end{aligned} \tag{4.32}$$

It follows from (4.6), (4.27) and (4.32) that

$$[dist(\Theta(U),\Theta(V))]^p\leqslant\beta[dist(U,V)]^p\Rightarrow dist(\Theta(U),\Theta(V))\leqslant\sqrt{\beta}\,dist(U,V),\qquad\forall\,U,V\in\Gamma, \tag{4.33}$$

where β satisfies $0<\beta<1$. This shows that $\Theta:\Gamma\to\Gamma$ is a contraction mapping such that there exists the fixed point $U=\begin{pmatrix}U_1\\U_2\end{pmatrix}$ of Θ in Γ, which implies that $\begin{pmatrix}U_1\\U_2\end{pmatrix}$ is the solution of the system (4.1), satisfying $e^{\alpha t}\Big\|\begin{pmatrix}U_1\\U_2\end{pmatrix}\Big\|^p\to0$, $t\to+\infty$ so that $e^{\alpha t}\Big\|\begin{pmatrix}u-u_*\\v-v_*\end{pmatrix}\Big\|^p\to0,\ t\to+\infty$. Therefore, the equilibium point $(u_*,v_*)^T$ system (2.6) is globally exponential stability in the pth moment.

5. Numerical examples

Example 5.1. Set

$$M_1=M_2=2,\ a_2=b_1=b_2=c_1=c_2=d=1,\ a_1=2, \tag{5.1}$$

which means that the conditions (H1) and (H2) hold, and direct computation yields

$$u_* = 1 = v_*, \ \alpha_1 = 0, \ \alpha_2 = 6 = \beta_1, \ \beta_2 = 2. \tag{5.2}$$

Particularly, $u_* = 1 = v_*$ doesn't contradict the boundedness condition (H1). Moreover, set

$$Q = \begin{pmatrix} 0.62 & 0 \\ * & 0.83 \end{pmatrix}, \tag{5.3}$$

and

$$\Omega = [0, 1] \times [0, 1], \tag{5.4}$$

then [1, Remark 1] tells that $\lambda_1 = \pi^2$. Direct computation yields

$$\lambda_1 Q - \begin{pmatrix} \alpha_1 & \frac{1}{2}(\alpha_2 + \beta_1) \\ * & \beta_2 \end{pmatrix} = \begin{pmatrix} 6.1192 & -6.0000 \\ -6.0000 & 6.1918 \end{pmatrix} > 0,$$

which means that the condition (3.1) holds.

According to Theorem 3.1, $(1, 1)^T$ is the unique stationary solution of the ecosystem (2.1). Moreover, Theorem 3.2 tells that the unique stationary solution $(1, 1)^T$ of the ecosystem (2.1) is globally asymptotically stable .

Example 5.2. Suppose (5.1)-(5.4) hold, and then $\lambda_1 = \pi^2$. Set

$$P = \begin{pmatrix} 0.2 & 0 \\ * & 0.3 \end{pmatrix}, \quad B_k \equiv \begin{pmatrix} 0.5 & 0 \\ * & 0.6 \end{pmatrix}, \quad k \in \mathbb{N}. \tag{5.5}$$

And hence $\sup\limits_{k\in\mathbb{N}} \lambda_{\max} B_k^2 = 0.36$, and the condition (3.17) holds. Let $\gamma = 3$, and then the condition (3.18) holds.

Let

$$\lambda = 0.5, \quad t_k - t_{k-1} \equiv 0.5, \ k \in \mathbb{N},$$

Direct computation yields

$$c = \lambda_{\max}\Big[-2\lambda_1 Q + 3P + 2\begin{pmatrix} \alpha_1 & \frac{1}{2}(\alpha_2 + \beta_1) \\ * & \beta_2 \end{pmatrix} + \gamma e^{\lambda\tau}\lambda_{\max} P\Big] = 1.5949 > 0.$$

Set $\sigma = 1.6$, and then $\sigma - \lambda = 0.6 > c$. Moreover,

$$\ln(\sup_{k\in\mathbb{N}} \lambda_{\max} B_k^2) = -1.0217 < -0.5500 = -(\sigma - \lambda)(t_k - t_{k-1}), \ k \in \mathbb{N},$$

and the condition (3.19) is satisfied. Now all the conditions of Theorem 3.3 are satisfied, and Theorem 3.3 yields that the unique equilibrium point $(1, 1)^T$ of the system (2.6) is globally exponentially stable with convergence rate 25% .

On the other hand, if setting $B_k \equiv diag(0.1, 0.2)$ in Example 5.2, direct computation yields that $\frac{\lambda}{2} = 35\%$; if letting $t_k - t_{k-1} \equiv 0.8$, direct computation yields that $\frac{\lambda}{2} = 15\%$. Such comparisons are list as follows,

Table 1. Comparisons the influences on the convergence rate $\frac{\lambda}{2}$ under different pulse amplitude with the same other data

	Case 1	Case 2
Pulse amplitude	$B_k \equiv diag(0.5, 0.6)$	$B_k \equiv diag(0.1, 0.2)$
Pulse intensity	smaller	bigger
Pulse interval	$(t_{k+1} - t_k) \equiv 0.5$	$(t_{k+1} - t_k) \equiv 0.5$
Pulse frequency	same	same
Convergence rate $\frac{\lambda}{2}$	25%	35%

Table 2. Comparisons the influences on the convergence rate $\frac{\lambda}{2}$ under different pulse frequency with the same other data

	Case 1	Case 2
Pulse amplitude	$B_k \equiv diag(0.5, 0.6)$	$B_k \equiv diag(0.5, 0.6)$
Pulse intensity	same	same
Pulse interval	$(t_{k+1} - t_k) \equiv 0.5$	$(t_{k+1} - t_k) \equiv 0.8$
Pulse frequency	smaller	bigger
Convergence rate $\frac{\lambda}{2}$	25%	15%

Remark 6. Table 1 illuminates that the greater the pulse intensity, the faster the stability of the system. And Table 2 indicates that the more frequent the pulses, the faster the stability of the system.

6. Conclusions

Motivated mainly by some ideas and methods of the related literature [1, 2, 30, 36-39], the author uses variational methods to derive the uniqueness of the equilibrium point of ecosystem so that the global stability can be consider. After giving two global stability criteria, the author proposes numerical examples to illuminate the effectiveness of the theorems. Numerical examples and stability criteria tell that in the face of the epidemic situation, taking a certain frequency of positive and effective epidemic prevention measures is conducive to the stability and control of the epidemic situation. Particularly, the whole method of this paper is different from any existing literature.

Recently, a lot has been reported based on the application of reaction-diffusion system with integer and non-integer order([40-44]). For example, the authors of [40] provide the essential mathematical basis for computational studies of space fractional reaction-diffusion systems, from biological and numerical analysis perspectives, and the stability analysis has a lot of implications for understanding the various spatiotemporal and chaotic behaviors of the species in the spatial domain. How to use the similar methods and impulsive control to deal with the stabilization of fractional reaction-diffusion epidemic model? It is an interesting matter.

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