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Effects of uniform rotation and Cornell-type potential on KG-scalar particle in Kaluza-Klein theory

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Abstract

The non-inertial effects on spin-0 scalar particle that interacts with scalar potentials of Cornell-type in cylindrical system and Coulomb-type in the magnetic cosmic string space-time using Kaluza-Klein theory is analyzed. We show that the energy eigenvalue and eigenfunction depend on the global parameters characterizing the space-time, and the gravitational analogue of the Aharonov-Bohm effect for bound states is observed.

keywords: Kaluza-Klein theory, Relativistic wave equation, Aharonov-Bohm effect, special functions.

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1 Introduction

The relativistic quantum dynamics of spin-0 and spin-$\frac{1}{2}$ particle in the space-time background produced by topological defects using Kaluza-Klein theory (KKT) [1, 2, 3] have been investigated. Topological defects play an important role in various physical systems, such as, in condensed matter physics system [4, 5, 6, 7] where, topological defects analogue to cosmic strings appear in phase transitions in liquid crystals [8, 9]. Geometric quantum phases [10] is a quantum mechanical phenomena that describe phase shifts acquire by the wave-function of a quantum particle. A well-known such quantum phase

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is the Aharonov-Bohm effect [11, 12, 13] due to the presence of quantum flux produced by topological defects space-time. In the relativistic quantum system, this effect has investigated, such as, with scalar potential under torsion effects in cosmic string space-time [14, 15, 16, 17], bound states solution of spin-0 scalar particle in cosmic string space-time [18, 19]. In the context of Kaluza-Klein theory, this effect has studied with or without potential of various kind in the five-dimensional cosmic string space-time [20, 21, 22, 23, 24, 25], and in the five-dimensional Minkowski space-time [26, 27, 28].

Non-inertial eects related to rotation have been investigated in the relativistic quantum system, such as, on a Dirac particle [29], in relativistic Landau-He-McKellar-Wilkens quantization [30], on a neutral particle [31], on the Dirac oscillator [32], on a scalar field in the space-times with a space-like dislocation and a spiral dislocation [33], on scalar boson in cosmic string space-time [34], on spin-0 scalar particle in cosmic string space-time [35], on DKP equation with a magnetic cosmic string [36], on scalar field in the space-time with a magnetic screw dislocation [37], on the Dirac oscillator in cosmic string space-time [38], and KG-oscillator in cosmic string space-time with a spacelike dislocation [39].

In this work, we solve KG-equation with a scalar potential of Cornell-type in cylindrical system using Kaluza-Klein theory under the effects of rotation and analyze a relativistic analogue of the Aharonov-Bohm effect. Subsequently, we analyze the same relativistic system using a Coulomb-type scalar potential.

2 spin-0 scalar particle in rotating magnetic cosmic string space-time with potential

In the context of Kaluza-Klein theory [1, 2, 3], the metric with a magnetic quantum flux (Φ) passing along the symmetry axis of the string in five-

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dimension is given by [21]

$$ds^2 = -dt^2 + dr^2 + \alpha^2\, r^2\, d\phi^2 + dz^2 + (dy + \kappa\, A_\phi d\phi)^2, \tag{1}$$

where the gauge field given by

$$A_\phi = \kappa^{-1}\, \frac{\Phi}{2\pi} \tag{2}$$

such that $\vec{B} = \vec{\nabla} \times \vec{A} = -\kappa^{-1}\, \Phi\, \delta^2(\vec{r})$ [21]. Here $y = x^4$ is the fifth spatial coordinate having ranges $0 < y < 2\,\pi\, a$ where, a is the radius of the compact dimension of y, and κ is the Kaluza constant [21]. The parameter $\alpha = (1 - 4\,\mu)$ [40, 41] where, μ is the linear mass density of the string. In gravitation and cosmology, we assume the values of the parameter α are in the ranges $0 < \alpha < 1$.

