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Derivation of Coulomb's law based on a mechanical model of electromagnetic field and a spherical source and sink model of electric charges

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We suppose that vacuum is filled with a kind of continuously distributed matter which may be called the $\Omega(1)$ substratum, or the electromagnetic aether. Suppose that the time scale of a macroscopic observer is very large compares to the the Maxwelllian relaxation time of the $\Omega(1)$ substratum. Thus, the macroscopic observer concludes that the $\Omega(1)$ substratum behaves like a Newtonian-fluid. Inspired by H. A. Lorentz, we speculate that electric charges may be extremely small hard spherical sources or spherical sinks with finite radii. Based on the spherical source and spherical sink model of electric charges, we derive Coulomb's law of interactions between static electric charges in vacuum. Further, we derive a reduced form of the Lorentz's force law for static electric charges in vacuum.

Keywords: Coulomb's Law; spherical source; spherical sink; electromagnetic aether; Lorentz's force law; hydrodynamics; fluid mechanics; vacuum mechanics.

I. INTRODUCTION

Coulomb's law of interactions between static electric charges in vacuum can be written as [1]

$$\mathbf{F}_{12} = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{r^2}\hat{\mathbf{r}}_{12}, \tag{1}$$

where q_1 and q_2 are the electric quantities of two electric charges, r is the distance between the two electric charges, ϵ_0 is the dielectric constant of vacuum, $\mathbf{F}_{12}$ is the force exerted on the electric charge with electric quantity q_2 by the electric charge with electric quantity q_1, $\hat{\mathbf{r}}_{12}$ denotes the unit vector directed outward along the line from the electric charge with electric quantity q_1 to the electric charge with electric quantity q_2.

The main purpose of this manuscript is to derive Coulomb's law of interactions between static electric charges in vacuum by means of fluid mechanics based on spherical source and spherical sink model of particles.

The motive of this manuscript is to seek a mechanism of Coulomb's law. The reasons why new mechanical interpretations of Coulomb's law are interesting may be summarized as follows.

Firstly, Coulomb's law is an elementary law in physics and play various roles in the fields of electromagnetism, electrodynamics, quantum mechanics, cosmology and thermodynamics, etc. [2, 3]. From the point view of reductionism, the fundamental importance of Coulomb's law in all branches of physics urges the reductionists to provide it a proper mechanical interpretation.

Secondly, the mechanism of this action-at-a-distance Coulomb's law remains an unsolved problem in physics for more than 200 years after the law was put forth by Coulomb in 1785 [2–6]. A satisfactory mechanical interpretation in the framework of Descartes' scientific research program [5] is interesting.

Thirdly, although the Maxwell's theory of electromagnetic phenomena is a field theory [1], the concept of field is different from that of continuum mechanics [7, 8, 8–10] because of the absence of a continuum. Thus, the Maxwell's theory can only be regarded as a phenomenological theory. New mechanical interpretations of Coulomb's law may help us to establish a field theory of electromagnetic phenomena [1, 11].

Fourthly, there exist some inconsistencies and inner difficulties in the classical electrodynamics [8, 12–14]. New theories of Coulomb's law may help to resolve such difficulties.

Finally, one of the tasks of physics is the unification of the four fundamental interactions in the universe. New theories of interactions between static electric charges may shed some light on this puzzle.

To conclude, it seems that new considerations on Coulomb's law is needed.

In this manuscript, we show that Coulomb's law of interactions between static electric charges may be derived based on a mechanical model of vacuum and a spherical source and sink model of electric charges.

II. A BRIEF INTRODUCTION OF A MECHANICAL MODEL OF ELECTROMAGNETIC FIELD

Maxwell's equations in vacuum can be written as [1]

$$\nabla \cdot \mathbf{E} = \frac{\rho_e}{\epsilon_0}\,, \tag{2}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}\,, \tag{3}$$

$$\nabla \cdot \mathbf{B} = 0\,, \tag{4}$$

$$\frac{1}{\mu_0}\nabla \times \mathbf{B} = \mathbf{j}_e + \epsilon_0\frac{\partial \mathbf{E}}{\partial t}\,, \tag{5}$$

where $\mathbf{E}$ is the electric field vector, $\mathbf{B}$ is the magnetic induction vector, ρ_e is the density field of electric charges, $\mathbf{j}_e$ is the electric current density, ϵ_0 is the dielectric constant of vacuum, μ_0 is magnetic permeability of vacuum,

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t is time, $\nabla = \mathbf{i}\partial/\partial x + \mathbf{j}\partial/\partial y + \mathbf{k}\partial/\partial z$ is the Hamilton operator.

In 1846, W. Thomson compared electric phenomena with elasticity. He pointed out that the elastic displacement $\mathbf{u}$ of an incompressible elastic solid is a possible analogy to the vector electromagnetic potential $\mathbf{A}$ [5]. In 1845–1862, G. G. Stokes suggested that the electromagnetic aether might behave like a glue-water jelly [15–17].

Following G. G. Stokes, we proposed a visco-elastic continuum model of vacuum in 2008 [18]. Maxwell's equations (2-5) in vacuum are derived by methods of continuum mechanics based on a mechanical model of vacuum and a source or sink flow model of electric charges [18]. The $\Omega(1)$ substratum behaves as a visco-elastic continuum [18]. Maxwell's equations approximate the macroscopic behavior of the $\Omega(1)$ particles, in analogy to the way that classical elastic mechanics approximates the macroscopic behavior of the atoms of solid materials. We briefly review this mechanical model [18] of electromagnetic field in this section.

Assumption 1 *Suppose that vacuum is filled with a kind of continuously distributed matter, which may be called the $\Omega(1)$ substratum. Suppose that all the mechanical quantities of the $\Omega(1)$ substratum under consideration, such as the density, displacements, strains, stresses, etc., are piecewise continuous functions of space and time. Further, we suppose that the material points of the $\Omega(1)$ substratum remain be in one-to-one correspondence with the material points before a deformation appears.*

In order to describe the deformation of the $\Omega(1)$ substratum, we introduce a Cartesian coordinate system $\{0, x, y, z\}$ or $\{0, x_1, x_2, x_3\}$ which is attached to the $\Omega(1)$ substratum.

