 doi:10.20944/preprints202101.0544.v2

Gravitation with Cosmological Term, Expansion of the Universe as Uniform Acceleration in Clifford Coordinates

A. Kritov[1]

Abstract

The paper presents a novel approach to the cosmological constant problem by the use of the Clifford algebras of space $Cl_{3,0}$ and anti-space $Cl_{0,3}$ with a particular focus on the paravector representation, emphasizing the fact that both algebras have a center represented just by two coordinates. Since the paravector representation allows assigning the scalar element of grade 0 to the time coordinate, we consider the relativity in such two-dimensional spacetime for a uniformly accelerated frame with the constant acceleration $3H_0c$. Using the Rindler coordinate transformations in two-dimensional spacetime and then applying it to Minkowski coordinates, we obtain the FLRW metric, which in the case of the Clifford algebra of space $Cl_{3,0}$ corresponds to the anti-de Sitter (AdS) flat ($k = 0$) case, the negative cosmological term and an oscillating model of the universe. The approach with anti-Euclidean Clifford algebra $Cl_{0,3}$ leads to the de Sitter model with the positive cosmological term and the exact form of the scale factor used in modern cosmology.

Keywords: Clifford algebras; Cl(3,0), Cl(0,3); two dimensional spacetime; time-volume coordinates; constant uniform acceleration $3Hc$; Rindler coordinates; FLRW metric; scale factor; AdS and de Sitter models.

The problem of the cosmological or Λ term is a long standing subject since being the Einstein's "biggest blunder". Nowadays the cosmological constant is widely used in the frame of the standard model of cosmology as observational data in the past decades strongly suggests that our universe has a positive cosmological constant. However, a major outstanding problem is still that most quantum field theories predict a huge value for the quantum vacuum [30, 31]. Hence, the cosmological constant remained a subject of theoretical and empirical interest, and the explanation of this small but positive value is an outstanding theoretical challenge.

The cosmological constant is commonly attributed to a "dark energy" or the density of the quantum vacuum Ω_Λ. The presented approach provides an alternative view on the nature of the cosmological constant and its appearance in the metric as the cosmological term without requiring a presence of a "kind of negative energy". Starting from the Clifford algebras analysis, which reflects the intrinsic property of spacetime, the model derives the scale factors used in modern cosmology for the AdS and de Sitter models by considering the uniform acceleration in two dimensions given by the center of respective Clifford algebra. As shown, connecting the model's constant parameters to a central point mass leads also to the static forms of the SadS and the SdS metrics for gravitation in the spherically symmetric case. The cosmological term appears as the *relativistic effect* due to the Rindler coordinate transformations. The full correspondence requires the value for such uniform acceleration to be $\alpha = 3H_0c$.

The paper also refers to the author's recent work [15], which introduced the unified two-dimensional space-time model by conjecture that it consists of time and spatial-volume coordinates. It introduced a few important parameters, which are used throughout. The paper further formalizes the approach based on the symmetry of underlying Clifford algebras [1, 3, 4, 17, 22].

The paper is structured as follows. Section 1 reviews the basic properties of the Pauli algebra $Cl_{3,0}$, its paravector representation, the center, and the structure of corresponding spacetimes given by the quadratic forms. Section 2 gives an outlook on the relativity in two-dimensional spacetime of the center of algebra $Cl_{3,0}$ based on the obtained quadratic form's invariance. Section 3 applies the result to (1+3) spacetime, and using the Rindler coordinate transformations obtained in the previous section, derives the scale factor of the FLRW metric. Section 4, along with a brief review of the algebra $Cl_{0,3}$, quickly re-applies the approach to $Cl_{0,3}$. Section 5 aims to align the obtained results with the observed value of the cosmological deceleration parameter.

Note: The Hubble constant $3H_0^2 = c^2\Lambda$ is a *constant*. Denoted Clifford algebras $Cl_{p,q}$ and the quaternions $\mathbb{H}$ are over the reals.

[1]ORCID 0000-0003-3226-3676.
Lomonosov Moscow State University*, Faculty of Physics.
The contact: akritov@gmail.com

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 5 February 2021 doi:10.20944/preprints202101.0544.v2

1. Clifford Algebra of Space $Cl_{3,0}$

The Pauli algebra $Cl_{3,0}(\mathbb{R})$ describes the structure of Euclidean $\mathbb{R}^3$ space. $Cl_{3,0}$ is eight-dimensional algebra with the basis that can be given by Pauli matrices as follows:

$$\begin{array}{cccc|cccc} e_0 & e_1 & e_2 & e_3 & \sigma_0 & \sigma_1 & \sigma_2 & \sigma_3 \\ e_{123} & e_{23} & e_{13} & e_{12} & i\sigma_0 & i\sigma_1 & i\sigma_2 & i\sigma_3 \end{array} ,$$

