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Correction of the Koenig formula for the kinetic energy of a rotating solid

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Abstract: An exact solution is obtained for the kinetic energy in the general case of the spatial motion of solids with arbitrary rotation, which differs from the Koenig formula by three additional terms with centrifugal moments of inertia. The description of motion in the Lagrange form and the superposition principle are used, which provides a geometric summation of the velocities and accelerations of the joint motions in the Lagrange form for any particle at any time. The integrand function in the equation for kinetic energy is represented by the sum of the identical velocity components of the joint plane-parallel motions. The moments of inertia in the Koenig formula do not change during movement and can be calculated from the current or initial state of the body. The centrifugal moments change and turn to 0 when rotating relative to the main central axes only for bodies with equal main moments of inertia, for example, for a ball. In other cases, the difference in the main moments of inertia leads to cyclic changes in the kinetic energy with the possible manifestation of precession and nutation, the amplitude of which depends on the angular velocities of rotation of the body. An example of using equations for a robot with one helical and two rotational kinematic pairs is given.

Keywords: kinetic energy, Lagrange variables, the principle of superposition of motions, polar, axial and centrifugal moments of inertia.

1. Introduction

Robots for various purposes are widely used in performing basic and auxiliary operations, moving workpieces and tools along a given trajectory with the required speed and positioning accuracy. Robotization is one of the main factors of industrial development [1-2].

Currently, the most common methods of designing and studying robotic systems are focused on the use of matrix equations and differ in the choice of local coordinate systems [3-4]. The force calculation is usually performed using the kinetostatics method [5-6], which allows us to consider a mobile system as conditionally stationary, in quasi-static equilibrium.

These methods lead to large amounts of calculations with frequent repetition of the same type of procedures, the use of high-performance computing devices that use algorithms for parallel data processing in multiprocessor systems. Application software packages for modeling the work of robots are widely used [7-8]. Due to the complexity of the mathematical apparatus and numerical methods of calculation, it is difficult to assess the influence of auxiliary factors on the results of dynamic analysis of mechanisms.

The problem of ensuring the necessary accuracy of positioning, calculation of force factors in kinematic pairs and dangerous sections of links remain relevant. The search for methods aimed at improving the applied mathematical apparatus, methods for calculating the kinetic energy of the links of mechanisms with the transition from numerical methods of analysis to analytical ones continues [9-10].

Kinetic energy is a mandatory and usually the main component of the motion of absolutely rigid bodies. This requires special responsibility when choosing methods for determining it, for example, in the dynamic analysis of mobile robots and other mechanisms, the complexity of which increases with the development of technological progress. The most popular formula for the spatial motion of solids is the approximate Koenig formula [11-12]

$$E_k = 0,5mv_C^2 + 0,5(\theta_{x,t}^2 J_C^x + \theta_{y,t}^2 J_C^y + \theta_{z,t}^2 J_C^z)\,, \tag{1}$$

where $\theta_{i,t}$, J_C^i is the angular velocities and moments of inertia of the body relative to the central axes, was published in 1751 [12]. The basis for formula (1) is the theorem on the equality of the kinetic energy of a material system to the sum of the energy of the portable translational motion together with the center of mass C and the energy of the rotational motion relative to the coordinate axes moving together with the center of mass.

The aim of the work is 1) to obtain an accurate equation for calculating the kinetic energy of rotating solids, which is acceptable for the dynamic analysis of mechanisms with a developed structure, such as robot manipulators; 2) to estimate the error of equation (1) for various variants of bodies and features of their movement.

Consider the rotation of a body in the x–y plane with respect to the fixed pole P(α_P, β_P) in the space of Lagrange variables [13-14]. When denoting the current coordinates $x_i \in (x, y, z)$, we will use as the Lagrange variables $\alpha_p \in (\alpha, \beta, \gamma)$ the initial coordinates of the particles $\alpha_p = x_p|_{t=0}$. The initial position of an arbitrary point $M(\alpha, \beta)$ is determined by the angle $(\theta_z)_0$ of inclination of the straight line *PM* with respect to the x-axis and the distance between the points $|PM|=L=const$

$$\alpha = \alpha_P + L\cos(\theta_z)_0\,, \qquad \beta = \beta_P + L\sin(\theta_z)_0\,. \tag{2}$$

After turning the straight-line *PM* by an angle $\Delta\theta_x = \theta_x - (\theta_x)_0$, the coordinates of the point can be determined by the same relations,

$$x = \alpha_P + L\cos\theta_z = \alpha_P + L\cos[(\theta_z)_0 + \Delta\theta_z]\,, \qquad y = \beta_P + L\sin[(\theta_z)_0 + \Delta\theta_z]\,.$$

Excluding from these equations the length L and the initial value of the angle $(\theta_x)_0$ using equations (2), we obtain a system of equations of motion,

$$x = \alpha_P + (\alpha - \alpha_P)\cos\Delta\theta_z - (\beta - \beta_P)\sin\Delta\theta_z\,, \qquad y = \beta_P + (\alpha - \alpha_P)\sin\Delta\theta_z + (\beta - \beta_P)\cos\Delta\theta_z\,, \qquad z = \gamma \tag{3}$$

and the corresponding speeds

$$x_t = -\theta_{z,t}(y - \beta_P)\,, \qquad y_t = \theta_{z,t}(x - \alpha_P)\,, \qquad z_t = 0\,. \tag{4}$$

The lower index *t* when denoting the coordinates $x_{i,t}$ and angles of rotation $\theta_{i,t}$ corresponds to the differentiation of the corresponding functions $dx_i / dt \equiv x_{i,t}$, $d\theta_i / dt \equiv \theta_{i,t}$.