Assumption 2 *Suppose that the material of the $\Omega(1)$ substratum under consideration is homogeneous, that is*

$$\frac{\partial \rho_1}{\partial x} = \frac{\partial \rho_1}{\partial y} = \frac{\partial \rho_1}{\partial z} = \frac{\partial \rho_1}{\partial t} = 0, \tag{6}$$

where ρ_1 is the density of the $\Omega(1)$ substratum.

Assumption 3 *Suppose that the deformation processes of the $\Omega(1)$ substratum are isothermal. So we neglect the thermal effects. Suppose that the deformation processes are not influenced by the gradient of the stress tensor. Suppose that the material of the $\Omega(1)$ substratum under consideration is isotropic. Supposee that the deformation of the $\Omega(1)$ substratum under consideration is small. Suppose that there are no initial stress and strain in the body under consideration.*

We introduce the following assumption [18].

Assumption 4 *Suppose the constitutive relation of the $\Omega(1)$ substratum satisfies the following relationships*

$$\frac{de_{ij}}{dt} = \frac{1}{2\eta} s_{ij} + \frac{1}{2W}\frac{ds_{ij}}{dt}, \tag{7}$$

where e_{ij} is the strain deviator, t is time, s_{ij} is the stress deviator, η is the dynamic viscocity, W is the shear modulus.

We call the materials behaving like Eq. (7) as Maxwell liquids since J. C. Maxwell established such a constitutive relation in 1868 [19–22]. We introduce the following definition of Maxwellian relaxation time τ [18]

$$\tau = \frac{\eta}{W}\,. \tag{8}$$

Using Eq. (8), Eq. (7) can be written as [18]

$$\frac{s_{ij}}{\tau} + \frac{ds_{ij}}{dt} = 2W\frac{de_{ij}}{dt}\,. \tag{9}$$

The vectorial form of the equation of momentum conservation of the $\Omega(1)$ substratum can be written as [9, 23–27]

$$W\nabla^2\mathbf{u} + (W+\lambda)\nabla(\nabla\cdot\mathbf{u}) + \mathbf{f} = \rho_1\frac{\partial^2\mathbf{u}}{\partial t^2}, \tag{10}$$

where $\mathbf{u}$ is the displacement, λ is Lamé constant, $\mathbf{f}$ is the volume force density exerted on the $\Omega(1)$ substratum, $\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2$ is the Laplace operator.

Let T_0 be the characteristic time of a macroscopic observer of an electric charge. We may suppose that the observer's time scale T_0 is very large comparing to the the Maxwelllian relaxation time τ. So the Maxwelllian relaxation time τ is a relatively small number and the stress deviator s_{ij} changes very slowly. Thus, the second term in the left side of Eq.(9) may be neglected. According to this macroscopic observer, the constitutive relation of the $\Omega(1)$ substratum may be written as

$$s_{ij} = 2\eta\frac{de_{ij}}{dt}. \tag{11}$$

Therefore, the observer concludes that the $\Omega(1)$ substratum behaves like the Newtonian-fluid. We introduce the following definition of point source and sink [28]. Suppose that there exist a singularity at a point $P_0 = (x_0, y_0, z_0)$ in a continuum. If the velocity field of the singularity at a point $P = (x, y, z)$ is

$$\mathbf{v}(x,y,z,t) = \frac{Q}{4\pi r^2}\,\hat{\mathbf{r}}, \tag{12}$$

where $r = \sqrt{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2}$, $\hat{\mathbf{r}}$ is the unit vector directed outward along the line from the singularity to this point $P = (x, y, z)$, we call such a singularity a point source in the case of $Q > 0$ or a point sink in the case of $Q < 0$. Here Q is called the strength of the source or sink.

For the case of continuously distributed point sources or sinks, it is useful to introduce a definition for the volume density ρ_s of point sources or sinks. The definition is

$$\rho_s = \lim_{\triangle V\to 0}\frac{\triangle Q}{\triangle V}\,, \tag{13}$$

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where $\triangle V$ is a small volume, $\triangle Q$ is the sum of the strengthes of all the point sources or sinks in the volume $\triangle V$.

The idea that all microscopic particles are sink flows in a fluidic substratum has been proposed by many researchers in the history, for instance, J. C. Maxwell ([29], p. 243), B. Riemann ([30], p. 507), H. Poincaré ([31], p. 171), J. C. Taylor ([32], p. 431-436). Therefore, we suppose that all the electric charges in the universe are the sources or sinks in the $\Omega(1)$ substratum [18]. We define such a source as a negative electric charge. We define such a sink as a positive electric charge. The electric charge quantity q_e of an electric charge is defined as [18]

$$q_e = -\,k_Q \rho_1 \, Q \,, \tag{14}$$

where k_Q is a positive dimensionless constant.

For the case of continuously distributed electric charges, it is useful to introduce the following definition of the volume density ρ_e of electric charges [18]

$$\rho_e = \lim_{\triangle V \to 0} \frac{\triangle q_e}{\triangle V} \,, \tag{15}$$

where $\triangle V$ is a small volume, $\triangle q_e$ is the sum of the strengthes of all the electric charges in the volume $\triangle V$.

Using Eq. (13-15), we have [18]

$$\rho_e = -\,k_Q \rho_1 \rho_s . \tag{16}$$

According to Eq. (14) and Eq. (12), the masses bearing positive electric charges are changing since the strength of a sink evaluates the volume of the $\Omega(1)$ substratum entering the sink per unit of time. Therefore, the equation of mass conservation of the $\Omega(1)$ substratum can be written as [18]

$$\nabla \cdot \mathbf{v} = -\frac{\rho_e}{k_Q \rho_1}, \tag{17}$$

where $\mathbf{v}$ is the velocity field of the $\Omega(1)$ substratum, which is defined by $\mathbf{v} = \partial \mathbf{u}/\partial t$.