where e_0 is scalar, $(e_1,\ e_2,\ e_3)$ is vector, (e_{23}, e_{13}, e_{12}) is bivector, and e_{123} is a volume element (tri-vector or multivector of grade 3). The algebra is isomorphic to two-by-two matrices with complex entries $Cl_{3,0} \cong Mat(2,\mathbb{C})$ and has two subalgebras: the even subalgebra $Cl^0_{3,0} \cong \mathbb{H}$ of quaternions, and the center $Cen(Cl_{3,0}) \cong \mathbb{C}$ therefore

$$Cl_{3,0} \cong \mathbb{C} \otimes \mathbb{H}\,, \tag{1}$$

where a quaternion corresponds to elements of grades 0 and 2. The algebra $Cl_{3,0}$ has a unique property. As an algebra of the structure of Euclidean $\mathbb{R}^3$ space, it also describes four-dimensional Minkowski spacetime. It can be understood in two ways. The first is that $Cl_{3,0}$ is isomorphic to even subalgebras $Cl^0_{3,1} \cong Cl^0_{1,3}$ [1, 22]. The second is that $Mat(2,\mathbb{C})$ can be normalized to $SL(2,\mathbb{C})$ which is the classical spin homomorphism and double-cover of the proper Lorentz group $SO(1,3)^+$, and it is also homomorphic with $SO(3,\mathbb{C})$, which is isomorphic to complexified quaternions $\mathbb{H}(\mathbb{C})$.

Such isomorphism of the algebra with 1+3 spacetime allows to consider *paravector* representation of $Cl_{3,0}$ utilized in physics as the algebra of physical space (APS). Because the basis of $Cl_{3,0}$ precisely corresponds to two copies of 4-dimensional vector spaces, all eight orthogonal coordinates can be split into two sets of Minkowski and anti-Minkowski spacetimes, spanned by two 4-vectors given by the upper and lower raw as follows:

$$\begin{array}{c|ccc} x_0 & x_1 & x_2 & x_3 \\ \hline x_{123} & x_{23} & x_{13} & x_{12} \end{array} , \tag{2}$$

where $x_\mu = (x_0, x_1, x_2, x_3)$ is the "usual" Minkowski four-vector (x_0 is time) or paravector in the algebra of physical space (APS), and $\overline{x}_\mu = (\overline{x}_0, \overline{x}_1, \overline{x}_2, \overline{x}_3) = (x_{123}, x_{23}, x_{13}, x_{12})$ is its "anti-Minkowskian" counterpart. Notably, such an anti-Minkowski 4-vector $\overline{x}$ is built only from the spatial components of the the 4-vector x. Using the basis of Pauli matrices we obtain the canonical representation of 4-vectors by the matrices

$$g = x^k \sigma_k\,, \quad \overline{g} = \overline{x}^k\, i\sigma_k\,, \quad k = 0,1,2,3\,; \quad g, \overline{g} \in SL(2,\mathbb{C}) \tag{3}$$

with the Einstein convention for summation. Two such vector spaces $\mathbb{R}^{1,3}$ have quadratic forms with reverse signatures

$$\begin{aligned} det(g) &= \ x_0^2 - x_1^2 - x_2^2 - x_3^2\,, \\ det(\overline{g}) &= -\overline{x}_0^2 + \overline{x}_1^2 + \overline{x}_2^2 + \overline{x}_3^2\,. \end{aligned} \tag{4}$$

Thus, in paravector representation of the algebra $Cl_{3,0}$ a scalar value or grade 0 element is assigned to the time coordinate of Minkowski spacetime, and the volume element represents its "anti-Minkowskian" or complexified counterpart.

Another essential feature of this algebra is that it has the center $Cen(Cl_{3,0}) \cong \mathbb{C}$. By definition, the center of Clifford algebra consists of elements that commute with all elements of algebra. In case of $Cl_{3,0}$ these two elements are given by two basis elements of grade 0 and 3 i.e by e_0 and e_{123} [1, 17], explicitly

$$e_0 = \sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad e_{123} = i\sigma_0 = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}.$$

Hence, in the case of paravector representation the center of $Cl_{3,0}$ consists of two coordinates given by *time* and *volume* elements that form subalgebra $\mathbb{C}$ of $Cl_{3,0}$. To maintain the accordance with the previous work [15] we denote these coordinates as $t = x_0$, $\eta = \overline{x}_0 = x_{123}$. The two-dimensional spacetime of $Cen(Cl_{3,0})$ is spanned by a vector $e_0 t + e_{123}\eta$ that represents a complex number u and its quadratic form are respectively

$$u = t + i\eta \in \mathbb{C}\,, \quad |u|^2 = uu^* = t^2 + \eta^2\,, \quad (t, \eta \in \mathbb{R})\,. \tag{5}$$

The center of Clifford algebra $Cen(Cl_{3,0})$ is the subalgebra $\mathbb{C}$, and is a two-dimensional spacetime of plane.

2. Physical Relativity and Uniform Acceleration in Two Dimensions of $Cen(Cl_{3,0})$

Geometry in two dimensions of $\mathbb{R}^2$ is trivial from a mathematical point of view as it represents the Euclidean plane. However, because the coordinate x_0 is considered the time coordinate, the prime interest of physics, in this case, is the relativity of motion in frames of reference in such spacetime.