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For plane-parallel movements in other planes, the equations can be obtained by using a circular substitution:

when rotating relative to the *x*-axis:

$$x=\alpha, \qquad x_t=0,$$
$$y=\beta_P+(\beta-\beta_P)\cos\Delta\theta_x-(\gamma-\gamma_P)\sin\Delta\theta_x, \qquad y_t=-\theta_{x,t}(z-\gamma_P), \tag{5}$$
$$z=\gamma_P+(\beta-\beta_P)\sin\Delta\theta_x+(\gamma-\gamma_P)\cos\Delta\theta_x, \qquad z_t=\theta_{x,t}(y-\beta_P),$$

when rotating relative to the *y*-axis:

$$x=\alpha_P+(\alpha-\alpha_P)\cos\Delta\theta_y+(\gamma-\gamma_P)\sin\Delta\theta_y, \qquad x_t=\theta_{y,t}(z-\gamma_P),$$
$$y=\beta, \qquad y_t=0, \tag{6}$$
$$z=\gamma_P+(\gamma-\gamma_P)\cos\Delta\theta_y-(\alpha-\alpha_P)\sin\Delta\theta_y, \qquad z_t=-\theta_{y,t}(x-\alpha_P).$$

The lower index when indicating the angle of rotation of the body θ_i indicates the direction of the axis of rotation passing through the pole *P* parallel to the axes of the observer's coordinate system, relative to which the body rotates. Equations (3) – (6) will be used as the initial ones in all further transformations.

For spatial movements, systems of type (6) can be obtained using the superposition principle [15], which allows us to consider complex spatial processes as the simultaneous realization of several plane-parallel movements. Since the Euler and Lagrange variables coincide in the initial state $x_i(\alpha_p, t=0)=\alpha_i$, the joint of movements is reduced to replacing the Lagrange variables of the external motion $x_i^{ex}=x_i^{ex}(\alpha_p, t)$ with expressions for the corresponding Euler variables of the internal motion $x_i^{in}=x_i^{in}(\alpha_p, t)$. The equations of joint motion $x_i=x_i(\alpha_p, t)$ in their sequential or simultaneous flow coincide with the equations of external motion after replacing the Lagrange variables with the corresponding equations for the Euler variables of internal motion,

$$x_i(\alpha_p, t)=x_i^{ex}(x_i^{in}(\alpha_p, t), t). \tag{7}$$

External and internal movements are similar to figurative and relative ones in classical mechanics, but differ by the mandatory use of a single coordinate system. The principle allows its repeated application without violating the rule of geometric addition of the velocities and accelerations of the joint motion at each point and at each moment of time [15–16].

For example, to obtain the equations of motion with rotation relative to a moving pole, one should superimpose on the rotations (3) – (6) the translational displacement of the pole *P*,

$$x=\alpha+x_P(t)-\alpha_P, \qquad y=\beta+y_P(t)-\beta_P, \qquad z=\gamma+z_P(t)-\gamma_P. \tag{8}$$

As a result, the first term (Lagrangian coordinate) in each of the equations (3) – (6) is replaced by the current (Eulerian) coordinate

$$x = x_P + (\alpha - \alpha_P)\cos\Delta\theta_z - (\beta - \beta_P)\sin\Delta\theta_z, \quad y = y_P + (\alpha - \alpha_P)\sin\Delta\theta_z + (\beta - \beta_P)\cos\Delta\theta_z, \quad z = \gamma + z_P - \gamma_P, \tag{9}$$

$$x = \alpha + x_P - \alpha_P, \quad y = y_P + (\beta - \beta_P)\cos\Delta\theta_x - (\gamma - \gamma_P)\sin\Delta\theta_x, \quad z = z_P + (\beta - \beta_P)\sin\Delta\theta_x + (\gamma - \gamma_P)\cos\Delta\theta_x,$$

$$x = x_P + (\alpha - \alpha_P)\cos\Delta\theta_y + (\gamma - \gamma_P)\sin\Delta\theta_y, \quad y = \beta + y_P - \beta_P, \quad z = z_P + (\gamma - \gamma_P)\cos\Delta\theta_y - (\alpha - \alpha_P)\sin\Delta\theta_y.$$

Each system of 3 equations corresponds to a motion with 4 degrees of freedom: 3 translational movements along the coordinate axes of the pole P and the rotation of the body $\Delta\theta_i$ relative to the corresponding axis.

Equations (3) – (9) are sufficient to determine any kinematic characteristics of the motion of absolutely rigid bodies, including the kinetic energy, which, by definition, for a body with volume V, density ρ and mass m, is determined by the integral [11–12]

$$E_k = 0,5\rho\int_V v^2\delta V = 0,5\int_m v^2\delta m = 0,5\int_m (x_t^2 + y_t^2 + z_t^2)\delta m. \tag{10}$$

Taking into account the different nature of the arguments when describing motion in the Lagrange form, in equation (10) and below, the operator δ is used to denote infinitesimal increments of arguments and functions in the space of Lagrange variables, including when integrating with respect to mass δm. The d operator characterizes infinitesimal changes in functions over time, as used above for the velocity components $dx_i / dt \equiv x_{i,t}$.