To introduce a uniform rotation in the above space-time, considering the transformation $\phi \to \phi + \omega\, t$ [31, 32, 33, 34, 35, 36, 37, 38, 39, 42, 43] into the line-element (1), where ω is the velocity of constant rotation of the rotating frame, which gives us the following line element

$$\begin{aligned} ds^2 &= -[1 - \omega^2\, (\alpha^2\, r^2 + \kappa^2\, A_\phi^2)]\, dt^2 + dr^2 + dz^2 + dy^2 + 2\,\kappa\, A_\phi\, dy\, d\phi \\ &+ (\alpha^2\, r^2 + \kappa^2\, A_\phi^2)\, d\phi^2 + 2\,\omega\,\kappa\, A_\phi\, dy\, dt + 2\,(\alpha^2\, r^2 + \kappa^2\, A_\phi^2)\,\omega\, dt\, d\phi, \end{aligned} \tag{3}$$

By consequence of the rotating frame and in order to make the component g_{00} remain negative, we extract the information on the radial coordinate $0 \leq r < \frac{\sqrt{1-\kappa^2\, A_\phi^2\,\omega^2}}{\alpha\,\omega}$, that is, the radial coordinate in the background dened by the metric (3) is restrict to the range above. We can note that, in addition to the velocity of rotation of the uniformly rotating frame, the above inequality is determined by the parameter related to the quantum ux Φ of the KKT. We can also note that, for $r > \frac{\sqrt{1-\kappa^2\, A_\phi^2\,\omega^2}}{\alpha\,\omega}$, we have that the particle is placed outside of the light-cone [44].

The relativistic quantum dynamics of spin-0 scalar particle with a scalar potential $S(r)$ by modifying the mass term $m \to m + S(r)$ in five-dimensional

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space-time is described by

$$\left[\frac{1}{\sqrt{-g}}\,\partial_M(\sqrt{-g}\,g^{MN}\,\partial_N)-(m+S)^2\right]\Psi=0, \tag{4}$$

where $M,N=0,1,2,3,4$ with $g=-\alpha^2\,r^2$ is the determinant of the metric tensor with g^{MN} its inverse.

By considering the line-element (3) into the Eq. (4), we obtain the following differential equation:

$$\left[-\left(\frac{\partial}{\partial t}-\omega\,\frac{\partial}{\partial\phi}\right)^2+\frac{\partial^2}{\partial r^2}+\frac{1}{r}\,\frac{\partial}{\partial r}+\frac{1}{\alpha^2\,r^2}\left(\frac{\partial}{\partial\phi}-\kappa\,A_\phi\,\frac{\partial}{\partial y}\right)^2+\frac{\partial^2}{\partial z^2}+\frac{\partial^2}{\partial y^2}\right]\Psi$$
$$=(m+S)^2\,\Psi. \tag{5}$$

Since the line-element (2) is independent of t,ϕ,z,y. One can choose the following ansatz for the function Ψ as:

$$\Psi(t,r,\phi,z,y)=e^{i\,(-E\,t+l\,\phi+k\,z+q\,y)}\,\psi(r), \tag{6}$$

where E is the total energy of the particle, $l=0,\pm 1,\pm 2,..\in\mathbf{Z}$, and k,q are constants.

Substituting the ansatz (6) into the Eq. (5), we obtain the following equation:

$$\left[\frac{d^2}{dr^2}+\frac{1}{r}\,\frac{d}{dr}+(E+l\,\omega)^2-k^2-q^2-\frac{(l-\kappa\,q\,A_\phi)^2}{\alpha^2\,r^2}-(m+S)^2\right]\psi(r)=0. \tag{7}$$

Case A : Interactions with Cornell-type potential

The Cornell-type potential consists of linear plus Coulomb-like term is a particular case of the quark-antiquark interaction [45, 46]. The Coulomb potential is responsible at small distances or short range interactions and linear potential leads to confinement of quark. This type of potential is given by [17]

$$S(r)=\frac{\eta_c}{r}+\eta_L\,r \tag{8}$$

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where η_c, η_L are the potential parameters.