Because the plane has the quadratic form (5) with the signature (+,+) the invariant interval in such spacetime is given by[2]

$$ds^2 = c^2dt^2 + d\eta^2 \,. \tag{6}$$

Hence to obtain new rules for the coordinate transformations, it is necessary to substitute all hyperbolic functions involved in Lorentz transformations with their trigonometric counterparts in $SO(2)$. For instance, the coordinate velocity and two-velocity for a moving frame become

$$u = \frac{d\eta}{dt} = c\tan(\beta)\,, \qquad u = (c\cos(\beta), c\sin(\beta))\,.$$

It can be noted that the coordinate velocity can be infinite[3], however, the proper velocity (measured with the observer's clock) is limited by unity.

The Rindler or Kottler-Møller coordinates are essential type of coordinate transformations in special relativity [10, 20, 23]. These are the coordinate transformations in the case of the uniformly accelerated frame and are derived by integrating the Lorentz transformations with initial conditions using the evident relation $\beta = \left(\frac{\alpha\tau}{c}\right)$ where β is the rapidity, α is the *proper* uniform acceleration, and τ is the observer's proper time. Because of the invariant (6) in the spacetime of $Cen(Cl_{3,0})$ these differential transformations are given by the trigonometric counterparts

$$dt = \gamma d\tau = \cos\left(\frac{\alpha\tau}{c}\right) d\tau\,, \tag{7}$$

$$d\eta = v(t)dt = c\tan\left(\frac{\alpha\tau}{c}\right) dt\,.$$

These expressions provide the coordinate transformations between an observer A located at the center of the spherical volume at rest and the frame B with coordinate $\eta(t)$ of the spherically expanding volume moving with velocity $u = \frac{dV}{dt}$. The integration of these two with respect to proper time with the initial synchronization of the clocks at the beginning ($t = \tau = 0$ in $\eta = 0$) leads to the Rindler coordinate transformations

$$t = \frac{c}{\alpha}\sin\left(\frac{\alpha\tau}{c}\right)\,, \tag{8}$$

and the coordinate distance from the origin of motion in terms of proper time is

$$\eta = \frac{c^2}{\alpha}\left(1 - \cos\left(\frac{\alpha\tau}{c}\right)\right) = 2\eta_0\sin^2\left(\frac{\alpha\tau}{2c}\right)\,, \quad \eta_0 = \frac{c^2}{\alpha}\,, \tag{9}$$

where the pre-factor η_0 is the Rindler horizon parameter, which has an important role in the relativity of uniform acceleration. Notably, the expressions demonstrate the significance of the property of the Rindler transformations in the spacetime of $Cen(Cl_{3,0})$ which is the equivalence of the uniform acceleration to the *harmonic oscillation* of volume coordinate.

3. The Scale Factor and AdS Model for $Cl_{3,0}$

In the case of spherical symmetry that is considered further, the coordinate $\eta = x_{123}$ in Clifford algebra $Cl_{3,0}$ clearly represents a spherical volume. Model [15] provided an explicit form for the relation that preserves physical dimensionality $\eta = VA^{-1}$. Since both $\eta(\tau)$ and $V(\tau)$ are variable, the factor A is a constant having the physical dimension of the area. As the parameter has to be a fixed constant then, without loss of generality, we may connect it by direct proportionality to a central point mass m, as later shown. Hence, η has the dimension of length.

The Friedmann–Lemaître–Robertson–Walker (FLRW) is the metric for a comoving frame with time-dependent spatial component given by the scale factor $a(\tau)$. In the flat case this is $ds^2 = -d\tau^2 + a(\tau)^2\left(dR^2 + R^2d\Omega\right)$. As the FLRW metric tensor is diagonal, then the tetrad transformation from the Minkowski to the FLRW metric is given by the square root of the metric tensor as

$$h_\mu{}^a = \mathrm{diag}\left(1, a, Ra, Ra\sin(\theta)\right) \quad \text{with} \quad r = Ra\,, \quad \text{and} \quad g_{\mu\nu} = h_\mu{}^a h_\nu{}^b n_{ab}\,, \tag{10}$$

where R is the comoving (fixed) distance, and r is the coordinate distance [13]. In fact, the tetrad is simply a Jacobian matrix whose determinant defines a volume element $d\eta = dV = a^3\,4\pi R^2 dR$, therefore

$$a = \left(\frac{V}{V_R}\right)^{1/3}\,, \quad \text{where} \quad V_R = \frac{4\pi}{3}R^3 \quad \text{and} \quad V = \frac{4\pi}{3}r^3\,. \tag{11}$$

[2]The physical dimensionality of η is distance as discussed in the next section.

[3]The coordinate η implies spatial volume. Even though the case may correspond to super-luminal universe expansion in the coordinate distance. As discussed later, we can only measure our proper time in a local frame.