2. Correction of the Koenig formula

The exact integration of equation (10) can be performed for plane-parallel motion with rotation relative to one axis, for example, z, when over the entire volume of the body $z_t = 0$. Taking into account the equations of motion (9), we transform the right part to the form

$$E_k = 0,5m[(x_t)_P^2 + (y_t)_P^2] - \theta_{z,t}\int_m [(x_t)_P(y - y_P) - (y_t)_P(x - x_P)]\delta m + 0,5\theta_{z,t}^2\int_m [(x - x_P)^2 + (y - y_P)^2]\delta m. \tag{11}$$

For integration in the second term, we use the concept of the center of mass $C(x_C, y_C)$, whose coordinates [11–12] define the equations

$$x_C m = \int_m x\delta m, \qquad y_C m = \int_m y\delta m. \tag{12}$$

Since x_C and y_C remain the same for all particles, they can be extracted from the integral and written instead of (11)

$$E_k = 0,5m[(x_t)_P^2 + (y_t)_P^2] - \theta_{z,t}m[(x_t)_P(y_C - y_P) - (y_t)_P(x_C - x_P)] + 0,5\theta_{z,t}^2\int_m [(x - x_P)^2 + (y - y_P)^2]\delta m. \tag{13}$$

The integral remaining on the right side in solid mechanics is called the moment of inertia in the rotational motion of the body, by analogy with the mass m in the translational motion [11-12]. Taking into account the features of the more general case of motion, in this paper we will call it, by analogy with the terminology of the theory of elasticity [17-18], the polar moment of inertia, as more fully reflecting its geometric meaning, while maintaining the designation accepted in solid mechanics J_P

$$J_P^z = \int_m [(x - x_P)^2 + (y - y_P)^2]\delta m . \tag{14}$$

As a result, instead of (13) we get

$$E_k = 0,5m(x_t^2 + y_t^2)_P + 0,5\theta_{z,t}^2 J_P - \theta_{z,t} m[(x_t)_P(y_C - y_P) - (y_t)_P(x_C - x_P)] . \tag{15}$$

The position of the center of mass depends on the configuration of the body and is determined by equations (12). The position of the pole can be selected arbitrarily. If the center of mass is chosen as the pole, the last term turns to 0 and the formula (15) takes the simplest form

$$E_k = 0,5m(x_t^2 + y_t^2)_C + 0,5\theta_{z,t}^2 J_C^z . \tag{16}$$

Equation (16) can be extended to any spatial translational motion with rotation relative to one axis, since the z_t component is the same for all particles of the body,

$$E_k = 0,5\int_m (x_t^2 + y_t^2 + z_t^2)\delta m = 0,5[\int_m (x_t^2 + y_t^2)\delta m + \int_m (z_t^2)_C \delta m] = 0,5mv_C^2 + 0,5\theta_{z,t}^2 J_C^z .$$

The result can be written in invariant form for any axis

$$E_k = 0,5mv_C^2 + 0,5\theta_{i,t}^2 J_C^i , \tag{17}$$

but it cannot be used for the spatial motion of solids with rotation relative to 2 or 3 axes due to the lack of a method for determining the moment of inertia relative to the instantaneous axis of rotation J_C^i when its position changes.

The exact analytical dependence of the kinetic energy in the spatial motion of a rigid body with arbitrary rotation can be obtained by representing the integral (10) as the sum of three integrals, in each of which the integrals represent the sum of the eponymous velocity components of the joint plane-parallel motions

$$E_k = 0,5\int_m v^2 \delta m = 0,5\int_m (x_t^2 + y_t^2 + z_t^2)\delta m =$$

$$= 0,5\int_m (x'_t + x''_t + x'''_t)^2 \delta m + 0,5\int_m (y'_t + y''_t + y'''_t)^2 \delta m + 0,5\int_m (z'_t + z''_t + z'''_t)^2 \delta m , \tag{18}$$

where the strokes indicate the velocity components from the three joint plane-parallel movements with a common pole: 1 stroke – rotation relative to the x-axis, 2 strokes-for the y-axis, 3 strokes-for the z-axis. This corresponds to the rule of geometric addition of velocities when joint movements, it is performed using the superposition principle (7).

To reduce the mathematical entries, we first consider the rotation of the body relative to the three coordinate axes passing through the fixed pole $P(\alpha_P, \beta_P, \gamma_P)$. To the result obtained we add the energy of the translational motion of the body together with the pole, as provided for in Koenig's theorem [11–12].

From the above equations (3) – (6), it follows that in each integral of equation (18), one of the terms turns to 0

$$E_k = 0{,}5\int_m (x_t'' + x_t''')^2 \delta m + 0{,}5\int_m (y_t' + y_t''')^2 \delta m + 0{,}5\int_m (z_t' + z_t'')^2 \delta m \,. \quad (19)$$

For the first integral with the velocity components along the *x*-axis, taking into account equations (3) – (6), we obtain (the superscript with the designation E_k^i shows the direction of the velocity component in this integral)

$$E_k^x = 0{,}5\int_m \{\theta_{z,t}^2 (y-\beta_P)^2 + \theta_{y,t}^2 (z-\gamma_P)^2 - 2\theta_{y,t}\theta_{z,t}(y-\beta_P)(z-\gamma_P)\}\delta m \,. \quad (20)$$

Since the angular velocities $\theta_{i,t}$ are the same for all the particles of the body, they can be extracted from the integral

$$E_k^x = 0{,}5\theta_{z,t}^2 \int_m (y-\beta_P)^2 \delta m + 0{,}5\theta_{y,t}^2 \int_m (z-\gamma_P)^2 \delta m - \theta_{y,t}\theta_{z,t}\int_m (y-\beta_P)(z-\gamma_P)\delta m \,. \quad (21)$$

The integrals remaining in the right part refer to the axial $(I_i^j)_P$ and centrifugal $(I_{ij}^k)_P$ moments of inertia [17–18]

$$(I_y^z)_P = \int_m (y-\beta_P)^2 \delta m \,, \qquad (I_z^y)_P = \int_m (z-\gamma_P)^2 \delta m \,, \qquad (I_{yz})_P = \int_m (y-\beta_P)(z-\gamma_P)\delta m \,. \quad (22)$$