Substituting eqs. (3) and (8) into the Eq. (7), we obtain the following equation:

$$\left[\frac{d^2}{dr^2} + \frac{1}{r}\,\frac{d}{dr} + \lambda - \frac{j^2}{r^2} - \eta_L^2\,r^2 - \frac{a}{r} - b\,r\right]\,\psi(r) = 0, \tag{9}$$

where

$$\begin{aligned} \lambda &= (E + l\,\omega)^2 - k^2 - q^2 - m^2 - 2\,\eta_c\,\eta_L, \\ j &= \sqrt{\frac{(l - \frac{q\,\Phi}{2\pi})^2}{\alpha^2} + \eta_c^2}, \\ a &= 2\,m\,\eta_c, \\ b &= 2\,m\,\eta_L. \end{aligned} \tag{10}$$

Introducing a new variable $\rho = \sqrt{\eta_L}\,r$, Eq. (9) becomes

$$\left[\frac{d^2}{d\rho^2} + \frac{1}{\rho}\,\frac{d}{d\rho} + \zeta - \frac{j^2}{\rho^2} - \rho^2 - \frac{\eta}{\rho} - \theta\,\rho\right]\,\psi(\rho) = 0, \tag{11}$$

where

$$\zeta = \frac{\lambda}{\eta_L} \quad , \quad \eta = \frac{a}{\sqrt{\eta_L}} \quad , \quad \theta = \frac{b}{\eta_L^{\frac{3}{2}}}. \tag{12}$$

Suppose the possible solution to Eq. (11) is

$$\psi(\rho) = \rho^j\,e^{-\frac{1}{2}\,(\rho+\theta)\,\rho}\,H(\rho). \tag{13}$$

Substituting the solution Eq. (13) into the Eq. (11), we obtain

$$H''(\rho) + \left[\frac{\gamma}{\rho} - \theta - 2\,\rho\right]\,H'(\rho) + \left[-\frac{\beta}{\rho} + \Theta\right]\,H(\rho) = 0, \tag{14}$$

where

$$\begin{aligned} \gamma &= 1 + 2\,j, \\ \Theta &= \zeta + \frac{\theta^2}{4} - 2\,(1 + j), \\ \beta &= \eta + \frac{\theta}{2}\,(1 + 2\,j). \end{aligned} \tag{15}$$

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Equation (14) is the biconfluent Heun's differential equation [23, 24, 14, 47, 48] and $H(\rho)$ is the Heun polynomials.

The above equation (14) can be solved by the Frobenius method. We consider the power series solution [49]

$$H(\rho)=\sum_{i=0}^{\infty} c_i\,\rho^i \tag{16}$$

Substituting the above power series solution into the Eq. (14), we obtain the following recurrence relation for the coefficients:

$$c_{n+2}=\frac{1}{(n+2)(n+2+2\,j)}\left[\{\beta+\theta\,(n+1)\}\;c_{n+1}-(\Theta-2\,n)\,c_n\right]. \tag{17}$$

And the various coefficients are

$$c_1=\left(\frac{\eta}{\gamma}+\frac{\theta}{2}\right)\,c_0,$$
$$c_2=\frac{1}{4\,(1+j)}\,[(\beta+\theta)\;c_1-\Theta\,c_0]. \tag{18}$$

We must truncate the power series by imposing the following two conditions [23, 24, 14, 26, 27, 28]:

$$\Theta = 2\,n, \quad (n=1,2,...)$$
$$c_{n+1} = 0. \tag{19}$$

By analyzing the condition $\Theta=2\,n$, we get the following second degree expression of the energy eigenvalues $E_{n,l}$:

$$\frac{\lambda}{\eta_L}+\frac{\theta^2}{4}-2\,(1+j)=2\,n$$
$$\Rightarrow \quad E_{n,l}=-\omega\,l\pm\sqrt{k^2+q^2+2\,\eta_L\left(n+1+\eta_c+\sqrt{\frac{(l-\frac{q\,\Phi}{2\pi})^2}{\alpha^2}+\eta_c^2}\right)} \tag{20}$$