The coordinate η represents the value of the spherical volume $V = A\eta$, and we can also indicate the point $\eta_{\rm R}$ that corresponds to such fixed volume $V_R = A\eta_{\rm R}$ hence

$$a = \left(\frac{\eta}{\eta_{\rm R}}\right)^{1/3} . \tag{12}$$

The substitution of η from (9) results in the scale factor of the FLRW metric

$$a(\tau) = \left(\frac{2c^2}{\alpha\eta_{\rm R}}\right)^{1/3} \left[\sin\left(\frac{\alpha\tau}{2c}\right)\right]^{2/3} . \tag{13}$$

To demonstrate that such a form of the scale factor corresponds to the Schwarzschild-AdS (SAdS) case, we analyze the recession velocity of a point located at the spherical shell. In accordance with (9), consider an expanding spherically symmetric volume and its derivative with proper time as

$$V(\tau) = A\,\eta(\tau) = A\,\frac{c^2}{\alpha}\left(1 - \cos\left(\frac{\alpha\tau}{c}\right)\right) , \quad \dot{V} = \frac{dV}{d\tau} = A\,c\sin(\frac{\alpha\tau}{c}) . \tag{14}$$

Now we may express the derivative $\dot{V}$ in terms of r:

$$\dot{V}(r) = \pm A\,c\left(1 - \left(1 - \frac{\alpha}{A\,c^2}V\right)^2\right)^{1/2} = \pm A\,c\left(\frac{2\alpha}{c^2 A}V - \frac{\alpha^2}{c^4 A^2}V^2\right)^{1/2} = \pm\left(\frac{8\pi A\alpha}{3}r^3 - \frac{16\pi^2\alpha^2}{9c^2}r^6\right)^{1/2} . \tag{15}$$

This allows to obtain the recession and free-fall velocities at distance r from the center of the sphere using the apparent relation for spherical volume element expansion

$$v(r) = \frac{\dot{V}}{4\pi r^2} = \pm\left(\frac{A\alpha}{6\pi r} - \frac{\alpha^2}{9c^2}r^2\right)^{1/2} , \quad \alpha_1 > 0 . \tag{16}$$

Using the ratios of the model, respectively given by (23) and (4) in [15] as

$$A = \beta\,V_m\frac{H_0}{c} , \quad \text{where} \quad V_m := m\left(k\frac{3H_0^2}{8\pi G}\right)^{-1} , \tag{17}$$

where k, β are constant parameters of the model, and A is set with direct proportionality to the central mass. The substitution leads to

$$v(r) = \pm\left(\frac{2Gm}{r} - \frac{\Lambda c^2}{3}r^2\right)^{1/2} , \qquad (3H_0^2 = c^2\Lambda > 0) \tag{18}$$

along with the requirements

$$\alpha = 3H_0 c , \qquad \frac{3k}{2\beta} = 1 . \tag{19}$$

In order to eliminate the dependency of the scale factor on a central mass the model[4] sets

$$\eta_{\rm R} = \frac{k}{\beta}\frac{c}{H_0} ,$$

therefore, with the use of (19) the scale factor (13) takes the form

$$a(\tau) = \left[\sin\left(\frac{3}{2}H_0\tau\right)\right]^{2/3} , \tag{20}$$

where the pre-factor becomes the unity. Since the form of the scale factor depends only on the fundamental constants and proper time, it can be considered the *intrinsic property of space*.

Furthermore, the gravitational potential $\phi = \frac{1}{2}v^2$ for (18) corresponds to the SAdS gravity with negative cosmological term. The explicit path to the static metric can be given as follows. The velocity is also $v(r) = \dot{a}R$, where $a(\tau)$ is (13), and $\dot{a}$ is the derivative with respect to proper time. With the use of the tetrad

$$f_\mu{}^a = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -va^{-1} & a^{-1} & 0 & 0 \\ 0 & 0 & a^{-1} & 0 \\ 0 & 0 & 0 & a^{-1} \end{pmatrix} , \tag{21}$$

[4]From $\eta_{\rm R} = k\eta_m$, which is the definition for an arbitrary R given by (9), and $\eta_m = \frac{c}{\beta H_0}$ as given by (24) in [15].

which is the equivalent of the following coordinates change

$$dR = -v\frac{d\tau}{a} + \frac{dr}{a}\,, \quad R = \frac{r}{a}\,, \quad \text{and} \quad d\tau = dt\,, \tag{22}$$

the FLRW metric is transformed into the Gulfstrand-Painlevé (GP) metric [13]. The diagonalization of the GP metric tensor leads to its static form [2,12], which in the case of the obtained velocity (18) is the Schwarzschild-AdS metric. On the correspondence of static and non-static metrics, see also [21].

Notably, if one takes the derivative in (14) but with coordinate time instead, the resulting expression for $v(r)$ has the same form, but the cosmological term removed $\Lambda = 0$. It implies that the expansion can be measured only in the comoving frame, that is by an observer attached to the comoving point of the expanding universe, which becomes apparent, taking into account the transformation of the time coordinate (8). The approach suggests that we only evaluate and *measure our proper time* attached to our local frame of reference, that is, at the point of the expanding universe, but not in coordinate time. Thus, the coordinate time (8) becomes somewhat abstract and not directly related to our observations and measurements.

