Chebyshev Type Inequalities for the Riemann-Liouville Variable-Order Fractional Integral Operator

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Abstract

This paper presents Chebyshev Type inequalities for the Riemann-Liouville variable-order fractional integral operator using two synchronous functions on the set of real numbers. It is the first result of its kind in the current literature using variable-order Riemann-Liouville fractional integral operator. Some special cases for the result obtained in the paper are discussed.

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1. Introduction

Fractional integral inequalities are very important in establishing the uniqueness of solutions for certain fractional differential equations and integral equations. They also provide upper and lower bounds for the solutions of fractional boundary value problems. Generally speaking, without inequalities, the advance of differential and integral equations would not be at its present stage. For more works on fractional integral (or differential) inequalities, one may refere the book [1](and the references therein) and the papers [2],[3],[4],[5],[6](and the references therein). In 2009, using the Riemann-Liouville constant-order fractional integral operator, Belarbi and Dahmani [2] established some new integral inequalities for the Chebyshev functional [7] in the case of two synchronous functions. In 2010, Dahmani [10] used the Riemann-Liouville fractional integral to present recent results on fractional integral inequalities. By considering the extended Chebyshev functional in the case of synchronous functions, he established two main results. The first one deals with some inequalities using one fractional parameter. The second result concerns others inequalities using two fractional

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parameters. In 2011, Dahmani, Mechouar and Braham [3]using the Riemann-Liouville fractional integral operator they established some integral results related to Chebyshev's functional in the case of differentiable functions whose derivatives belong to the space $L_p([0,\infty[)$. On extensions and generalizations of some of Chebyshev type inequalities in the consequent years until 2021, we refer the reader to see [11], [6], [5], [9], [12], [13], [14], [15], [16], [17]. All papers in the current literature of inequalities related to Chebyshev functional are on constant-order fractional integral operator. In this paper, we established new kind of Chebyshev type inequality using Riemann-Liouville variable-order fractional integral operator.

The following three Theorems are established by Belarbi and Dahmani(see [2]). In our main result they are special cases.

Theorem 1. *Let f and g be two synchronous functions on $[0,\infty)$. Then for all $t > 0$, $\alpha > 0$, we have:*

$$
{}^{RL}_0 I^\alpha_t (fg)(t) \geq \frac{\Gamma(\alpha+1)}{t^\alpha} \left({}^{RL}_0 I^\alpha_t f(t) \right) \left({}^{RL}_0 I^\alpha_t g(t) \right) \tag{1}
$$

Theorem 2. *Let f and g be two synchronous functions on $[0,\infty)$. Then for all $t > 0$, $\alpha > 0$, $\beta > 0$ we have:*

$$
\frac{t^\alpha}{\Gamma(\alpha+1)} {}^{RL}_0 I^\beta_t (fg)(t) + \frac{t^\beta}{\Gamma(\beta+1)} {}^{RL}_0 I^\alpha_t (fg)(t)
$$

$$
\geq \left({}^{RL}_0 I^\alpha_t f(t) \right) \left({}^{RL}_0 I^\beta_t g(t) \right) + \left({}^{RL}_0 I^\beta_t f(t) \right) \left({}^{RL}_0 I^\alpha_t g(t) \right) \tag{2}
$$

Theorem 3. *Let $(f_i)_{i=1,2,\cdots,k}$ be k positive increasing functions on $[0,\infty)$. Then for any $t > 0$, $\alpha > 0$, we've*

$$
{}^{RL}_0 I^\alpha_t \left(\prod_{i=1}^k f_i \right)(t) \geq \left[{}^{RL}_0 I^\alpha_t (1) \right]^{(1-k)} \prod_{i=1}^k {}^{RL}_0 I^\alpha_t f_i(t) \tag{3}
$$

2. Preliminaries

Throughout this paper, we use the following definitions.

Definition 1. *Two functions f and g are said to be synchronous on $[a,b]$ if*

$$
(f(x) - f(y))(g(x) - g(y)) \geq 0 \text{ for any } x, y \in [a,b]
$$

Definition 2. *Given $\Re(z) > 0$, we define the gamma function, $\Gamma(z)$, as*

$$
\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt
$$

$\Gamma(z)$ *is a holomorphic function in $\Re(z) > 0$.*

In the following definition of Riemann-Liouville variable-order fractional integral we used the notation RL stands for Riemann-Liouville.