For greater certainty, the upper index of the axial moments of inertia $(I_i^j)_P$ indicates the direction of the normal to the corresponding plane, which coincides with the axis of rotation indicated by the index of angular velocity $\theta_{i,t}$ in the term under consideration. For centrifugal moments of inertia, where the multipliers are two angular velocities, there are no upper indices. Lowercase subscripts at moments of inertia $(I_{ij}^k)_P$ in Equations (22) and further indicate the variables in the corresponding integrand functions. Equation (21) takes the form

$$E_k^x = 0{,}5\theta_{z,t}^2 (I_y^z)_P + 0{,}5\theta_{y,t}^2 (I_z^y)_P - \theta_{y,t}\theta_{z,t}(I_{yz})_P \,. \quad (23)$$

For the other two integrals of equation (19), converting them using equations (5) and (6) to the form (20), we find

$$E_k^y = 0{,}5\theta_{z,t}^2 (I_x^z)_P + 0{,}5\theta_{x,t}^2 (I_z^x)_P - \theta_{x,t}\theta_{z,t}(I_{xz})_P \,, \quad (24)$$

$$E_k^z = 0{,}5\theta_{x,t}^2 (I_y^x)_P + 0{,}5\theta_{y,t}^2 (I_x^y) - \theta_{y,t}\theta_{x,t}(I_{xy})_P \,, \quad (25)$$

Summing up the right-hand sides of equations (23) – (25), we get

$$E_k = 0{,}5\theta_{z,t}^2 (I_x^z + I_y^z)_P + 0{,}5\theta_{y,t}^2 (I_x^y + I_z^y)_P + 0{,}5\theta_{x,t}^2 (I_y^x + I_z^x)_P - \theta_{y,t}\theta_{z,t}(I_{yz})_P - \theta_{x,t}\theta_{z,t}(I_{xz})_P - \theta_{y,t}\theta_{x,t}(I_{xy})_P \,, \quad (26)$$

where

$$(I_x^y)_P = (I_x^z)_P = \int_m (x-\alpha_P)^2 \delta m \,, \quad (I_y^x)_P = (I_y^z)_P = \int_m (y-\beta_P)^2 \delta m \,, \qquad (I_z^x)_P = (I_z^y)_P = \int_m (z-\gamma_P)^2 \delta m \,,$$

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$$(I_{xy})_P = \int_m (x-\alpha_P)(y-\beta_P)\delta m\,, \quad (I_{yz})_P = \int_m (y-\beta_P)(z-\gamma_P)\delta m\,, \quad (I_{zx})_P = \int_m (x-\alpha_P)(z-\gamma_P)\delta m\,. \tag{27}$$

The sum of the axial moments of inertia (27) in the first three terms of equation (26) determines the squares of the distances between the projections of the point and the pole in different planes. When a rigid body rotates, they remain constant, as follows from the equations of motion (3) – (6), and the resulting polar moments of inertia do not change, for example

$$J_P^z = \int_m [(x-\alpha_P)^2 + (y-\beta_P)^2]\delta m = \int_m [(\alpha-\alpha_P)^2 + (\beta-\beta_P)^2]\delta m = const\,. \tag{28}$$

When using the notation generally accepted in solid state mechanics (14)

$$\begin{aligned}(I_y^z)_P + (I_x^z)_P &= \int_m [(x-\alpha_P)^2 + (y-\beta_P)^2]\delta m = J_P^z\,,\\ (I_z^y)_P + (I_x^y)_P &= \int_m [(x-\alpha_P)^2 + (z-\gamma_P)^2]\delta m = J_P^y\,,\\ (I_y^x)_P + (I_z^x)_P &= \int_m [(z-\gamma_P)^2 + (y-\beta_P)^2]\delta m = J_P^x\,,\end{aligned} \tag{29}$$

instead of (26) we get

$$E_k = 0{,}5\theta_{z,t}^2 J_P^z + 0{,}5\theta_{y,t}^2 J_P^y + 0{,}5\theta_{x,t}^2 J_P^x - \theta_{y,t}\theta_{z,t}(I_{yz})_P - \theta_{x,t}\theta_{z,t}(I_{xz})_P - \theta_{y,t}\theta_{x,t}(I_{xy})_P\,. \tag{30}$$

By analogy with equation (1), the upper index of the moments of inertia J_P^i corresponds to the axis relative to which it should be determined. The subscript indicates the point through which the axis of rotation should pass $i \in (x,y,z)$.

Equation (30) does not contain any restrictions and is applicable to any variants of the spatial motion of bodies with arbitrary rotation. The last three terms with centrifugal moments of inertia distinguish the result (30) from the Koenig formula (1). In contrast to the polar moments of inertia (29) in the first three terms, they change with the rotation of the body and affect the values of the kinetic energy.