Thereby, Clifford algebra $Cl_{3,0}$ corresponds to the Schwarzschild-AdS gravity with the negative cosmological term and naturally requires the oscillatory model of periodic universe expansion. For instance, from (20) the half-period (from zero to zero) of the oscillation is

$$T = \frac{4\pi c}{\alpha} = \frac{4\pi}{3H_0} \tag{23}$$

measured in proper time. The amplitude of such oscillation can also be obtained using the coordinate distance $r = Ra$ and the explicit form for $\eta_R \to R$.

4. Clifford Algebra of Anti-space $Cl_{0,3}$

The Clifford algebra of "anti-Euclidean" space $Cl_{0,3}$ is also an eight-dimensional algebra, and is *not isomorphic* to $Cl_{3,0}$ (APS). The quadratic form is negatively defined. The algebra is isomorphic to split-biquaternions and is also an algebra of alternions $\mathbb{A}_4$ [25,26]. The basis can be given using the quaternion units [17]

$$\begin{array}{cccc|cccc} e_0 & e_1 & e_2 & e_3 & (1,\ 1) & (i,-i) & (j,-j) & (k,-k) \\ e_{123} & e_{23} & e_{13} & e_{12} & (1,-1) & (i,\ i) & (j,\ j) & (k,\ k) \end{array}\,, \tag{24}$$

where (a, b) denotes the diagonal two-by-two matrices. The upper raw represents the basis for the quaternions $\mathbb{H}$, and the lower is its "split-complexification" thereby

$$Cl_{0,3} \cong \mathbb{D} \otimes \mathbb{H}\,, \qquad Cl_{0,3} \cong \mathbb{H} \oplus \mathbb{H}\,. \tag{25}$$

Contrary to the previous case, there is no trivial correspondence to Minkowski spacetime since quaternions have isometry to the Euclidean space[5]. The upper raw in the basis represents quaternion h_1, and the lower raw is split-complexified quaternion jh_2 (a split-complex number is $c = a+jb \in \mathbb{D}$: $\mathrm{j}^2 = 1, \mathrm{j} \neq \pm 1, a, b \in \mathbb{R}$. $\mathbb{D} = \mathbb{R}\oplus\mathbb{R}$, and it is algebra of $Cl_{1,0}$.) Because of (25) the algebra $Cl_{0,3}$ is also $\mathbb{H}^{1,1}$. Hence, an eight-dimensional vector can be written as

$$h = h_1 + jh_2\,.$$

Using the complex conjugate with respect to j the vector norm is

$$\begin{gathered} |h|^2 = hh^* = (h_1 + jh_2)(h_1 - jh_2) = |h_1|^2 - |h_2|^2 = \\ x_0^2 + x_1^2 + x_2^2 + x_3^2 - x_{123}^2 - x_{23}^2 - x_{13}^2 - x_{12}^2\,. \end{gathered} \tag{26}$$

It gives the sum of two Minkowski *paravectors* $(x_0, x_{23}, x_{13}, x_{12})$, and (x_{123}, x_1, x_2, x_3) with reverse signatures, and implies that, for example, the first Minkowski four-vector can be built from the scalar and three bi-vectors of $Cl_{0,3}$. However, it must be stressed that such accordance to Minkowski four-vectors can be considered only in terms of vector space, but not in terms of algebra. Using such paravector representation, we can also map x_0 to the time coordinate and x_{123} to a volume as in the previous APS case.

The center of $Cl_{0,3}$ is $\mathbb{D}$ yielding two dimensional (Minkowski) spacetime $\mathbb{R}^{1,1}$. Such spacetime is spanned by two-vector represented by split-complex number

$$u = t + j\eta \in \mathbb{D}\,, \quad |u|^2 = uu^* = t^2 - \eta^2\,, \quad (t, \eta \in \mathbb{R})\,. \tag{27}$$

[5] Early attempts using a quaternion as the four-vector can be attributed to Minkowski himself, who considered this as "too narrow and clumsy for the purposes" [28]. Since Silberstein's time, there have been many attempts from physicists to map quaternions to 4-vectors. Stephen Hawking's proposal [5,11] to introduce imaginary time ($t \to it$) is one of such attempt because it also indirectly complexifies the quaternion. The *impossibility* of the direct mapping of a Minkowski 4-vector to a quaternion originates from the fact that the Lie group $SU(2)$ requires complexification over $\mathbb{C}$.