Now, we impose additional recurrence condition $c_{n+1} = 0$ to find the individual energy levels and wave-functions one by one as done in [23, 24, 14]. For $n = 1$, we have $\Theta = 2$ and $c_2 = 0$ which implies from Eq. (18)

$$c_1 = \frac{2}{\beta+\theta}\, c_0 \Rightarrow \left(\frac{\eta}{1+2\,j} + \frac{\theta}{2}\right) = \frac{2}{\beta+\theta}$$

$$\eta_{1\,L}^3 - \frac{a^2}{2\,(1+2\,j)}\,\eta_{1\,L}^2 - a\,b\,(\frac{1+j}{1+2\,j})\,\eta_{1\,L} - \frac{b^2}{8}\,(3+2\,j) = 0 \tag{21}$$

a constraint on the potential parameter $\eta_{1\,L}$. This algebraic third degree equation has at least one real root and it is exactly this solution that gives us the allowed value of the potential η_L for the lowest state of the system and one can obtain first degree polynomial solution to $H(\rho)$ for the radial mode $n = 1$.

Therefore, the ground state energy level for $n = 1$ is given by

$$E_{1,l} = -\omega\, l \pm \sqrt{k^2 + q^2 + 2\,\eta_{1\,L}\left(2 + \eta_c + \sqrt{\frac{(l-\frac{q\,\Phi}{2\,\pi})^2}{\alpha^2} + \eta_c^2}\right)}\,. \tag{22}$$

And the radial wave-function is

$$\psi_{1,l} = \rho^{\sqrt{\frac{(l-\frac{q\,\Phi}{2\pi})^2}{\alpha^2}+\eta_c^2}}\, e^{-\frac{1}{2}\left(\frac{2\,m}{\sqrt{\eta_{1\,L}}}+\rho\right)\rho}\,(c_0 + c_1\,\rho)\,, \tag{23}$$

where

$$c_1 = \frac{1}{\sqrt{\eta_{1\,L}}}\left[\frac{2\,m\,\eta_c}{\left(1 + 2\sqrt{\frac{(l-\frac{q\,\Phi}{2\pi})^2}{\alpha^2} + \eta_c^2}\right)} + m\right] c_0. \tag{24}$$

Then by substituting the real solution of $\eta_{1\,L}$ from Eq. (21) into the equation (22) together with (23)–(24), one can obtained the ground state energy level and corresponding eigenfunction for the radial mode $n = 1$.

Case B : Interactions with Coulomb-type potential

We consider $\eta_L \to 0$ into the scalar potential S. Thus the Coulomb potential is given by

$$S(r) = \frac{\eta_c}{r}, \tag{25}$$

This kind of potential has used to study position-dependent mass systems [23, 27, 50, 51] in the relativistic quantum mechanics.

The radial wave-equations Eq. (9) becomes

$$\left[\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} + \bar{\lambda} - \frac{j^2}{r^2} - \frac{a}{r}\right]\psi(r) = 0, \tag{26}$$

where $\bar{\lambda} = (E + \omega\, l)^2 - k^2 - q^2 - m^2$.

The above equation can be expressed as [52]

$$\psi''(r) + \frac{1}{r}\psi'(r) + \frac{1}{r^2}(-\xi_1\, r^2 + \xi_2\, r - \xi_3)\,\psi(r) = 0 \tag{27}$$

where

$$\xi_1 = -\bar{\lambda} \quad , \quad \xi_2 = -a \quad , \quad \xi_3 = j^2. \tag{28}$$