To estimate the possible range of changes in the centrifugal moments of inertia in equation (30), we use the initial values of the axial and centrifugal moments of inertia, which remain unchanged during the movement of the body. Their values follow from equations (27), if the current coordinates $x_i \in (x,y,z)$ are equated with the initial ones $\alpha_p \in (\alpha,\beta,\gamma)$

$$(I_\alpha^z)_P = (I_\alpha^y)_P = \int_m (\alpha-\alpha_P)^2\delta m\,, \quad (I_\beta^x)_P = (I_\beta^z)_P = \int_m (\beta-\beta_P)^2\delta m\,, \quad (I_\gamma^x)_P = (I_\gamma^y)_P = \int_m (\gamma-\gamma_P)^2\delta m\,, \tag{31}$$

$$(I_{\alpha\beta})_P = \int_m (\alpha-\alpha_P)(\beta-\beta_P)\delta m\,, \quad (I_{\beta\gamma})_P = \int_m (\beta-\beta_P)(\gamma-\gamma_P)\delta m\,, \quad (I_{\gamma\alpha})_P = \int_m (\alpha-\alpha_P)(\gamma-\gamma_P)\delta m\,. \tag{32}$$

For the lower indices in the left part of equations (31) – (32), Lagrange variables $\alpha_P \in (\alpha,\beta,\gamma)$ are used to distinguish the moments of inertia in the initial state from the current ones (27).

As noted above, the polar moments of inertia (28) remain unchanged when the body rotates. Using the equations of motion (3) – (6), the system (27) for the centrifugal moments of inertia $(I_{ij})_P$ is converted to the form

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$$(I_{xy})_P = \int_m (x-\alpha_P)(y-\beta_P)\delta m = 0{,}5\sin(2\Delta\theta_z)\int_m [(\alpha-\alpha_P)^2-(\beta-\beta_P)^2]\delta m + \cos(2\Delta\theta_z)\int_m (\alpha-\alpha_P)(\beta-\beta_P)\delta m\,,$$

$$(I_{yz})_P = \int_m (y-\beta_P)(z-\gamma_P)\delta m = 0{,}5\sin(2\Delta\theta_x)\int_m [(\beta-\beta_P)^2-(\gamma-\gamma_P)^2]\delta m + \cos(2\Delta\theta_x)\int_m (\beta-\beta_P)(\gamma-\gamma_P)\delta m\,,$$

$$(I_{zx})_P = \int_m (x-\alpha_P)(z-\gamma_P)\delta m = 0{,}5\sin(2\Delta\theta_y)\int_m [(\gamma-\gamma_P)^2-(\alpha-\alpha_P)^2]\delta m + \cos(2\Delta\theta_y)\int_m (\alpha-\alpha_P)(\gamma-\gamma_P)\delta m\,,$$

or, taking into account the notation for the initial values (31) – (32),

$$(I_{xy})_P = \int_m (x-\alpha_P)(y-\beta_P)\delta m = 0{,}5\sin(2\Delta\theta_z)[(I_\alpha^z)_P-(I_\beta^z)_P]+\cos(2\Delta\theta_z)(I_{\alpha\beta})_P\,,$$

$$(I_{yz})_P = 0{,}5\sin(2\Delta\theta_x)[(I_\beta^x)_P-(I_\gamma^x)_P]+\cos(2\Delta\theta_x)(I_{\beta\gamma})_P\,, \qquad (33)$$

$$(I_{zx})_P = 0{,}5\sin(2\Delta\theta_y)[(I_\gamma^y)_P-(I_\alpha^y)_P]+\cos(2\Delta\theta_y)(I_{\gamma\alpha})_P\,.$$

The obtained equations coincide with the known relations for changing the centrifugal moments of inertia of solids when the coordinate axes are rotated [17-18]. Consequently, the last three terms in equation (30) take into account the change in the centrifugal moments of inertia relative to the axes of the observer's coordinate system due to the rotation of the body. The share of kinetic energy determined by the terms with the centrifugal moments of inertia (33) is affected by the features of rotation relative to all axes. If the rotation occurs relative to only 2 axes, for example y and z, equation (30) takes the form

$$E_k = 0{,}5\theta_{z,t}^2 J_P^z + 0{,}5\theta_{y,t}^2 J_P^y - \theta_{y,t}\theta_{z,t}(I_{yz})_P\,, \qquad (34)$$

where, due to $\Delta\theta_x$ = 0, the remaining centrifugal moment $(I_{yz})_P$ does not change when the body rotates, as do the polar moments of inertia (29),

$$(I_{yz})_P = \int_m (y-\beta_P)(z-\gamma_P)\delta m = \int_m (\beta-\beta_P)(\gamma-\gamma_P)\delta m\,.$$

If the rotation occurs relative to only one axis, equation (30) is converted to the form (15) for a fixed pole (the first and third terms are converted to 0)

$$E_k = 0{,}5\theta_{z,t}^2 (I_x^z + I_y^z)_P = 0{,}5\theta_{z,t}^2 J_P^z\,. \qquad (35)$$

When rotating with respect to three axes, use equation (30) with centrifugal moments (33).

If we joint the pole *P* with the center of mass *C*, equation (30) takes the form

$$E_k = 0{,}5(\theta_{z,t}^2 J_C^z + \theta_{y,t}^2 J_C^y + \theta_{x,t}^2 J_C^x) - \theta_{y,t}\theta_{z,t}(I_{yz})_C - \theta_{x,t}\theta_{z,t}(I_{xz})_C - \theta_{y,t}\theta_{x,t}(I_{xy})_C\,, \qquad (36)$$

where for the polar moments of inertia instead of (29) we get

$$J_C^z = \int_m [(x-\alpha_C)^2+(y-\beta_C)^2]\delta m\,, \quad J_C^y = \int_m [(x-\alpha_C)^2+(z-\gamma_C)^2]\delta m\,, \quad J_C^x = \int_m [(y-\beta_C)^2+(z-\gamma_C)^2]\delta m\,. \qquad (37)$$