The momentum of a volume element $\triangle V$ of the $\Omega(1)$ substratum containing continuously distributed electric charges, and moving with an average speed $\mathbf{v}_e$, changes. Therefore, the equation of momentum conservation Eq. (10) of the $\Omega(1)$ substratum should be written as [18]

$$W\nabla^2\mathbf{u} + (W+\lambda)\nabla(\nabla\cdot\mathbf{u}) + \mathbf{f} = \rho_1 \frac{\partial^2 \mathbf{u}}{\partial t^2} - \frac{\rho_e \mathbf{v}_e}{k_Q} . \tag{18}$$

In order to simplify Eq. (18), we may introduce the following assumption [18].

Assumption 5 *Suppose that the $\Omega(1)$ substratum is almost incompressible, i.e., we suppose that the volume change coefficient θ is a sufficient small quantity and varies very slowly in the space so that it can be treated as $\theta = 0$.*

Based on Assumption 5, we have $\nabla\cdot\mathbf{u} = \theta = 0$. Therefore, the vectorial form of the equation of momentum conservation Eq. (18) reduces to the following form [18]

$$W\nabla^2\mathbf{u} + \mathbf{f} = \rho_1 \frac{\partial^2 \mathbf{u}}{\partial t^2} - \frac{\rho_e \mathbf{v}_e}{k_Q} . \tag{19}$$

According to the Stokes-Helmholtz resolution theorem [23, 24], there exist a scalar function ψ and a vector function $\mathbf{R}$ such that $\mathbf{u}$ is represented by

$$\mathbf{u} = \nabla\psi + \nabla\times\mathbf{R}. \tag{20}$$

We introduce the definitions [18]

$$\nabla\phi = k_E \frac{\partial}{\partial t}(\nabla\psi)\,, \quad \mathbf{A} = k_E \nabla\times\mathbf{R}\,, \tag{21}$$

$$\mathbf{E} = -\,k_E \frac{\partial \mathbf{u}}{\partial t}\,, \quad \mathbf{B} = k_E \nabla\times\mathbf{u}\,, \tag{22}$$

where ϕ is the scalar electromagnetic potential, $\mathbf{A}$ is the vector electromagnetic potential, $\mathbf{E}$ is the electric field intensity, $\mathbf{B}$ is the magnetic induction, k_E is a positive dimensionless constant.

From Eq. (20), Eq. (21) and Eq. (22), we have [18]

$$\mathbf{E} = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t}, \quad \mathbf{B} = \nabla\times\mathbf{A}, \tag{23}$$

$$\nabla\times\mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}\,, \tag{24}$$

$$\nabla\cdot\mathbf{B} = 0\,. \tag{25}$$

Noticing $\nabla\cdot\mathbf{u} = 0$, $\nabla\cdot\mathbf{A} = 0$ and $\mathbf{f} = 0$, Eq. (19) can be written as [18]

$$\frac{k_Q W}{k_E}\,\nabla\times\mathbf{B} = \frac{k_Q\rho_1}{k_E}\frac{\partial \mathbf{E}}{\partial t} + \rho_e \mathbf{v}_e\,. \tag{26}$$

We introduce the following definitions [18]

$$\mathbf{j}_e = \rho_e \mathbf{v}_e\,, \qquad \epsilon_0 = \frac{k_Q \rho}{k_E}\,, \qquad \frac{1}{\mu_0} = \frac{k_Q G}{k_E}\,. \tag{27}$$

Using Eqs. (27), Eq. (26) becomes [18]

$$\frac{1}{\mu_0}\,\nabla\times\mathbf{B} = \mathbf{j}_e + \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}\,. \tag{28}$$

Noticing Eq. (22) and Eq. (27), Eq. (17) becomes [18]

$$\nabla\cdot\mathbf{E} = \frac{\rho_e}{\epsilon_0}\,. \tag{29}$$

Eq. (24), Eq. (25), Eq. (28) and Eq. (29) coincide with Maxwell's equations (2–5). Thus, Maxwell's equations (2–5) are derived based on this mechanical model of vacuum and the singularity model of electric charges [18].

III. A SPHERICAL SOURCE AND SPHERICAL SINK MODEL OF ELECTRIC CHARGES

Eq.(11) is the constitutive relation of a Newtonian-fluid. It is known that the motion of an incompressible Newtonian-fluid is governed by the Navier–Stokes equations (refer to, for instance, [33–38]),

$$\rho_1 \left[\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} \right] = -\nabla p - \eta \nabla^2 \mathbf{v}, \tag{30}$$

where $\mathbf{v}$ is the velocity field of the fluid, p is the pressure field, ρ_1 is the density field, η is the dynamic viscosity coefficient, t is time.

The definition of Reynolds number Re of a fluid field is

$$Re = \frac{\rho_0 U_0 L_0}{\eta_0}, \tag{31}$$

where ρ_0 is the characteristic density, U_0 is the characteristic velocity, L_0 is the characteristic length, η_0 is the characteristic dynamic viscosity coefficient.

Assumption 6 *We speculate that the characteristic velocity U_0 of an electric charge is so high compares to the characteristic dynamic viscosity coefficient η_0 of the $\Omega(1)$ substratum that the Reynolds number Re of the fluid field is a large number.*

Under this assumption, we may treat the $\Omega(1)$ substratum as an inviscid incompressible fluid when we study the motion of electric charges. Therefore, according to Assumption 6, the motion of the $\Omega(1)$ substratum is governed by the Euler equations [33–38]

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = -\frac{1}{\rho_1}\nabla p. \tag{32}$$

If there exist a velocity field which is continuous and finite at all points of the space, with the exception of individual isolated points, then these isolated points are called velocity singularities usually. Point spherical sources and spherical sinks are examples of singularities.