Definition 3. *(see[8]) Let $\alpha : [a,b] \times [a,b] \longrightarrow (0,\infty)$, the left Riemann-Liouville fractional integral of order $\alpha(.,.)$ for function $f(t)$ is defined by*

$$
{}^{RL}_a I_t^{\alpha(.,.)} f(t) = \int_a^t \frac{(t-s)^{\alpha(t,s)-1}}{\Gamma(\alpha(t,s))} f(s)ds, \ t > a
\tag{4}
$$

3. Main result

In this section, we introduce one inequality for two synchronous functions on $\mathbb{R}$. This inequality is for Riemann-Liouville two variable-order of fractional integral operator which is defined as (4). From this inequality, four important inequality produced as a corollary by assuming monotonocity and differentiability of the two synchronous functions.

Theorem 4. *Let f and g be two synchronous functions on $\mathbb{R}$. Then for all $a, c \in \mathbb{R}$, $t > a$, $s > c$, and $\alpha, \beta : [a,b] \times [a,b] \longrightarrow (0,\infty)$, we've*

$$
\left({}^{RL}_a I_t^{\alpha(.,.)}(fg)(t) \right)\left({}^{RL}_c I_s^{\beta(.,.)}(1) \right) + \left({}^{RL}_a I_t^{\alpha(.,.)}(1) \right)\left({}^{RL}_c I_s^{\beta(.,.)}(fg)(s) \right)
$$
$$
\geq \left({}^{RL}_a I_t^{\alpha(.,.)}f(t) \right)\left({}^{RL}_c I_s^{\beta(.,.)}g(s) \right) + \left({}^{RL}_a I_t^{\alpha(.,.)}g(t) \right)\left({}^{RL}_c I_s^{\beta(.,.)}f(s) \right)
\tag{5}
$$

Proof. Since f and g are synchronous on $\mathbb{R}$, then for all $x, y \in \mathbb{R}$, we've

$$
(f(x) - f(y))(g(x) - g(y)) \geq 0
$$

this implies

$$
f(x)g(x) + f(y)g(y) \geq f(x)g(y) + f(y)g(x)
\tag{6}
$$

Now multiply inequality (6) by $(t-x)^{\alpha(t,x)-1}/\Gamma(\alpha(t,x))$ and integrate from a to t with respect to x, that is,

$$
\int_a^t \frac{(t-x)^{\alpha(t,x)-1}}{\Gamma(\alpha(t,x))} f(x)g(x)dx + \int_a^t \frac{(t-x)^{\alpha(t,x)-1}}{\Gamma(\alpha(t,x))} f(y)g(y)dx
$$
$$
\geq \int_a^t \frac{(t-x)^{\alpha(t,x)-1}}{\Gamma(\alpha(t,x))} f(x)g(y)dx + \int_a^t \frac{(t-x)^{\alpha(t,x)-1}}{\Gamma(\alpha(t,x))} f(y)g(x)dx
\tag{7}
$$

which means

$$
\left({}^{RL}_a I_t^{\alpha(.,.)}(fg)(t) \right) + f(y)g(y)\left({}^{RL}_a I_t^{\alpha(.,.)}(1) \right)
$$
$$
\geq g(y)\left({}^{RL}_a I_t^{\alpha(.,.)}f(t) \right) + f(y)\left({}^{RL}_a I_t^{\alpha(.,.)}g(t) \right)
\tag{8}
$$

3

Now multiply inequality (8) by $(s-y)^{\beta(s,y)-1}/\Gamma(\beta(s,y))$ and integrate from c to s with respect to y, that is,

$$
\int_c^s \frac{(s-y)^{\beta(s,y)-1}}{\Gamma(\beta(s,y))} \left({}_a^{RL}I_t^{\alpha(.,.)}(fg)(t) \right) dy + \int_c^s \frac{(s-y)^{\beta(s,y)-1}}{\Gamma(\beta(s,y))} f(y)g(y) \left({}_a^{RL}I_t^{\alpha(.,.)}(1) \right) dy
$$

$$
\geq \int_c^s \frac{(s-y)^{\beta(s,y)-1}}{\Gamma(\beta(s,y))} g(y) \left({}_a^{RL}I_t^{\alpha(.,.)}f(t) \right) dy \qquad (9)
$$

$$
+ \int_c^s \frac{(s-y)^{\beta(s,y)-1}}{\Gamma(\beta(s,y))} f(y) \left({}_a^{RL}I_t^{\alpha(.,.)}g(t) \right) dy
$$

which means

$$
\left({}_a^{RL}I_t^{\alpha(.,.)}(fg)(t) \right) \left({}_c^{RL}I_s^{\beta(.,.)}(1) \right) + \left({}_a^{RL}I_t^{\alpha(.,.)}(1) \right) \left({}_c^{RL}I_s^{\beta(.,.)}(fg)(s) \right)
$$

$$
\geq \left({}_a^{RL}I_t^{\alpha(.,.)}f(t) \right) \left({}_c^{RL}I_s^{\beta(.,.)}g(s) \right) + \left({}_a^{RL}I_t^{\alpha(.,.)}g(t) \right) \left({}_c^{RL}I_s^{\beta(.,.)}f(s) \right)
$$

and this completes the proof. □

Remark 1. *The inequality (5) is reversed if the functions are asynchronous on $\mathbb{R}$ (that is, $(f(x) - f(y))(g(x) - g(y)) \leq 0$, for any $x, y \in (-\infty, \infty)$).*