For centrifugal moments, instead of (33), the equations are valid

$$(I_{xy})_C = 0{,}5\sin(2\Delta\theta_z)[(I_\alpha)_C-(I_\beta)_C]+\cos(2\Delta\theta_z)(I_{\alpha\beta})_C\,,$$

$$(I_{yz})_C = 0{,}5\sin(2\Delta\theta_x)[(I_\beta)_C-(I_\gamma)_C]+\cos(2\Delta\theta_x)(I_{\beta\gamma})_C\,, \qquad (38)$$

$$(I_{zx})_C = 0{,}5\sin(2\Delta\theta_y)[(I_\gamma)_C-(I_\alpha)_C]+\cos(2\Delta\theta_y)(I_{\alpha\gamma})_C\,.$$

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where instead of (31) – (32) for the central moments of inertia in the initial state, we have

$$(I_\alpha^z)_C = (I_\alpha^y)_C = \int_m (\alpha - \alpha_C)^2 \delta m\,, \qquad (I_\beta^x)_C = (I_\beta^z)_C = \int_m (\beta - \beta_C)^2 \delta m\,, \qquad (I_\gamma^x)_C = (I_\gamma^y)_C = \int_m (\gamma - \gamma_C)^2 \delta m\,,$$

$$(I_{\alpha\beta})_C = \int_m (\alpha - \alpha_C)(\beta - \beta_C)\delta m \qquad (I_{\beta\gamma})_C = \int_m (\beta - \beta_C)(\gamma - \gamma_C)\delta m \qquad (I_{\gamma\alpha})_C = \int_m (\alpha - \alpha_C)(\gamma - \gamma_C)\delta m\,.$$

To calculate the kinetic energy, taking into account the translational motion the center of mass, equations (30) and (36), in accordance with Koenig's theorem [11–12], are converted to the form

$$E_k = 0{,}5 v_P^2 m + 0{,}5(\theta_{z,t}^2 J_P^z + \theta_{y,t}^2 J_P^y + \theta_{x,t}^2 J_P^x) - \theta_{y,t}\theta_{z,t}(I_{yz})_P - \theta_{x,t}\theta_{z,t}(I_{xz})_P - \theta_{y,t}\theta_{x,t}(I_{xy})_P\,, \tag{39}$$

$$E_k = 0{,}5 v_C^2 m + 0{,}5(\theta_{z,t}^2 J_C^z + \theta_{y,t}^2 J_C^y + \theta_{x,t}^2 J_C^x) - \theta_{y,t}\theta_{z,t}(I_{yz})_C - \theta_{x,t}\theta_{z,t}(I_{xz})_C - \theta_{y,t}\theta_{x,t}(I_{xy})_C\,. \tag{40}$$

The axial moments of inertia determine the equations (29) or (37), the centrifugal (33) or (38).

3. Example of applying equations

Let us consider the application of the obtained equations on the example of a robot manipulator with two rotational and one helical kinematic pairs for moving a load of mass m [1, 6]. The block diagram of a robot with 3 independent drives and the rotation of the links relative to the intersecting axes is shown in Fig. 1.

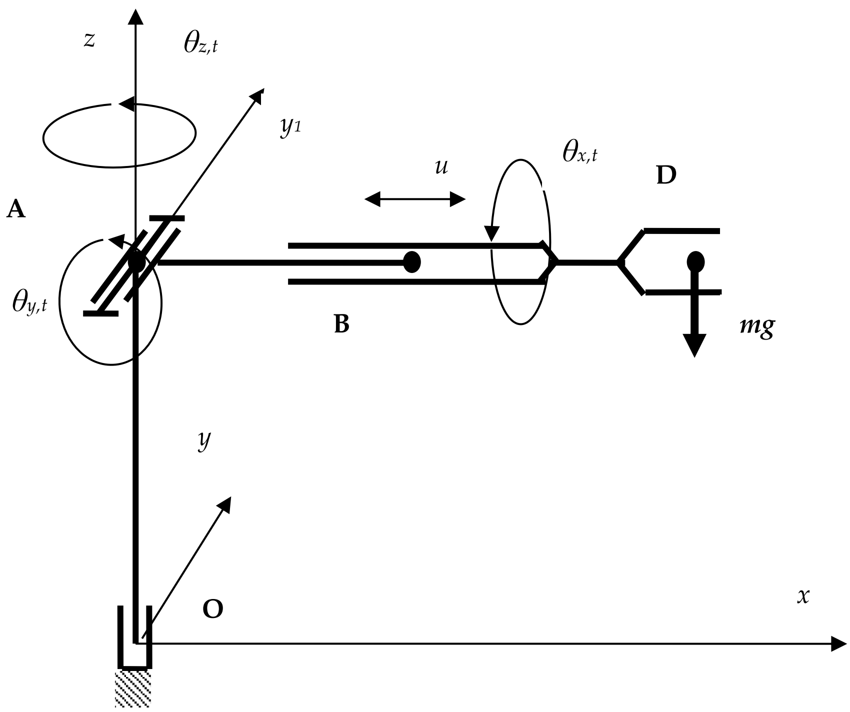

Fig. 1. Block diagram of a three-link robot with two rotational and one helical kinematic pairs in the initial position

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Link 1 is fixed to the rack by means of a rotational kinematic pair, which allows the link to rotate relative to the z axis passing through the origin. The link particles change their position in the observer's space in accordance with equations (3), which, taking into account the accepted coordinate system and the position of the pole, take the form

$$x = \alpha\cos\Delta\theta_z - \beta\sin\Delta\theta_z, \qquad y = \alpha\sin\Delta\theta_z + \beta\cos\Delta\theta_z, \qquad z = \gamma. \qquad (41)$$