In 1892 [39], Lorentz established an electromagnetic theory in order to derive the Fresnel convection coefficient. There are only two types of entities in Lorentz's theory: movable electrons and a stagnant aether. To avoid singularities, the electrons were not designed to be singularities as Larmor's electrons in the aether field, but were extremely small hard spheres with a finite radius.

Inspired by Lorentz [39], we speculate that electric charges may not be singularities, but may be extremely small hard spherical sources or spherical sinks with finite radii. Thus, we introduce a mechanical model of spherical sources and spherical sinks with finite radii in fluids.

Definition 7 *Suppose that there exist a hard sphere with a finite radius a at point $P_0 = (x_0, y_0, z_0)$. If the velocity field near the hard sphere at point $P = (x, y, z)$ is*

$$\mathbf{v}(x, y, z, t) = \frac{Q}{4\pi r^2}\hat{\mathbf{r}}, \tag{33}$$

where $r = \sqrt{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2}, r \geq a$, is the distance between the point P_0 and the point P, $\hat{\mathbf{r}}$ denotes the unit vector directed outward along the line from the point P_0 to the point P, then we call this hard sphere a spherical source if $Q > 0$ or a spherical sink if $Q < 0$. Q is called the strength of the spherical source or the spherical sink.

For convenience, we may regard a spherical sink as a negative spherical source. Suppose that a static spherical source with strength Q and a radius a locates at the origin $(0, 0, 0)$. In order to calculate the volume leaving the source per unit time, we enclose the source with an arbitrary spherical surface S with a radius $b > a$. A calculation shows that

$$\oiint_S \mathbf{v} \cdot \mathbf{n} dS = \oiint_S \frac{Q}{4\pi b^2}\hat{\mathbf{r}} \cdot \mathbf{n} dS = Q, \tag{34}$$

where $\mathbf{n}$ denotes the unit vector directed outward along the line from the origin of the coordinates to the field point(x, y, z).

Eq. (34) shows that the strength Q of a spherical source or spherical sink evaluates the volume of the fluid leaving or entering a control surface per unit time.

Based on the definitions of spherical sources and spherical sinks, we introduce the following model of electric charges.

Assumption 8 *Suppose that all the electric charges in the universe are small hard spherical sources or spherical sinks with finite radii in the $\Omega(1)$ substratum. We define a spherical source as a negative electric charge. We define a spherical sink as a positive electric charge. The electric charge quantity of an electric charge is defined as*

$$q_e = -k_Q \rho_1 Q, \tag{35}$$

where ρ_1 is the density of the $\Omega(1)$ substratum, k_Q is a positive dimensionless constant, Q is called the strength of the spherical source or spherical sink.

A calculation shows that the mass m of a electric charge is changing with time as

$$\frac{dm}{dt} = -\rho_1 Q = \frac{q_e}{k_Q}, \tag{36}$$

where q_e is the electric charge quantity of the electric charge.

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IV. FORCES ACTING ON SPHERICAL SOURCES AND SPHERICAL SINKS IN IDEAL FLUIDS

Suppose the velocity field $\mathbf{v}$ of an ideal fluid is irrotational, then we have [33–38, 40],

$$\mathbf{v} = \nabla \phi, \tag{37}$$

where ϕ is the velocity potential.

It is known that the equation of mass conservation of an ideal fluid becomes Laplace's equation [33–38, 40],

$$\nabla^2 \phi = 0, \tag{38}$$

where ϕ is velocity potential.

Using spherical coordinates(r, θ, φ), a general form of solution of Laplace's equation Eq. (38) can be obtained by sepatation of variables as [38]

$$\phi(r,\theta) = \sum_{l=0}^{\infty} \left(A_l r^l + \frac{B_l}{r^{l+1}} \right) P_l(\cos\theta), \tag{39}$$

where A_l and B_l are arbitrary constants, $P_l(x)$ are Legendre's function of the first kind which is defined as

$$P_l(x) = \frac{1}{2^l l!} \frac{\mathrm{d}^l}{\mathrm{d}x^l} (x^2 - 1)^l. \tag{40}$$

From Eq. (33) and Eq. (39), we see that the velocity potential $\phi(r, \theta)$ of a spherical source or spherical sink is a solution of Laplace's equation (38).

Proposition 9 *Suppose that (1) the velocity field $\mathbf{v}$ of a fluid is irrotational, i.e., we have $\mathbf{v} = \nabla \phi$, where ϕ is the velocity potential; (2) there is an arbitrary closed surface S fixed in the space without any bodies or singularities inside S; (3) the velocity field $\mathbf{v}$ is continuous in the closed surface S. Then, we have*

$$\frac{\mathrm{D}}{\mathrm{D}t} \oint\!\!\!\oint_S \rho_1 \phi \mathbf{n} dS = \frac{\partial}{\partial t} \oint\!\!\!\oint_S \rho_1 \phi \mathbf{n} dS + \oint\!\!\!\oint_S \rho_1 \mathbf{v}(\mathbf{v} \cdot \mathbf{n}) dS. \tag{41}$$

where $\mathrm{D}/\mathrm{D}t$ represents the material derivative in the lagrangian system.

For the proof of Proposition 9, refer to, for instance, Appendix 2 in [41].