The energy eigenvalues $E_{n',l}$ is given by:

$$(2\,n' + 1)\sqrt{\xi_1} - \xi_2 + 2\sqrt{\xi_1\,\xi_3} = 0$$
$$\Rightarrow E_{n',l} = -\omega\, l \pm m\sqrt{1 + \frac{k^2}{m^2} + \frac{q^2}{m^2} - \frac{\eta_c^2}{\left(n' + \frac{1}{2} + \sqrt{\frac{(l-\frac{q\,\Phi}{2\pi})^2}{\alpha^2} + \eta_c^2}\right)^2}} \tag{29}$$

where $n' = 0, 1, 2, ...$

The wave-function is given by

$$\psi_{n',l}(r) = |N|_{n',l}\, r^{\sqrt{\frac{(l-\frac{q\,\Phi}{2\pi})^2}{\alpha^2}+\eta_c^2}}\, e^{-\sqrt{k^2+q^2+m^2-(E_{n',l}+\omega\, l)^2}\, r} \times L_{n'}^{\left(2\sqrt{\frac{(l-\frac{q\,\Phi}{2\pi})^2}{\alpha^2}+\eta_c^2}\right)}(2\sqrt{k^2+q^2+m^2-(E_{n',l}+\omega\, l)^2}\, r), \tag{30}$$

where $|N|_{n',l}$ is the normalization constant and $L_{n'}^{(2\,j)}(r)$ is the generalized Laguerre polynomials.

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In both cases we have observed that the angular momentum number l is shifted, $l \to l_0 = \frac{1}{\alpha}(l - \frac{q\,\Phi}{2\,\pi})$, an effective angular quantum number. Therefore, the relativistic energy eigenvalues and eigenfunction depends on the geometric quantum phase [13]. Thus, we have that, $E_{n,l}(\Phi + \Phi_0) = E_{n,l\mp\tau}(\Phi)$, where $\Phi_0 = \pm\frac{2\pi}{q}\,\tau$ with $\tau = 0, 1, 2, ..$ This dependence of the relativistic energy level on the geometric quantum phase gives us a relativistic analogue of the Aharonov-Bohm effect for bound states. Furthermore, due to the effects of rotation, a contribution to the relativistic energy levels that gives rise to a Sagnac-type effect [29, 53, 54, 55] is observed in both cases.

3 Conclusions

We have investigated rotating effects on scalar particle subject to Cornell-type potential in cylindrical system and subsequently with Coulomb-type scalar potential in the magnetic cosmic string space-time using Kaluza-Klein theory. We have started our discussion in all cases through the restriction on the radial coordinate r that arises from the uniformly rotating frame and the topology of the space-time. We have obtained the bound states solution of the quantum system and analyze the effects on the eigenvalue and eigenfunction. In **Case A**, we have considered a Cornell-type scalar potential into the system and arrived the biconfluent Heun's differential equation form. Finally using the power series expansion method, we have solved the differential equation and obtained the energy eigenvalues and eigenfunction by truncating the power series solution. Then analyzing the recurrence condition $c_{n+1} = 0$ for each radial mode, for example, the radial mode $n = 1$, we have evaluated the ground state energy level (22) and the eigenfunction (23)–(24) and others are in the same way. One can see that the lowest state is defined by the radial mode $n = 1$ instead of $n = 0$. This effect arises due to the presence of Cornell-type potential in the quantum system. In **Case B**, we have considered a Coulomb-type scalar potential and obtained the energy eigenvalues (29) and eigenfunction (30) by using the Nikiforov-Uvarov method.

Equations (20) and (29) gives us the spectrum of energy of a scalar particle subject to a Cornell-type and Coulomb-type scalar potential, respectively in the magnetic cosmic string space-time in a uniformly rotating frame. The contributions to the relativistic energy levels that stem from the topology of the magnetic cosmic string are given by the effective angular momentum, $l \to l_0 = \frac{(l - \frac{q\,\Phi}{2\pi})}{\alpha}$. Since there is no interaction between the scalar particle and the topological defects, the presence of the effective angular momentum in the relativistic energy levels means that there exists an analogue of the AharonovBohm effect. Besides, we can observe a Sagnac-type effect by the presence of the coupling between the angular velocity ω and the angular momentum quantum number l.

Data Availability

There is no data associated with this manuscript.

Conflict of Interests

Author declares that there is no conflict of interests regarding publication this manuscript.

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