Remark 2. *The inequality (5) gives inequality (1) if $a = c = 0$, $s = t$, $\alpha(.,.) = \beta(.,.) = \alpha$(constant).*

Remark 3. *The inequality (5) gives inequality (2) if $a = c = 0$, $s = t$, $\alpha(.,.) = \alpha$(constant) and $\beta(.,.) = \beta$(constant).*

Remark 4. *The inequality (5) gives inequality (3) if $a = c = 0$, $s = t$, $\alpha(.,.) = \beta(.,.) = \alpha$(constant), $\prod_{i=1}^{k-1} f_i(t) = g(t)$, $f_k(t) = f(t)$ where $(f_i)_{i=1,2,\cdots,k}$ are k positive increasing functions on $[0, \infty)$.*

Corollary 1. *Let f and g be two functions defined on $\mathbb{R}$. For all $a, c \in \mathbb{R}$, $t > a$, $s > c$, $\alpha, \beta : [a, b] \times [a, b] \longrightarrow (0, \infty)$. Suppose that f is increasing, g is differentiable and there exists a real numbers $m_1 := \inf_{t \geq a} g'(t)$ and $m_2 := \inf_{s \geq c} g'(s)$. Then we've*

$$
\left({}_a^{RL}I_t^{\alpha(.,.)}(fg)(t) \right) \left({}_c^{RL}I_s^{\beta(.,.)}(1) \right) + \left({}_a^{RL}I_t^{\alpha(.,.)}(1) \right) \left({}_c^{RL}I_s^{\beta(.,.)}(fg)(s) \right)
$$

$$
\geq \left({}_a^{RL}I_t^{\alpha(.,.)}f(t) \right) \left({}_c^{RL}I_s^{\beta(.,.)}(g(s) \right) + \left({}_a^{RL}I_t^{\alpha(.,.)}(g(t) \right) \left({}_c^{RL}I_s^{\beta(.,.)}f(s) \right)
$$

$$
+ m_2 \left[\left({}_a^{RL}I_t^{\alpha(.,.)}(1) \right) \left({}_c^{RL}I_s^{\beta(.,.)}(sf(s)) \right) - \left({}_a^{RL}I_t^{\alpha(.,.)}f(t) \right) \left({}_c^{RL}I_s^{\beta(.,.)}(s) \right) \right] \qquad (10)
$$

$$
+ m_1 \left[\left({}_a^{RL}I_t^{\alpha(.,.)}(tf(t)) \right) \left({}_c^{RL}I_s^{\beta(.,.)}(1) \right) - \left({}_a^{RL}I_t^{\alpha(.,.)}(t) \right) \left({}_c^{RL}I_s^{\beta(.,.)}f(s) \right) \right]
$$

Proof. Define $h(t) = g(t) - m_1 t$, $h(s) = g(s) - m_2 s$. As we can see h is differentiable and it is increasing on $\mathbb{R}$. It is also easy to verify that f and h are synchronous on $\mathbb{R}$. From inequality (5) replace $g(t)$ and $g(s)$ by $h(t) = g(t) - m_1 t$ and $h(s) = g(s) - m_2 s$ respectively. Then from Theorem (4), we've

$$\left({}^{RL}_a I_t^{\alpha(.,.)}(fh)(t) \right)\left({}^{RL}_c I_s^{\beta(.,.)}(1) \right) + \left({}^{RL}_a I_t^{\alpha(.,.)}(1) \right)\left({}^{RL}_c I_s^{\beta(.,.)}(fh)(s) \right)$$

$$\geq \left({}^{RL}_a I_t^{\alpha(.,.)} f(t) \right)\left({}^{RL}_c I_s^{\beta(.,.)} h(s) \right) + \left({}^{RL}_a I_t^{\alpha(.,.)} h(t) \right)\left({}^{RL}_c I_s^{\beta(s)} f(s) \right)$$