Link 2 rotates relative to the y1 axis, using the drive in a kinematic pair with the point $A(\alpha_A = 0,\ 0,\ \gamma_A = H)$,

$$x = \alpha_A + (\alpha - \alpha_A)\cos\Delta\theta_y + (\gamma - \gamma_A)\sin\Delta\theta_y,\ y = \beta,\ z = \gamma_A + (\gamma - \gamma_A)\cos\Delta\theta_y - (\alpha - \alpha_A)\sin\Delta\theta_y. \quad (42)$$

Links 2 and 3 are connected by a helical kinematic pair at point $B(\alpha_B = L_1, 0, \gamma_B = H)$. The drive provides translational and rotational movement of the link 3 relative to the link 2

$$x = \alpha + x_B - \alpha_B, \quad y = \beta_B + (\beta - \beta_B)\cos\Delta\theta_x - (\gamma - \gamma_B)\sin\Delta\theta_x, \quad z = \gamma_B + (\beta - \beta_B)\sin\Delta\theta_x + (\gamma - \gamma_B)\cos\Delta\theta_x. \qquad (43)$$

The translational movement u determines the pitch of the screw pair h and the angle of rotation of the link $\Delta\theta_x$

$$x_B - \alpha_B = u = h\Delta\theta_x / (2\pi).$$

In accordance with equation (7), for the equations of combined motion of link 2 with the external motion of link 1, the Lagrange variables in equations (41) should be replaced by the corresponding equations (42)

$$(x)_2 = [\alpha_A + (\alpha - \alpha_A)\cos\Delta\theta_y + (\gamma - \gamma_A)\sin\Delta\theta_y]\cos\Delta\theta_z - \beta\sin\Delta\theta_z,$$
$$(y)_2 = [\alpha_A + (\alpha - \alpha_A)\cos\Delta\theta_y + (\gamma - \gamma_A)\sin\Delta\theta_y]\sin\Delta\theta_z + \beta\cos\Delta\theta_z, \qquad (44)$$
$$(z)_2 = \gamma_A + (\gamma - \gamma_A)\cos\Delta\theta_y - (\alpha - \alpha_A)\sin\Delta\theta_y.$$

The time derivatives determine the components of the particle velocity

$$(x_t)_2 = -\theta_{z,t}\{[\alpha_A + (\alpha - \alpha_A)\cos\Delta\theta_y + (\gamma - \gamma_A)\sin\Delta\theta_y]\sin\Delta\theta_z + \beta\cos\Delta\theta_z\} - \\ -\theta_{y,t}[(\alpha - \alpha_A)\sin\Delta\theta_y - (\gamma - \gamma_A)\cos\Delta\theta_y]\cos\Delta\theta_z,$$
$$(y_t)_2 = \theta_{z,t}\{[\alpha_A + (\alpha - \alpha_A)\cos\Delta\theta_y + (\gamma - \gamma_A)\sin\Delta\theta_y]\cos\Delta\theta_z - \beta\sin\Delta\theta_z\} - \\ -\theta_{y,t}\{[(\alpha - \alpha_A)\sin\Delta\theta_y - (\gamma - \gamma_A)\cos\Delta\theta_y]\sin\Delta\theta_z\}, \qquad (45)$$
$$(z_t)_2 = -\theta_{y,t}[(\gamma - \gamma_A)\sin\Delta\theta_y + (\alpha - \alpha_A)\cos\Delta\theta_y].$$

Using the superposition rule (7) again, taking into account the internal motion (43) and external (44), with the coordinates of $\beta_A = \beta_B$ and $\gamma_A = \gamma_B$ equal, we obtain for link 3 a rotation relative to 3 axes

$$(x)_3 = \{\alpha_A + (\alpha + u - \alpha_A)\cos\Delta\theta_y + [(\beta - \beta_B)\sin\Delta\theta_x + (\gamma - \gamma_B)\cos\Delta\theta_x]\sin\Delta\theta_y\}\cos\Delta\theta_z - \\ -[\beta_B + (\beta - \beta_B)\cos\Delta\theta_x - (\gamma - \gamma_B)\sin\Delta\theta_x]\sin\Delta\theta_z,$$
$$(y)_3 = \{\alpha_A + (\alpha + u - \alpha_A)\cos\Delta\theta_y + [(\beta - \beta_B)\sin\Delta\theta_x + (\gamma - \gamma_B)\cos\Delta\theta_x]\sin\Delta\theta_y\}\sin\Delta\theta_z + \\ +[\beta_B + (\beta - \beta_B)\cos\Delta\theta_x - (\gamma - \gamma_B)\sin\Delta\theta_x]\cos\Delta\theta_z, \qquad (46)$$

$$(z)_3 = \gamma_A + [(\beta - \beta_B)\sin\Delta\theta_x + (\gamma - \gamma_B)\cos\Delta\theta_x]\cos\Delta\theta_y - (\alpha + u - \alpha_A)\sin\Delta\theta_y .$$

After time differentiation, we find the velocity components, for example, for $(x_t)_3$

$$\begin{aligned}(x_t)_3 = {} & u_t \cos\Delta\theta_y \cos\Delta\theta_z - \theta_{y,t}(\alpha + u - \alpha_A)\sin\Delta\theta_y \cos\Delta\theta_z + \theta_{x,t}[(\beta - \beta_B)\cos\Delta\theta_x - (\gamma - \gamma_B)\sin\Delta\theta_x]\sin\Delta\theta_y \cos\Delta\theta_z + \\ & + \theta_{y,t}[\gamma_B + (\beta - \beta_B)\sin\Delta\theta_x + (\gamma - \gamma_B)\cos\Delta\theta_x - \gamma_A]\cos\Delta\theta_y \cos\Delta\theta_z - \\ & - \theta_{z,t}\{\alpha_A + (\alpha + u - \alpha_A)\cos\Delta\theta_y + [(\beta - \beta_B)\sin\Delta\theta_x + (\gamma - \gamma_B)\cos\Delta\theta_x]\sin\Delta\theta_y\}\sin\Delta\theta_z + \\ & + \theta_{x,t}[(\beta - \beta_B)\sin\Delta\theta_x + (\gamma - \gamma_B)\cos\Delta\theta_x]\sin\Delta\theta_z - \theta_{z,t}[\beta_B + (\beta - \beta_B)\cos\Delta\theta_x - (\gamma - \gamma_B)\sin\Delta\theta_x]\cos\Delta\theta_z\end{aligned} \quad (47)$$