Theorem 10 *Suppose that (1) there exist an ideal fluid (2) the ideal fluid is irrotational and barotropic, (3) the density ρ_1 is homogeneous, that is $\partial\rho_1/\partial x = \partial\rho_1/\partial y = \partial\rho_1/\partial z = \partial\rho_1/\partial t = 0$, (4) there are no external body forces exerted on the fluid, (5) the fluid is unbounded and the velocity of the fluid at the infinity is approaching to zero. Suppose a spherical source or spherical sink is stationary and is immersed in the ideal fluid. Then, there is a force*

$$\mathbf{F}_Q = \rho_1 Q \mathbf{v}_0 + \frac{4\pi\rho_1 a^3}{3} \frac{\partial \mathbf{v}_0}{\partial t} \tag{42}$$

exerted on the spherical source or the spherical sink by the fluid, where ρ_1 is the density of the fluid, Q is the strength of the spherical source or the spherical sink, a is the radius of the spherical source or the spherical sink, $\mathbf{v}_0$ is the velocity of the fluid at the location of the spherical source induced by all means other than the spherical source itself.

Proof. Only the proof of the case of a spherical source is needed. Let us select the coordinates $\{x, y, z\}$ or $\{x_1, x_2, x_3\}$ that is attached to the static fluid at the infinity.

We set the origin of the coordinates at the center of the spherical source. Let S_1 denotes the spherical surface of the spherical source. We surround the spherical source by an arbitrary spherical surface S_2 with radius R centered at the center of the spherical source. The outward unit normal to the spherical surface S_1 and S_2 is denoted by $\mathbf{n}$. Let $\tau(t)$ denotes the mass system of fluid enclosed in the volume between the surface S_1 and the surface S_2 at time t.

Let $\mathbf{F}_Q$ denotes the hydrodynamic force exerted on the spherical source by the mass system τ. Then according to Newton's third law, a reacting force of the force $\mathbf{F}_Q$ must act on the the fluid enclosed in the mass system τ. Let $\mathbf{F}_1 = -\mathbf{F}_Q$ denotes this reacting force acted on the mass system τ by the spherical source through the surface S_1. Let $\mathbf{F}_2$ denotes the hydrodynamic force exerted on the mass system τ due to the pressure distribution on the surface S_2. Let $\mathbf{K}$ denotes momentum of the mass system τ.

Applying Newton's second law of motion to the mass system τ, we have

$$\frac{\mathrm{D}\mathbf{K}}{\mathrm{D}t} = \mathbf{F}_1 + \mathbf{F}_2 = -\mathbf{F}_Q + \mathbf{F}_2, \tag{43}$$

where $\mathrm{D}/\mathrm{D}t$ represents the material derivative in the lagrangian system (see, for instance, [33–38]).

In order to calculate $\mathbf{F}_Q$, we calculate $\mathrm{D}\mathbf{K}/\mathrm{D}t$ and $\mathbf{F}_2$ respectively. The expressions of the momentum $\mathbf{k}$ is

$$\mathbf{K} = \iiint_\tau \rho_1 \mathbf{v} dV, \tag{44}$$

where the integral is a volume integral, $\mathbf{n}$ denotes the unit vector directed outward along the line from the origin of the coordinates to the field point(x, y, z).

The expressions of the force $\mathbf{F}_2$ is

$$\mathbf{F}_2 = \oint\!\!\!\oint_{S_2} -p\mathbf{n} dS, \tag{45}$$

where the integral is a surface integral.

Since the velocity field is irrotational, we have the following relation

$$\mathbf{v} = \nabla \phi, \tag{46}$$

where ϕ is the velocity potential.

According to Ostrogradsky–Gauss theorem (see, for instance, [33–35, 37, 38]) and using Eq. (46), we have

$$\begin{aligned}\iiint_\tau \rho_1 \mathbf{v} dV &= \iiint_\tau \rho_1 \nabla\phi dV \\ &= \oiint_{S_2} \rho_1\phi\mathbf{n}dS - \oiint_{S_1}\rho_1\phi\mathbf{n}dS. \end{aligned} \quad (47)$$

Using Eq. (44) and Eq. (47), we have

$$\frac{\mathrm{D}\mathbf{K}}{\mathrm{D}t} = \frac{\mathrm{D}}{\mathrm{D}t}\left[\oiint_{S_2}\rho_1\phi\mathbf{n}dS - \oiint_{S_1}\rho_1\phi\mathbf{n}dS\right]. \quad (48)$$

Applying Proposition 9 to the first integral in Eq. (48), we have

$$\frac{\mathrm{D}}{\mathrm{D}t}\oiint_{S_2}\rho_1\phi\mathbf{n}dS = \frac{\partial}{\partial t}\oiint_{S_2}\rho_1\phi\mathbf{n}dS + \oiint_{S_2}\rho_1\mathbf{v}(\mathbf{v}\cdot\mathbf{n})dS. \quad (49)$$

Putting Eq. (49) into Eq. (48), we have

$$\begin{aligned}\frac{\mathrm{D}\mathbf{K}}{\mathrm{D}t} &= \frac{\partial}{\partial t}\oiint_{S_2}\rho_1\phi\mathbf{n}dS + \oiint_{S_2}\rho_1\mathbf{v}(\mathbf{v}\cdot\mathbf{n})dS \\ &- \frac{\mathrm{D}}{\mathrm{D}t}\oiint_{S_1}\rho_1\phi\mathbf{n}dS. \end{aligned} \quad (50)$$

Now, we calculate $\mathbf{F}_2$. According to Lagrange–Cauchy integral (see, for instance, [33–35, 37, 38]), we have

$$\frac{\partial\phi}{\partial t} + \frac{(\nabla\phi)^2}{2} + \frac{p}{\rho_1} = f(t), \quad (51)$$

where $f(t)$ is an arbitrary function of time t. Since the velocity $\mathbf{v}$ of the fluid at the infinity is approaching to zero, and noticing Eq. (39), $\phi(t)$ must be of the following form

$$\phi(r,\theta,t) = \sum_{l=0}^{\infty}\frac{B_l(t)}{r^{l+1}}P_l(\cos\theta), \quad (52)$$

where $B_l(t), l \geq 0$ are functions of time t.