this implies

$$\left({}^{RL}_a I_t^{\alpha(.,.)}(f(t))(g(t) - m_1 t) \right)\left({}^{RL}_c I_s^{\beta(.,.)}(1) \right) + \left({}^{RL}_a I_t^{\alpha(.,.)}(1) \right)\left({}^{RL}_c I_s^{\beta(.,.)}(f(s))(g(s) - m_2 s) \right)$$

$$\geq \left({}^{RL}_a I_t^{\alpha(.,.)} f(t) \right)\left({}^{RL}_c I_s^{\beta(.,.)}(g(s) - m_2 s) \right) + \left({}^{RL}_a I_t^{\alpha(.,.)}(g(t) - m_1 t) \right)\left({}^{RL}_c I_s^{\beta(.,.)} f(s) \right)$$

this implies

$$\left({}^{RL}_a I_t^{\alpha(.,.)}(fg)(t) \right)\left({}^{RL}_c I_s^{\beta(.,.)}(1) \right) - m_1\left({}^{RL}_a I_t^{\alpha(.,.)}(tf(t)) \right)\left({}^{RL}_c I_s^{\beta(.,.)}(1) \right)$$

$$+ \left({}^{RL}_a I_t^{\alpha(.,.)}(1) \right)\left({}^{RL}_c I_s^{\beta(.,.)}(fg)(s) \right) - m_2\left({}^{RL}_a I_t^{\alpha(.,.)}(1) \right)\left({}^{RL}_c I_s^{\beta(.,.)}(sf(s)) \right)$$

$$= \left({}^{RL}_a I_t^{\alpha(.,.)}(f(t))(g(t) - m_1 t) \right)\left({}^{RL}_c I_s^{\beta(.,.)}(1) \right) + \left({}^{RL}_a I_t^{\alpha(.,.)}(1) \right)\left({}^{RL}_c I_s^{\beta(.,.)}(f(s))(g(s) - m_2 s) \right)$$

$$\geq \left({}^{RL}_a I_t^{\alpha(.,.)} f(t) \right)\left({}^{RL}_c I_s^{\beta(.,.)}(g(s) - m_2 s) \right) + \left({}^{RL}_a I_t^{\alpha(.,.)}(g(t) - m_1 t) \right)\left({}^{RL}_c I_s^{\beta(.,.)} f(s) \right)$$

$$= \left({}^{RL}_a I_t^{\alpha(.,.)} f(t) \right)\left({}^{RL}_c I_s^{\beta(.,.)}(g(s)) \right) - m_2\left({}^{RL}_a I_t^{\alpha(.,.)} f(t) \right)\left({}^{RL}_c I_s^{\beta(.,.)}(s) \right)$$

$$+ \left({}^{RL}_a I_t^{\alpha(.,.)}(g(t)) \right)\left({}^{RL}_c I_s^{\beta(.,.)} f(s) \right) - m_1\left({}^{RL}_a I_t^{\alpha(.,.)}(t) \right)\left({}^{RL}_c I_s^{\beta(.,.)} f(s) \right)$$

this implies

$$\left({}^{RL}_a I_t^{\alpha(.,.)}(fg)(t) \right)\left({}^{RL}_c I_s^{\beta(.,.)}(1) \right) - m_1\left({}^{RL}_a I_t^{\alpha(.,.)}(tf(t)) \right)\left({}^{RL}_c I_s^{\beta(.,.)}(1) \right)$$

$$+ \left({}^{RL}_a I_t^{\alpha(.,.)}(1) \right)\left({}^{RL}_c I_s^{\beta(.,.)}(fg)(s) \right) - m_2\left({}^{RL}_a I_t^{\alpha(.,.)}(1) \right)\left({}^{RL}_c I_s^{\beta(.,.)}(sf(s)) \right)$$

$$\geq \left({}^{RL}_a I_t^{\alpha(.,.)} f(t) \right)\left({}^{RL}_c I_s^{\beta(.,.)}(g(s)) \right) - m_2\left({}^{RL}_a I_t^{\alpha(.,.)} f(t) \right)\left({}^{RL}_c I_s^{\beta(.,.)}(s) \right)$$

$$+ \left({}^{RL}_a I_t^{\alpha(.,.)}(g(t)) \right)\left({}^{RL}_c I_s^{\beta(.,.)} f(s) \right) - m_1\left({}^{RL}_a I_t^{\alpha(.,.)}(t) \right)\left({}^{RL}_c I_s^{\beta(.,.)} f(s) \right)$$