Other speed components for link 3 are not given to reduce the length of the article. The correctness of writing the equations of motion (44) and (46) can be checked by the condition of the absence of deformation [14]

$$R = \delta V / \delta V_0 = |x_{i,\alpha_p}| = 1, \qquad x_{i,\alpha_p} \equiv \partial x_i / \partial \alpha_p .$$

The equations of motion of the particles of the transported load at point D depend on the features of its attachment to link 3. For clamping grippers, the equations coincide with those written above (46) for link 3, but require specification of the initial coordinates of the point under consideration, for example, to determine the coordinates of the center of mass of the load $C_m(\alpha_{Cm}, \beta_{Cm}, \gamma_{Cm})$.

Knowing the velocity components (45) and (47), the kinetic energy of the links can be found numerically [4-6], but it is more convenient to use the above analytical dependencies (39) – (40). The particles of link 1 perform plane-parallel motion, the kinetic energy of the link is determined by equation (35). The particles of link 2 rotate relative to two axes $\theta_{y,t} \neq \theta_{z,t} \neq 0$, $\theta_{x,t} = 0$, the kinetic energy is determined by equation (34). Link 3 with a movable pole B rotates relative to three axes, the kinetic energy is determined by equations (39) or (40).

To calculate the required drive power and forces in kinematic pairs, it is advisable to use the analysis of energy flows, as in lever mechanisms with closed kinematic chains [16, 19]. The sum of the rates of change of the kinetic $W_k = dE_k/dt$ and potential $W_p = dE_p/dt$ energy of moving bodies in the considered section of the mechanism should be equal to the power W_e coming from external sources,

$$W_e = W_k + W_p .$$

The choice of the pole in equation (46) is uniquely determined at the stage of kinematic analysis, since equations (3) – (6) assume its motion to be known. It must belong to the kinematic pairs connecting the slave and master links. However, when calculating generalized forces, it is more convenient to use equation (40) with the kinematic characteristics of the center of mass [16, 19]. For example, for link 3, the required power of external forces is determined by the equation

$$(W_e)_B = (W_k)_{C3} + (W_p)_{C3} + (W_k)_D + (W_p)_D ,$$

where

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$$(W_k)_{C3} = m(x_t x_{tt} + y_t y_{tt} + z_t z_{tt})_{C3} + (\theta_{x,t}\theta_{x,tt} J^y_{C3} + \theta_{y,t}\theta_{y,tt} J^y_{C3} + \theta_{z,t}\theta_{z,tt} J^z_{C3}) - \\ -d\{\theta_{y,t}\theta_{z,t}(J_{yz})_{C3} + \theta_{x,t}\theta_{z,t}(J_{xz})_{C3} + \theta_{y,t}\theta_{x,t}(J_{xy})_{C3}\}/dt ,$$

$$(W_p)_{C3} = m_{C3} g(z_t)_{C3} .$$

This is the most cumbersome formula for dynamic analysis of the robot in question, but when using it, the complexity of the calculation is significantly lower compared to the matrix options. This technique ensures the implementation of the law of conservation of energy on any part of the mechanism at any time.

4. Conclusions

Equations (39) and (40) determine the kinetic energy of a solid body in the most general case of spatial motion with arbitrary rotation relative to any axes and can be used for dynamic analysis of mechanisms of any complexity with known information about the geometric features of the links that allow us to determine the axial, polar and centrifugal moments of inertia, as well as the current values of linear and angular velocities [19].

The kinetic energy of the rotational motion of a solid depends on the shape of the body. Taking into account that the initial values of the moments of inertia during movement are preserved, it can be argued that the current values of the centrifugal moments (38), depending on the angle of rotation of the body $\Delta\theta_i$, change modulo, for example, for the moment of inertia $(I_{xy})_C$ from the initial value $(I_{\alpha\beta})_C$ to half the difference of the axial moments of inertia (in the initial state) $[(I^z_\alpha)_C - (I^z_\beta)_C]/2$.

Koenig's formula (1) provides correct results when the last three terms in equations (36) and (40) turn to 0. This is only possible for bodies with equal principal moments of inertia, such as a ball, when and any two orthogonal axes passing through the center of gravity are the principal central axes of inertia. Otherwise, the first terms in equations (38), due to the difference in the principal main central moments of inertia, will lead to cyclic changes in the kinetic energy, the possible manifestation of precession and nutation, the amplitude of which depends on the angular velocities of rotation of the body.

The use of formula (36) instead of the approximate (1) provides an increase in the accuracy of the results of the dynamic analysis of mechanisms. The error in calculating the kinetic energy and dynamic analysis according to the Koenig formula (1) will depend on the design features of the mechanism that determine the moments of inertia of the body, and the absolute values of the angular velocities of rotation of the links.

The structure of the obtained equations does not allow us to propose an alternative method for the analytical determination of the moment of inertia relative to the instantaneous axis of rotation J_C in equation (17), which can only be determined experimentally [11].

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