Thus, we have the following estimations at the infinity of the velocity field

$$\phi = O\left(\frac{1}{r}\right), \quad \frac{\partial\phi}{\partial t} = O\left(\frac{1}{r}\right), \quad r\to\infty, \quad (53)$$

where $\varphi(x) = O(\psi(x)), x \to a$ stands for $\overline{\lim}_{x\to a} \mid \varphi(x) \mid / \psi(x) = k, (0 \leq k < +\infty)$.

Applying Eq. (51) at the infinity and using Eq. (53), we have $\mid \mathbf{v} \mid \to 0$, $\partial\phi/\partial t \to 0$ and $p = p_\infty$, where p_∞ is a constant. Thus, $f(t) = p_\infty/\rho_1$. Therefore, according to (51), we have

$$p = p_\infty - \rho_1\frac{\partial\phi}{\partial t} - \frac{\rho_1(\mathbf{v}\cdot\mathbf{v})}{2}. \quad (54)$$

Using Eq. (45) and Eq. (54), we have

$$\mathbf{F}_2 = \oiint_{S_2}\rho_1\frac{\partial\phi}{\partial t}\mathbf{n}dS + \oiint_{S_2}\frac{\rho_1(\mathbf{v}\cdot\mathbf{v})\mathbf{n}}{2}dS. \quad (55)$$

Putting Eq. (50) and Eq. (55) into Eq. (43), we have

$$\begin{aligned}\mathbf{F}_Q &= \oiint_{S_2}\left[\frac{1}{2}\rho_1(\mathbf{v}\cdot\mathbf{v})\mathbf{n} - \rho_1\mathbf{v}(\mathbf{v}\cdot\mathbf{n})\right]dS \\ &+ \frac{\mathrm{D}}{\mathrm{D}t}\oiint_{S_1}\rho_1\phi\mathbf{n}dS. \end{aligned} \quad (56)$$

Since the radius R of the spherical surface S_2 is arbitrary, we may let R to be large enough. Applying the result (5.13) in [38], we have

$$\oiint_{S_2}\left[\frac{1}{2}\rho_1(\mathbf{v}\cdot\mathbf{v})\mathbf{n} - \rho_1\mathbf{v}(\mathbf{v}\cdot\mathbf{n})\right]dS = 0. \quad (57)$$

Thus, using Eq. (57), Eq. (56) becomes

$$\mathbf{F}_Q = \frac{\mathrm{D}}{\mathrm{D}t}\oiint_{S_1}\rho_1\phi\mathbf{n}dS. \quad (58)$$

Applying Proposition 9, Eq. (58) becomes

$$\mathbf{F}_Q = \frac{\partial}{\partial t}\oiint_{S_1}\rho_1\phi\mathbf{n}dS + \oiint_{S_1}\rho_1\mathbf{v}(\mathbf{v}\cdot\mathbf{n})dS. \quad (59)$$

For convenience, we introduce the following definitions

$$I_1 = \frac{\partial}{\partial t}\oiint_{S_1}\rho_1\phi\mathbf{n}dS, \quad (60)$$

$$I_2 = \oiint_{S_1}\rho_1\mathbf{v}(\mathbf{v}\cdot\mathbf{n})dS. \quad (61)$$

Thus, Eq. (59) becomes

$$\mathbf{F}_Q = I_1 + I_2. \quad (62)$$

Now we calculate the two terms in Eq. (62) respectively. Firstly, we calculate the integral I_1 in Eq. (60). Since the velocity field induced by the spherical source with strength Q is Eq. (33), then according to the superposition principle of velocity field of ideal fluids, the velocity $\mathbf{v}$ on the surface S_1 is

$$\mathbf{v} = \frac{Q}{4\pi a^2}\mathbf{n} + \mathbf{v}_0, \quad (63)$$

where $\mathbf{n}$ denotes the unit vector directed outward.

Since the velocity field $\mathbf{v}$ is irrotational, we have

$$\phi = -\frac{Q}{4\pi a} + \phi_0, \quad (64)$$

where ϕ_0 is the velocity potential respect to $\mathbf{v}_0$, i.e., $\mathbf{v}_0 = \nabla\phi_0$.

Since the density ρ_1 is homogeneous, we have

$$\frac{\partial\rho_1}{\partial t} = 0. \quad (65)$$

Noticing Eq. (65), Eq. (60) can be written as

$$I_1 = \oiint_{S_1}\rho_1\frac{\partial\phi}{\partial t}\mathbf{n}dS. \quad (66)$$

Suppose that $\partial Q/\partial t = 0$. Using Eq. (64), we have

$$\frac{\partial \phi}{\partial t} = \frac{\partial \phi_0}{\partial t} - \frac{1}{4\pi a}\frac{\partial Q}{\partial t} = \frac{\partial \phi_0}{\partial t}. \tag{67}$$

Using Eq. (67) and Ostrogradsky–Gauss theorem, Eq. (66) becomes

$$\begin{aligned} I_1 &= \frac{\partial}{\partial t}\oiint_{S_1} \rho_1\phi_0 \mathbf{n} dS \\ &= \frac{\partial}{\partial t}\iiint_{V_1} \rho_1 \nabla\phi_0 dV \\ &= \frac{\partial}{\partial t}\iiint_{V_1} \rho_1 \mathbf{v}_0 dV. \end{aligned} \tag{68}$$

We speculate that the radius a of the spherical source may be so small that the velocity $\mathbf{v}_0$ at any point of the spherical surface of the spherical source may be treated as a constant. Thus, Eq. (68) becomes

$$\begin{aligned} I_1 &= \frac{\partial(\rho_1\mathbf{v}_0)}{\partial t}\iiint_{V_1} dV \\ &= \frac{\partial(\rho_1\mathbf{v}_0)}{\partial t}\frac{4\pi a^3}{3} \\ &= \frac{4\pi\rho_1 a^3}{3}\frac{\partial \mathbf{v}_0}{\partial t}. \end{aligned} \tag{69}$$