this implies

$$
\begin{aligned}
\left({}_{a}^{RL}I_t^{\alpha(.,.)}(fg)(t) \right) &\left({}_{c}^{RL}I_s^{\beta(.,.)}(1) \right) + \left({}_{a}^{RL}I_t^{\alpha(.,.)}(1) \right) \left({}_{c}^{RL}I_s^{\beta(.,.)}(fg)(s) \right) \\
&\geq \left({}_{a}^{RL}I_t^{\alpha(.,.)}f(t) \right) \left({}_{c}^{RL}I_s^{\beta(.,.)}(g(s)) \right) + \left({}_{a}^{RL}I_t^{\alpha(.,.)}(g(t)) \right) \left({}_{c}^{RL}I_s^{\beta(.,.)}f(s) \right) \\
&+ m_2 \left[\left({}_{a}^{RL}I_t^{\alpha(.,.)}(1) \right) \left({}_{c}^{RL}I_s^{\beta(.,.)}(sf(s)) \right) - \left({}_{a}^{RL}I_t^{\alpha(.,.)}f(t) \right) \left({}_{c}^{RL}I_s^{\beta(.,.)}(s) \right) \right] \\
&+ m_1 \left[\left({}_{a}^{RL}I_t^{\alpha(.,.)}(tf(t)) \right) \left({}_{c}^{RL}I_s^{\beta(.,.)}(1) \right) - \left({}_{a}^{RL}I_t^{\alpha(.,.)}(t) \right) \left({}_{c}^{RL}I_s^{\beta(.,.)}f(s) \right) \right]
\end{aligned}
$$

and this completes the proof. $\qquad\square$

Corollary 2. *Let f and g be two functions defined on $\mathbb{R}$. For all $a, c \in \mathbb{R}$, $t > a$, $s > c$, $\alpha, \beta : [a,b] \times [a,b] \longrightarrow (0, \infty)$. Suppose that f is decreasing, g is differentiable and there exists a real numbers $M_1 := \sup\limits_{t \geq a} g'(t)$ and $M_2 := \sup\limits_{s \geq c} g'(s)$. Then we've*

$$
\begin{aligned}
\left({}_{a}^{RL}I_t^{\alpha(.,.)}(fg)(t) \right) &\left({}_{c}^{RL}I_s^{\beta(.,.)}(1) \right) + \left({}_{a}^{RL}I_t^{\alpha(.,.)}(1) \right) \left({}_{c}^{RL}I_s^{\beta(.,.)}(fg)(s) \right) \\
&\geq \left({}_{a}^{RL}I_t^{\alpha(.,.)}f(t) \right) \left({}_{c}^{RL}I_s^{\beta(.,.)}(g(s)) \right) + \left({}_{a}^{RL}I_t^{\alpha(.,.)}(g(t)) \right) \left({}_{c}^{RL}I_s^{\beta(.,.)}f(s) \right) \\
&+ M_2 \left[\left({}_{a}^{RL}I_t^{\alpha(.,.)}(1) \right) \left({}_{c}^{RL}I_s^{\beta(.,.)}(sf(s)) \right) - \left({}_{a}^{RL}I_t^{\alpha(.,.)}f(t) \right) \left({}_{c}^{RL}I_s^{\beta(.,.)}(s) \right) \right] \\
&+ M_1 \left[\left({}_{a}^{RL}I_t^{\alpha(.,.)}(tf(t)) \right) \left({}_{c}^{RL}I_s^{\beta(.,.)}(1) \right) - \left({}_{a}^{RL}I_t^{\alpha(.,.)}(t) \right) \left({}_{c}^{RL}I_s^{\beta(.,.)}f(s) \right) \right]
\end{aligned} \tag{11}
$$

Proof. Define $H(t) = g(t) - M_1 t$, $H(s) = g(s) - M_2 s$. As we can see H is differentiable and it is decreasing on $\mathbb{R}$. It is also easy to verify that f and H are synchronous on $\mathbb{R}$. From inequality (5) replace $g(t)$ and $g(s)$ by $H(t) = g(t) - M_1 t$ and $H(s) = g(s) - M_2 s$ respectively. Follow the same procedure as Corollary (1). $\qquad\square$