Thus, the spacetime invariant for the $Cen(Cl_{0,3})$ is

$$ds^2 = c^2 dt^2 - d\eta^2 \,. \tag{28}$$

It has signature (+-), which leads to the parameterization with the hyperbolic functions or with the group $SO(1,1)$. Therefore the Rindler coordinate transformations[6] take its classical form of Minkowski spacetime [10, 20, 23] where the coordinate time is

$$t = \frac{c}{\alpha} \sinh\left(\frac{\alpha\tau}{c}\right) \,, \tag{29}$$

and the coordinate distance from the origin of motion is

$$\eta = \frac{c^2}{\alpha}\left(\cosh\left(\frac{\alpha\tau}{c}\right) - 1\right) = 2\eta_0 \sinh^2\left(\frac{\alpha\tau}{2c}\right) \,, \quad \eta_0 = \frac{c^2}{\alpha} \,, \tag{30}$$

where the initial synchronization of the clocks at the beginning ($t = \tau = 0$ in $\eta = 0$) is considered. Thus, the approach has the same path as given above in Section 3; repeating it yields the scale factor of the FLRW metric as

$$a = \left[\sinh\left(\frac{\alpha\tau}{2c}\right)\right]^{2/3} \,, \qquad \alpha = 3H_0 c \,. \tag{31}$$

The scale factor coincidences with the one used in the standard cosmology for the current "dark energy dominated" epoch where it has the following form [27]

$$a(\tau) = \left(\frac{\Omega_m}{\Omega_\Lambda}\right)^{1/3} \left[\sinh\left(\sqrt{\Omega_\Lambda}\frac{3}{2}H_0\tau\right)\right]^{2/3} \,. \tag{32}$$

The full correspondence can be seen in the case of $\Omega_m = 1$, $\Omega_\Lambda = 1$, though, as shown in the next section, the model allows using an arbitrary Ω_m, though eliminating the necessity for the value of Ω_Λ (or by setting it to the unity).

In the similar way as in Section 3, because of (30) the spherical volume expansion in the case of $Cl_{0,3}$ becomes

$$V(\tau) = A\,\eta(\tau) = A\,\frac{c^2}{\alpha}\left(\cosh\left(\frac{\alpha\tau}{c}\right) - 1\right) \,, \quad \dot{V} = \frac{dV}{d\tau} = A\,c \sinh\left(\frac{\alpha\tau}{c}\right) \,. \tag{33}$$

Expressing the derivative $\dot{V}$ in terms of r yields

$$\dot{V}(r) = \pm A\,c\left(\left(1 + \frac{\alpha}{A\,c^2}V\right)^2 - 1\right)^{1/2} = \pm\left(\frac{8\pi A\alpha}{3}r^3 + \frac{16\pi^2\alpha^2}{9c^2}r^6\right)^{1/2} \,, \tag{34}$$

and in the same way as (16) with (17) leads to the recession and free-fall velocities

$$v(r) = \frac{\dot{V}}{4\pi r^2} = \pm\left(\frac{2Gm}{r} + \frac{\Lambda c^2}{3}r^2\right)^{1/2} \,, \qquad (3H_0^2 = c^2\Lambda > 0) \,, \tag{35}$$

where comparing to (18) the cosmological term has the opposite sign. Using the tetrad (21) one obtains the static metric of the Schwarzschild-de Sitter.

In contrast with the previous case, the Clifford algebra of anti-space $Cl_{0,3}$ is different from Minkowski structure of isomorphic algebras and Lie groups. For instance, it is isomorphic to even subalgebras $Cl^0_{4,0} \cong Cl^0_{0,4}$ [1]. In terms of Lie groups, as per (25), $Cl_{0,3}$ corresponds to the group $SO(3,\mathbb{D}) \cong SO(3) \times SO(3)$, and it is isomorphic to split-complexified $SU(2)$. The latter provides the isomorphism of $SU(2) \times SU(2) \cong SO(4)$.

5. Observed Cosmological Model via Measured Deceleration Parameter

By definition the deceleration parameter is

$$q_0 = -\frac{\ddot{a}a}{\dot{a}^2} \,. \tag{36}$$

The derivatives of the scale factor can be obtained using its relation to the coordinate acceleration, velocity and coordinate distance: $\alpha' = \ddot{a}R$, $v = \dot{a}R$, $r = Ra$. In this way,

$$q_0 = -\frac{r}{v^2}\frac{d}{dr}\left(\frac{v^2}{2}\right) \,. \tag{37}$$

[6] They are derived by the integration of $dt = \gamma d\tau$ and $d\eta = v(t)\gamma d\tau$, where $\gamma = \cosh\left(\frac{\alpha\tau}{c}\right)$ and $v(t) = c\tanh\left(\frac{\alpha\tau}{c}\right)$.

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 5 February 2021 doi:10.20944/preprints202101.0544.v2

The substitution of (18) leads to

$$q_0 = \frac{Gm + H_0^2 r^3}{2Gm - H_0^2 r^3} . \tag{38}$$

Furthermore, using (35) for the SdS case, and assuming uniformly distributed mass m within a sphere of radius r with density expressed in terms of the critical density

$$\Omega_M = \frac{\rho}{\rho_{cr}} , \qquad \rho_{cr} = \frac{3H_0^2}{8\pi G}$$

both cases are

$$\text{SAdS}: \quad q_0 = \frac{1}{2}\frac{\Omega_M + 2}{\Omega_M - 1} , \qquad \text{via} \;\; \text{SdS}: \quad q_0 = \frac{1}{2}\frac{\Omega_M - 2}{\Omega_M + 1} . \tag{39}$$

Recent observations [16, 18] indicate that the value of the deceleration parameter is negative and is on the order of $q_0 = -0.6$. The given expression for the SdS case of the model leads to a comparable value for the deceleration parameter by setting $\Omega_M = 0.3$ as noted in [13], which is in accordance with the standard model. Notably, the presented approach does not use a value of Ω_Λ to obtain the comparable result for q_0 as per the measurements.