Now we calculate the integral I_2 in Eq. (61). Noticing Eq. (63), we have

$$\begin{aligned} I_2 &= \rho_1 \oiint_{S_1}\left[\frac{Q^2}{16\pi^2 a^4}\mathbf{n} + \frac{Q}{4\pi a^2}\mathbf{v}_0 \right. \\ &\left. + \frac{Q}{4\pi a^2}(\mathbf{v}_0\cdot\mathbf{n})\mathbf{n} + (\mathbf{v}_0\cdot\mathbf{n})\mathbf{v}_0\right] dS. \end{aligned} \tag{70}$$

For convenience, we introduce the following definitions

$$\begin{aligned} J_1 &= \oiint_{S_1}\frac{\rho_1 Q^2}{16\pi^2 a^4}\mathbf{n} dS, \\ J_2 &= \oiint_{S_1}\frac{\rho_1 Q}{4\pi a^2}\mathbf{v}_0 dS, \\ J_3 &= \oiint_{S_1}\frac{\rho_1 Q}{4\pi a^2}(\mathbf{v}_0\cdot\mathbf{n})\mathbf{n} dS, \\ J_4 &= \oiint_{S_1}(\mathbf{v}_0\cdot\mathbf{n})\mathbf{v}_0 dS. \end{aligned} \tag{71}$$

Thus, Eq. (70) becomes

$$I_2 = J_1 + J_2 + J_3 + J_4. \tag{72}$$

We regard the velocity $\mathbf{v}_0$ at any point of the spherical surface S_1 as a constant. Thus, the four integral terms in Eq. (72) turns out to be

$$J_1 = 0, \quad J_2 = \rho_1 Q\mathbf{v}_0, \quad J_3 = 0, \quad J_4 = 0. \tag{73}$$

Thus, using Eq. (73), we have

$$I_2 = \rho_1 Q \mathbf{v}_0. \tag{74}$$

Putting Eq. (69) and Eq. (74) into Eq. (62), we obtain Eq. (42). □

Theorem 10 only considers the situation that the spherical sources or spherical sinks are at rest in fluids. Now we consider the case that the spherical sources or spherical sinks are moving in fluid.

Theorem 11 *Suppose that the assumptions (1),(2),(3),(4) and (5) in Theorem 10 are valid and a spherical source or a spherical sink is moving in the fluid with a velocity $\mathbf{v}_s$, then there is a force*

$$\mathbf{F}_Q = \rho_1 Q(\mathbf{v}_f - \mathbf{v}_s) + \frac{4\pi\rho_1 a^3}{3}\frac{\partial}{\partial t}(\mathbf{v}_f - \mathbf{v}_s) \tag{75}$$

is exerted on the spherical source or the spherical sink by the fluid, where ρ_1 is the density of the fluid, Q is the strength of the spherical source or the spherical sink, a is the radius of the spherical source or the spherical sink, $\mathbf{v}_f$ is the velocity of the fluid at the location of the source induced by all means other than the spherical source itself.

Proof. The velocity of the fluid relative to the spherical source at the location of the spherical source is $\mathbf{v}_f - \mathbf{v}_s$. Let us select the coordinates that is attached to the spherical source and set the origin of the coordinates at the center of the spherical source. Then Eq. (75) can be obtained following the same procedures in the proof of Theorem 10. □

Applying Theorem 11 to the situation that a spherical source or spherical sink is exposed to the velocity field of another spherical source or sink, we have the following result.

Corollary 12 *Suppose that the assumptions (1),(2),(3),(4) and (5) in Theorem 10 are valid and a spherical source or spherical sink with strength Q_2 is exposed to the velocity field of another static spherical source or spherical sink with strength Q_1, then the force $\mathbf{F}_{12}$ exerted on the spherical source or spherical sink with strength Q_2 by the velocity field of the spherical source or spherical sink with strength Q_1 is*

$$\begin{aligned} \mathbf{F}_{12} &= \rho_1 Q_2\left(\frac{Q_1}{4\pi r^2}\hat{\mathbf{r}}_{12} - \mathbf{v}_2\right) \\ &+ \frac{\rho_1 a_2^3}{3r^2}\frac{\partial\left(Q_1\hat{\mathbf{r}}_{12} - \mathbf{v}_2\right)}{\partial t}, \end{aligned} \tag{76}$$

where $\hat{\mathbf{r}}_{12}$ denotes the unit vector directed outward along the line from the spherical source or spherical sink with strength Q_1 to the spherical source or spherical sink with strength Q_2, a_2 is the radius of the spherical source or the spherical sink with strength Q_2, r is the distance between the two bodies, $\mathbf{v}_2$ is the moving velocity of the spherical source or spherical sink with strength Q_2.

If the spherical source with strength Q_2 is also static in the $\Omega(1)$ substratum, then Eq. (76) reduces to

$$\mathbf{F}_{12} = \frac{\rho_1 Q_1 Q_2}{3\pi r^2}\hat{\mathbf{r}}_{12} + \frac{\rho_1 a_2^3}{3r^2}\frac{\partial Q_1}{\partial t}\hat{\mathbf{r}}_{12}. \tag{77}$$

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V. DERIVATION OF COULOMB'S LAW OF INTERACTIONS BETWEEN STATIC ELECTRIC CHARGES IN VACUUM

Based on Assumption 8 and Assumption 6, we can apply Theorem 10 and Theorem 11 to study the motions of electric charges.

Theorem 13 *Suppose that a static electric charge with an electric charge quantity q_2 is exposed to the electric field of another static electric charge with an electric charge quantity q_1, then the force $\mathbf{F}_{12}$ exerted on the electric charge with electric charge quantity q_2 by the electric field of the electric charge with electric charge quantity q_1 is*

$$\begin{aligned}\mathbf{F}_{12} &= \frac{1}{k_E k_Q}\frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{r^2}\hat{\mathbf{r}}_{12} \\ &\quad - \frac{a_2^3}{3k_Q r^2}\frac{\partial q_1}{\partial t}\hat{\mathbf{r}}_{12}, \end{aligned} \tag{78}$$

where $\hat{\mathbf{r}}_{12}$ denotes the unit vector directed outward along the line from the electric charge with electric charge quantity q_1 to the electric charge with electric charge quantity q_2, a_2 is the radius of the electric charge with electric charge quantity q_2, r is the distance between the two electric charges, k_Q and k_E are two positive dimensionless constants.