Corollary 3. *Let f and g be two functions defined on $\mathbb{R}$. For all $a, c \in \mathbb{R}$, $t > a$, $s > c$, $\alpha, \beta : [a,b] \times [a,b] \longrightarrow (0, \infty)$. Suppose that g and f are differentiable and there exists a real*

numbers $m_1 := \inf\limits_{t \geq a} g'(t)$, $m_2 := \inf\limits_{s \geq c} g'(s)$, $m_3 := \inf\limits_{t \geq a} f'(t)$ *and* $m_4 := \inf\limits_{s \geq c} f'(s)$. *Then we've*

$$
\begin{aligned}
&\left({}^{RL}_{a} I_t^{\alpha(.,.)}(fg)(t) \right) \left({}^{RL}_{c} I_s^{\beta(.,.)}(1) \right) + \left({}^{RL}_{a} I_t^{\alpha(.,.)}(1) \right) \left({}^{RL}_{c} I_s^{\beta(.,.)}(fg)(s) \right) \\
&\geq \left({}^{RL}_{a} I_t^{\alpha(.,.)} f(t) \right) \left({}^{RL}_{c} I_s^{\beta(.,.)} g(s) \right) + \left({}^{RL}_{a} I_t^{\alpha(.,.)} g(t) \right) \left({}^{RL}_{c} I_s^{\beta(.,.)} f(s) \right) \\
&\quad + m_1 \left[\left({}^{RL}_{a} I_t^{\alpha(.,.)} t f(t) \right) \left({}^{RL}_{c} I_s^{\beta(.,.)}(1) \right) - \left({}^{RL}_{a} I_t^{\alpha(.,.)}(t) \right) \left({}^{RL}_{c} I_s^{\beta(.,.)} f(s) \right) \right] \\
&\quad + m_2 \left[\left({}^{RL}_{a} I_t^{\alpha(.,.)}(1) \right) \left({}^{RL}_{c} I_s^{\beta(.,.)}(s f(s)) \right) - \left({}^{RL}_{a} I_t^{\alpha(.,.)} f(t) \right) \left({}^{RL}_{c} I_s^{\beta(.,.)}(s) \right) \right] \\
&\quad + m_3 \left[\left({}^{RL}_{a} I_t^{\alpha(.,.)} t g(t) \right) \left({}^{RL}_{c} I_s^{\beta(.,.)}(1) \right) - \left({}^{RL}_{a} I_t^{\alpha(.,.)}(t) \right) \left({}^{RL}_{c} I_s^{\beta(.,.)} g(s) \right) \right] \\
&\quad + m_4 \left[\left({}^{RL}_{a} I_t^{\alpha(.,.)}(1) \right) \left({}^{RL}_{c} I_s^{\beta(.,.)}(s g(s)) \right) - \left({}^{RL}_{a} I_t^{\alpha(.,.)} g(t) \right) \left({}^{RL}_{c} I_s^{\beta(.,.)}(s) \right) \right] \\
&\quad - m_1 m_3 \left({}^{RL}_{a} I_t^{\alpha(.,.)}(t^2) \right) \left({}^{RL}_{c} I_s^{\beta(.,.)}(s) \right) - m_2 m_4 \left({}^{RL}_{a} I_t^{\alpha(.,.)}(1) \right) \left({}^{RL}_{c} I_s^{\beta(.,.)}(s^2) \right) \\
&\quad + \left(m_2 m_3 + m_1 m_4 \right) \left({}^{RL}_{a} I_t^{\alpha(.,.)}(t) \right) \left({}^{RL}_{c} I_s^{\beta(.,.)}(s) \right)
\end{aligned}
\tag{12}
$$

Proof. Define $h(t) = g(t) - m_1 t$, $h(s) = g(s) - m_2 s$, $k(t) = f(t) - m_3 t$ and $k(s) = f(s) - m_4 s$. As we can see h and k are differentiable and they are increasing on $\mathbb{R}$. It is also easy to verify that h and k are synchronous on $\mathbb{R}$. From inequality (5) replace g and f by h and k respectively, that is,

$$
\begin{aligned}
&\left({}^{RL}_{a} I_t^{\alpha(.,.)}(kh)(t) \right) \left({}^{RL}_{c} I_s^{\beta(.,.)}(1) \right) + \left({}^{RL}_{a} I_t^{\alpha(.,.)}(1) \right) \left({}^{RL}_{c} I_s^{\beta(.,.)}(kh)(s) \right) \\
&\geq \left({}^{RL}_{a} I_t^{\alpha(.,.)} k(t) \right) \left({}^{RL}_{c} I_s^{\beta(.,.)} h(s) \right) + \left({}^{RL}_{a} I_t^{\alpha(.,.)} h(t) \right) \left({}^{RL}_{c} I_s^{\beta(.,.)} k(s) \right)
\end{aligned}
$$