The SAdS case results in $q_0 < -1$ if $0 < \Omega_M < 1$, which is slightly out of range of the measured value for q_0. Thus, the current observations suggest that the Universe expands according to the SdS model of gravitation.

6. Discussion

The case of Clifford algebra of the physical space $Cl_{3,0}$ demonstrates conformity to the anti-de Sitter cosmology and the *oscillating* scale factor of the FLRW metric. Such a case appears natural because of the algebra's native isomorphism to Minkowski spacetime. For instance, electromagnetism has symmetry with $Cl_{3,0}$ algebra via a homomorphism $SO(3,\mathbb{C})$ group, where it can be represented by a complex vector $\mathbf{E} + i\mathbf{B}$. The complex numbers corresponding to *time* and *space volume* $u = t + i\eta$ points to a certain similarity. The complex numbers are known to induce both spatial rotations and oscillations. Another notable property of $Cl_{3,0}$ is that it allows the quick extrapolation of the results to the spacetime algebra (STA) using the isomorphism with $Cl^0_{1,3}$. As an example, for $Cl_{1,3}$ where the basis is the Dirac matrices, and the transformation is $\gamma_\mu \to \overline{\gamma}_a = \gamma_\mu h_\mu{}^a$ [9]. Hence, in the case of the FLRW metric (10) it yields $\overline{\gamma}_k = a\gamma_k$, $(k = 1, 2, 3)$.

Nature seems to prefer complex numbers over split-complex numbers; thus, the application of the Clifford algebra $Cl_{0,3}$ to the model possesses certain "unnaturalness." Nevertheless, since it provides a path to the de Sitter model, which is the basis for modern observational cosmology, therefore $Cl_{0,3}$, its isomorphic algebras and related groups, such as $SO(3,\mathbb{D})$ and $SO(4)$ require further study on their possible relation to gravitation. For instance, the symmetry with the group $SO(3,\mathbb{D})$ suggests that we may consider a similar split-complex vector, which due to the hyperbolic nature of the split-complex numbers, does not induce harmonic oscillations.

However, the reviewed model must be switched to the inverse in one case: if Hawking's imaginary time [5,11] is a valid concept. Since it would affect the core, precisely the quadratic form of the algebra's center, the imaginary time would lead to the fact that the results (13) and (31) must be interchanged. Since the paravector representation implies the quadratic forms (4) and (26), and the invariant forms for the center given by (6) and (28), therefore the transformations which involve a complexification of the time coordinate have to be further thoroughly studied in the application to the approach.

As noted in Section 3 the approach eliminates the cosmological term by the redefinition of proper time with the coordinate time. The disappearance of the Lambda term from the cosmological equations has been clearly demonstrated in the integrable Weyl geometry models such as the Scale Invariant Vacuum theory, and as seen in the MOND, where the constant of acceleration is also related to cH_0. Such correspondences can be a prospective topic for future research.

7. Conclusions

The Clifford algebras of space $Cl_{3,0}$ (APS) and anti-Space $Cl_{0,3}$ have a two-dimensional center that in paravector representation corresponds to time-volume coordinates. It is shown that the uniform acceleration in such two-dimensional spacetime induces the FLRW metric in a comoving frame with the exact expressions for the scale factor. The constant uniform acceleration has the value expressed by the fundamental constants $\alpha = 3H_0 c = c\sqrt{3\Lambda}$.

A new feature of the model is that it states the essence of the cosmological expansion due to the uniform acceleration in Clifford coordinates of $Cen(Cl_{3,0})$, and $Cen(Cl_{0,3})$. The case of Clifford algebra of the physical space $Cl_{3,0}$ demonstrates conformity to the anti-de Sitter cosmology and the oscillating scale factor of the FLRW metric. Clifford algebra $Cl_{0,3}$ leads to the de Sitter model with the positive cosmological term and the exact form of the scale factor used in modern cosmology.

The approach suggests that the appearance of the cosmological term is due to the relativistic effect, specifically given by the time coordinate transformations (8) and (29). Hence, it is attributed to the fact that we only evaluate and measure our proper time attached to our local comoving frame of reference, that is at the point of the expanding universe.

Moreover, as shown, in the application to a point mass, the uniform acceleration approach results in expressions (16) and (18) corresponding to the Schwarzschild-AdS and the SdS metrics for gravitation in the static coordinates. Since the present study is limited by the spherically symmetric case, a generalization to an arbitrary coordinate system using the tetrad formalism and Clifford coordinates can be a prospective topic for future research.