Proof. From Assumption 8, we have

$$Q_1 = -\frac{q_1}{k_Q\rho_1}, \quad Q_2 = -\frac{q_2}{k_Q\rho_1}, \tag{79}$$

where Q_1 and Q_2 are the strengthes of the electric charges respectively. From a definition in [18], we have

$$\epsilon_0 = \frac{k_Q\rho_1}{k_E}, \tag{80}$$

where ϵ_0 is the dielectric constant of vacuum. Putting Eq. (79) and Eq. (80) into Eq. (77), we obtain Eq. (78). □

If we ignore the second term in right side of Eq. (78), then we have

$$\mathbf{F}_{12} = \frac{1}{k_E k_Q}\frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{r^2}\hat{\mathbf{r}}_{12}. \tag{81}$$

Compare Eq. (81) with Eq. (1), it is natural for us to introduce the following assumption.

Assumption 14 *Suppose we have the following relation*

$$k_E k_Q = 1. \tag{82}$$

Based on Assumption 14, Eq. (81) has the same form of Coulomb's law (1) of interactions between static electric charges in vacuum.

Theorem 13 only states the force exerted on a static electric charge by the electric field of another static electric charge. We may generalize this result to the case of an static electric charge exposed to any electric field of the $\Omega(1)$ substratum.

We introduce the following definition [18]

$$\mathbf{E} = -k_E\frac{\partial \mathbf{u}}{\partial t}, \tag{83}$$

where $\mathbf{u}$ is the displacement of the visco-elastic aether, $\partial\mathbf{u}/\partial t$ is the velocity field of the $\Omega(1)$ substratum, $\mathbf{E}$ is the electric field intensity, k_E is a positive dimensionless constant.

Since the observer of an electric charge concludes that the $\Omega(1)$ substratum behaves as a Newtonian-fluid under his time scale, we may define the electric field intensity as the velocity field of the $\Omega(1)$ substratum. Therefore, we define the electric field intensity in the $\Omega(1)$ substratum as

$$\mathbf{E} = -k_E\mathbf{v}_{\Omega(1)}, \tag{84}$$

where $\mathbf{v}_{\Omega(1)}$ is the velocity field of the $\Omega(1)$ substratum at the location of an testing electric charge induced by all means other than the testing electric charge itself, $\mathbf{E}$ is the electric field intensity, k_E is a positive dimensionless constant.

Theorem 15 *Suppose that a static electric charge with an electric charge quantity q is exposed to an electric field $\mathbf{E}$ of the $\Omega(1)$ substratum, then the force $\mathbf{F}$ exerted on the electric charge by the electric field $\mathbf{E}$ is*

$$\mathbf{F} = q\mathbf{E} - \frac{4\pi a^3\epsilon_0}{3k_Q}\frac{\partial \mathbf{E}}{\partial t}, \tag{85}$$

where q is the electric charge quantity of the electric charge, a is the radius of the electric charge, ϵ_0 is the dielectric constant of vacuum, k_Q is a positive dimensionless constant, $\mathbf{E}$ is the electric field at the location of the electric charge induced by all means other than the electric charge itself.

Proof. From Assumption 8, we have

$$Q = -\frac{q}{k_Q\rho_1}, \tag{86}$$

where Q is the strength of the electric charge. Putting Eq. (84) and Eq. (86) into Eq. (42) and using Eq. (80) and Eq. (82), we obtain Eq. (85). □

If we ignore the second term in Eq. (85), then we have

$$\mathbf{F} = q\mathbf{E}. \tag{87}$$

The Lorentz's force law can be written as [1]

$$\mathbf{F} = q\mathbf{E} + q\mathbf{v}_e \times \mathbf{B}, \tag{88}$$

where q is the electric charge quantity of a electric charge, $\mathbf{F}$ is the force exerted on the electric charge, $\mathbf{v}_e$ is the velocity of the electric charge, $\mathbf{E}$ and $\mathbf{B}$ are the electric field intensity and the magnetic induction respectively.

Eq. (87) is a reduced form of the Lorentz's force law (88) for static electric charges in vacuum.

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VI. DISCUSSION

According to Eq. (84), the electric field intensity is a linear function of the velocity field of the $\Omega(1)$ substratum. Therefore, the superposition principle of electric fields of static electric charges in vacuum is deduced from the superposition theorem of the velocity field of fluids.

It is an interesting task to generalize this work further to describe the forces exerted on a electric charge moving with a velocity in a electromagnetic field.

VII. CONCLUSION

Following J. C. Maxwell, we suppose that vacuum is filled with a kind of continuously distributed matter which may be called the $\Omega(1)$ substratum, or the electromagnetic aether. Suppose that the time scale of a macroscopic observer's is very large compares to the the Maxwelllian relaxation time of the $\Omega(1)$ substratum. Thus, the macroscopic observer concludes that the $\Omega(1)$ substratum behaves like a Newtonian-fluid. Inspired by H. A. Lorentz, we speculate that electric charges may not be singularities, but may be extremely small hard spherical sources or spherical sinks with finite radii. Based on the spherical source and spherical sink model of electric charges, we derive Coulomb's law of interactions between static electric charges in vacuum. Further, we define the electric field intensity as a linear function of velocity field of the $\Omega(1)$ substratum at the location of an testing electric charge induced by all means other than the testing electric charge itself. Then, we obtain a reduced form of the Lorentz's force law for static electric charges in vacuum.

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