this implies

$$
\begin{aligned}
&\left({}^{RL}_{a} I_t^{\alpha(.,.)}(f(t) - m_3 t)(g(t) - m_1 t) \right) \left({}^{RL}_{c} I_s^{\beta(.,.)}(1) \right) \\
&\quad + \left({}^{RL}_{a} I_t^{\alpha(.,.)}(1) \right) \left({}^{RL}_{c} I_s^{\beta(.,.)}(f(s) - m_4 s)(g(s) - m_2 s) \right) \\
&\geq \left({}^{RL}_{a} I_t^{\alpha(.,.)}(f(t) - m_3 t) \right) \left({}^{RL}_{c} I_s^{\beta(.,.)}(g(s) - m_2 s) \right) \\
&\quad + \left({}^{RL}_{a} I_t^{\alpha(.,.)}(g(t) - m_1 t) \right) \left({}^{RL}_{c} I_s^{\beta(.,.)}(f(s) - m_4 s) \right)
\end{aligned}
$$

this implies

$$\left({}^{RL}_a I_t^{\alpha(.,.)}((fg)(t) - m_1 t f(t) - m_3 t g(t) + m_1 m_3 t^2) \right) \left({}^{RL}_c I_s^{\beta(.,.)}(1) \right)$$

$$+ \left({}^{RL}_a I_t^{\alpha(.,.)}(1) \right) \left({}^{RL}_c I_s^{\beta(.,.)}((fg)(s) - m_2 s f(s) - m_4 s g(s) + m_2 m_4 s^2) \right)$$

$$\geq \left({}^{RL}_a I_t^{\alpha(.,.)} f(t) - m_3 {}^{RL}_a I_t^{\alpha(.,.)}(t) \right) \left({}^{RL}_c I_s^{\beta(.,.)} g(s) - m_2 {}^{RL}_c I_s^{\beta(.,.)}(s) \right)$$

$$+ \left({}^{RL}_a I_t^{\alpha(.,.)} g(t) - m_1 {}^{RL}_a I_t^{\alpha(.,.)}(t) \right) \left({}^{RL}_c I_s^{\beta(.,.)} f(s) - m_4 {}^{RL}_c I_s^{\beta(.,.)}(s) \right)$$

this implies

$$\left({}^{RL}_a I_t^{\alpha(.,.)}(fg)(t) \right) \left({}^{RL}_c I_s^{\beta(.,.)}(1) \right) + \left({}^{RL}_a I_t^{\alpha(.,.)}(1) \right) \left({}^{RL}_c I_s^{\beta(.,.)}(fg)(s) \right)$$

$$\geq \left({}^{RL}_a I_t^{\alpha(.,.)} f(t) \right) \left({}^{RL}_c I_s^{\beta(.,.)} g(s) \right) + \left({}^{RL}_a I_t^{\alpha(.,.)} g(t) \right) \left({}^{RL}_c I_s^{\beta(.,.)} f(s) \right)$$

$$+ m_1 \left[\left({}^{RL}_a I_t^{\alpha(.,.)} t f(t) \right) \left({}^{RL}_c I_s^{\beta(.,.)}(1) \right) - \left({}^{RL}_a I_t^{\alpha(.,.)}(t) \right) \left({}^{RL}_c I_s^{\beta(.,.)} f(s) \right) \right]$$

$$+ m_2 \left[\left({}^{RL}_a I_t^{\alpha(.,.)}(1) \right) \left({}^{RL}_c I_s^{\beta(.,.)}(s f(s)) \right) - \left({}^{RL}_a I_t^{\alpha(.,.)} f(t) \right) \left({}^{RL}_c I_s^{\beta(.,.)}(s) \right) \right]$$

$$+ m_3 \left[\left({}^{RL}_a I_t^{\alpha(.,.)} t g(t) \right) \left({}^{RL}_c I_s^{\beta(.,.)}(1) \right) - \left({}^{RL}_a I_t^{\alpha(.,.)}(t) \right) \left({}^{RL}_c I_s^{\beta(.,.)} g(s) \right) \right]$$

$$+ m_4 \left[\left({}^{RL}_a I_t^{\alpha(.,.)}(1) \right) \left({}^{RL}_c I_s^{\beta(.,.)}(s g(s)) \right) - \left({}^{RL}_a I_t^{\alpha(.,.)} g(t) \right) \left({}^{RL}_c I_s^{\beta(.,.)}(s) \right) \right]$$

$$- m_1 m_3 \left({}^{RL}_a I_t^{\alpha(.,.)}(t^2) \right) \left({}^{RL}_c I_s^{\beta(.,.)}(s) \right) - m_2 m_4 \left({}^{RL}_a I_t^{\alpha(.,.)}(1) \right) \left({}^{RL}_c I_s^{\beta(.,.)}(s^2) \right)$$

$$+ \left(m_2 m_3 + m_1 m_4 \right) \left({}^{RL}_a I_t^{\alpha(.,.)}(t) \right) \left({}^{RL}_c I_s^{\beta(.,.)}(s) \right)$$