References

[1] Ablamowicz, E.; Baylis, W.E. et al. Lectures on Clifford (Geometric) Algebras and Applications, Springer Science+Business Media LLC, **2003**

[2] Christillin, P.; Morchio, G. Relativistic Newtonian gravitation. arXiv:1707.05187 [gr-qc], **2019**

[3] Clifford, W.K. Mathematical Papers by William Kingdon Clifford, R.Tucker, Macmillan and Co, London, **1882**, 266-276

[4] Clifford, W. K. On the Classification of Geometric Algebras, paper XLIII, in Mathematical Papers of W. K. Clifford, edited by R. Tucker, MacMillan, London, **1882**.

[5] Deltete, Robert J.; Guy, Reed A. Emerging from imaginary time, *Synthese*, vol. 108, p.185–203, **1996**

[6] Garling, D. J. H. Clifford Algebras: An Introduction, Cambridge University Press, **2011**

[7] Girard, Patric R. Quaternions, Clifford Algebras and Relativistic Physics, Birkhauser Verlag AG, Basel-Boston-Berlin, **2007**

[8] Greiter, M.; Schuricht, D. Imaginary in all directions: an elegant formulation of special relativity and classical electrodynamics. **2003** *Eur. J. Phys.* 24 397

[9] Gu, Y.Q. Space-Time Geometry and Some Applications of Clifford Algebra in Physics, *Advances in Appplied Clifford Algebras*, **2018**

[10] Hamilton, J. Dwaune The uniformly accelerated reference frame, *Am. J. Phys.*, 46(1), Jan. **1978**

[11] Hartle J. B.; Hawking S. W. Wave function of the Universe, Phys. Rev. D 28, 2960,1983

[12] Kritov, A. Approach to the Schwarzschild Metricwith via SL(2,R) Group Decomposition, *Progress in Physics*, **2020**, V. 16(2), 139-142.

[13] Kritov, A. From the FLRW to the Gravitational Dynamics, *Progress in Physics*, **2019** (vol. 15), issue 3, 145–147.

[14] Kritov, A. On the Fluid Model of the Spherically Symmetric Gravitational Field, *Progress in Physics*, **2019** (vol. 15), issue 2, 101–105.

[15] Kritov, A. Unified Two Dimensional Spacetime for the River Model of Gravity and Cosmology, *Progress in Physics*, **2019**, V. 15(3), 163-170.

[16] Lixin, Xu. et al. Reconstruction of Deceleration Parameters from Recent Cosmic Observations,arXiv:astro-ph/0701519v2, **2007**

[17] Lounesto, P. Clifford Algebras and Spinors, Cambridge University Press, 2nd edition, **2001**

[18] Mamon, Abdulla Al Constraints on a generalized deceleration parameter from cosmic chronometers, arXiv:1702.04916v2 [gr-qc], **2018**

[19] McCrea, W. H.; Minle, E. A. Newtonian Universes and the Curvuature of Space, *The Quarterly Journal of Mathematics*, Volume-5, Issue 1, 1934, Pages 73–80.

[20] Møller, C. The Theory of Relativity. p.75, Oxford Clarendon Press, **1955**

[21] Mitra, A. Interpretational conlicts between the static and non-static forms of the de Sitter metric. *Scientific Reports*, **2012**, v. 2, article 923.

[22] Porteus, I. Clifford Algebras and the Classical Groups, Cambridge University Press, **2005**, p.137

[23] Rindler W. Essential Relativity Special, General, and Cosmological, Revised Second Edition, Springler-Verlag New York, **1986**, p.156

[24] Porteous, I.R. Clifford Algebras and the Classical Groups, Cambridge Studies in Advanced Mathematics: 50, **2009**

[25] Rosenfeld, B. A.; Wiebe, B. Geometry of Lie Groups, Springer US, **1997**

[26] Rosenfeld, B. A. Non-Euclidean geometry, State Publishing House of Technical and Theoretical Literature, Moscow, **1955** (In Russian)

[27] Sazhin, M. V.; Sazhina, O.S.; Chadayammuri, U. The Scale Factor in the Universe with Dark Energy, arXiv: astro-ph.CO, 1109.2258v1, **2011**

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 5 February 2021 doi:10.20944/preprints202101.0544.v2

[28] Silberstein, L. Quaternionic Form of Relativity, *Phil. Mag.*, S. 6, Vol. 23, No. 137 **1912**, 790-809.

[29] Tipler, Frank J. Newtonian Cosmology Revisited, Mon. Not. R. Astron. Soc. 282,206-210 **1996**

[30] Vilenkin A., Cosmological constant problems and their solutions, arXiv:hep-th/0106083, **2001**.

[31] Weinberg, S. The cosmological constant problem, *Reviews of Modern Physics*, Vol. 61, No. 1, January **1989**

[32] Zeldovich , Ya. B. Hydrodynamics of the Universe, Ann. Rev. Fluid Mech. 9, **1977** pp.215–228.