$\square$

Corollary 4. *Let f and g be two functions defined on $\mathbb{R}$. For all $a, c \in \mathbb{R}$, $t > a$, $s > c$, $\alpha, \beta : [a,b] \times [a,b] \longrightarrow (0, \infty)$. Suppose that g and f are differentiable functions and there exists a real numbers $M_1 := \sup_{t \geq a} g'(t)$, $M_2 := \sup_{s \geq c} g'(s)$, $M_3 := \sup_{t \geq a} f'(t)$ and $M_4 := \sup_{s \geq c} f'(s)$.*

Then we've

$$
\begin{aligned}
&\left({}_a^{RL}I_t^{\alpha(.,.)}(fg)(t) \right)\left({}_c^{RL}I_s^{\beta(.,.)}(1) \right) + \left({}_a^{RL}I_t^{\alpha(.,.)}(1) \right)\left({}_c^{RL}I_s^{\beta(.,.)}(fg)(s) \right) \\
&\geq \left({}_a^{RL}I_t^{\alpha(.,.)}f(t) \right)\left({}_c^{RL}I_s^{\beta(.,.)}g(s) \right) + \left({}_a^{RL}I_t^{\alpha(.,.)}g(t) \right)\left({}_c^{RL}I_s^{\beta(.,.)}f(s) \right) \\
&\quad + M_1\left[\left({}_a^{RL}I_t^{\alpha(.,.)}tf(t) \right)\left({}_c^{RL}I_s^{\beta(.,.)}(1) \right) - \left({}_a^{RL}I_t^{\alpha(.,.)}(t) \right)\left({}_c^{RL}I_s^{\beta(.,.)}f(s) \right) \right] \\
&\quad + M_2\left[\left({}_a^{RL}I_t^{\alpha(.,.)}(1) \right)\left({}_c^{RL}I_s^{\beta(.,.)}(sf(s)) \right) - \left({}_a^{RL}I_t^{\alpha(.,.)}f(t) \right)\left({}_c^{RL}I_s^{\beta(.,.)}(s) \right) \right] \\
&\quad + M_3\left[\left({}_a^{RL}I_t^{\alpha(.,.)}tg(t) \right)\left({}_c^{RL}I_s^{\beta(.,.)}(1) \right) - \left({}_a^{RL}I_t^{\alpha(.,.)}(t) \right)\left({}_c^{RL}I_s^{\beta(.,.)}g(s) \right) \right] \\
&\quad + M_4\left[\left({}_a^{RL}I_t^{\alpha(.,.)}(1) \right)\left({}_c^{RL}I_s^{\beta(.,.)}(sg(s)) \right) - \left({}_a^{RL}I_t^{\alpha(.,.)}g(t) \right)\left({}_c^{RL}I_s^{\beta(.,.)}(s) \right) \right] \\
&\quad - M_1M_3\left({}_a^{RL}I_t^{\alpha(.,.)}(t^2) \right)\left({}_c^{RL}I_s^{\beta(.,.)}(s) \right) - M_2M_4\left({}_a^{RL}I_t^{\alpha(.,.)}(1) \right)\left({}_c^{RL}I_s^{\beta(.,.)}(s^2) \right) \\
&\quad + \left(M_2M_3 + M_1M_4 \right)\left({}_a^{RL}I_t^{\alpha(.,.)}(t) \right)\left({}_c^{RL}I_s^{\beta(.,.)}(s) \right)
\end{aligned}
\tag{13}
$$

Proof. Define $G(t) = g(t) - M_1 t$, $G(s) = g(s) - M_2 s$, $N(t) = f(t) - M_3 t$ and $N(s) = f(s) - M_4 s$. As we can see G and N are differentiable and they are decreasing on $\mathbb{R}$. It is also easy to verify that G and N are synchronous on $\mathbb{R}$. Follow the same procedure as Corollary (3). $\qquad\square$

4. Conclusion

We produced Chebyshev Type inequalities for the Riemann-Liouville variable-order fractional integral operator using two synchronous functions on the set of real numbers. It is the first result of its kind in the current literature using variable-order Riemann-Liouville fractional integral operator. Some special cases for the result obtained in the paper are discussed. The inequalities obtained are important to determine the uniqueness of solutions for some model of fractional differential equations which involves variable-order Riemann-Liouville fractional integral operator . Other inequalities of chebyshev type can also be produced via other definitions of variable-order fractional integral operator, forexample using Caputo definition and so on.

9

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