AN ALGEBRAIC INEQUALITY WITH APPLICATIONS TO CERTAIN CHEN INEQUALITIES

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ABSTRACT. We give a simple proof of the Chen inequality for the Chen invariant $\underbrace{\delta(2,...,2)}_{k \text{ terms}}$ of submanifolds in Riemannian space forms.

Keywords. Riemannian space form; submanifold; Chen invariants; Chen inequalities.
MSC: 53C40

1. Introduction

In [1], [2], B.-Y. Chen introduced a string of Riemannian invariants, known as *Chen invariants*, which are different in nature from the classical Riemannian invariants. He established sharp relationships between these invariants and the squared mean curvature for submanifolds in Riemannian space forms, known as *Chen inequalities* (see [2]).

The proof uses an algebraic inequality, discovered by B.-Y. Chen in [1].

In the present paper, we obtain a different algebraic inequality which allows us to give simple proofs of certain Chen inequalities.

2. Preliminaries

The theory of Chen invariants and Chen inequalities was initiated by B.-Y. Chen [1], [2].

Let (M, g) be an n-dimensional $(n \geq 2)$ Riemannian manifold and ∇ its Levi-Civita connection. One denotes by R the Riemannian curvature tensor field on M. For any $p \in M$ and $\pi \subset T_pM$ a plane section, the *sectional curvature* $K(\pi)$ of π is defined by $K(\pi) = R(e_1, e_2, e_1, e_2)$, where we use the convention $R(e_1, e_2, e_1, e_2) = g(R(e_1, e_2)e_2, e_1)$, with $\{e_1, e_2\}$ an orthonormal basis of π. Let $\{e_1, ..., e_n\}$ be an orthonormal basis of T_pM. The *scalar curvature* τ at p is given by

$$\tau(p) = \sum_{1 \leq i < j \leq n} K(e_i \wedge e_j),$$

where $K(e_i \wedge e_j)$ is the sectional curvature of the plane section spanned by e_i and e_j (other authors consider $\tau(p) = \sum_{1 \leq i \neq j \leq n} K(e_i \wedge e_j)$).

The *Chen first invariant* δ_M is defined by

$$\delta_M(p) = \tau(p) - \inf\{K(\pi) | \pi \subset T_pM \text{ plane section}\}.$$

The Chen invariant $\delta(2,2)$, given by

$$\delta(2,2)(p) = \tau(p) - \inf\{K(\pi_1) + K(\pi_2) | \pi_1, \pi_2 \subset T_pM \text{ orthogonal plane sections}\},$$

was studied in [3].

We shall consider the Chen invariant $\underbrace{\delta(2, ..., 2)}_{k \text{ terms}}$, denoted by $\delta^k(2, ..., 2)$, which is given by

$$\delta^k(2, ..., 2)(p) = \tau(p) - \inf\{K(\pi_1) + ... + K(\pi_k)\},$$

where $\pi_1, ..., \pi_k$ are mutually orthogonal plane sections at p.

Obviously, $\delta^1(2) = \delta_M$.

In the next section, we shall prove an algebraic inequality and study its equality case. As application we shall give a simple proof of the Chen inequality for the invariant $\delta^k(2, ..., 2)$.

3. An algebraic inequality

Proposition 3.1. *Let $k, n \in \mathbb{N}^*$, $n \geq 2k$, and $a_1, a_2, ..., a_n \in \mathbb{R}$. Then*

$$\sum_{1 \leq i < j \leq n} a_i a_j - \sum_{i=1}^{k} a_{2i-1} a_{2i} \leq \frac{n-k-1}{2(n-k)} \left(\sum_{i=1}^{n} a_i \right)^2.$$

Moreover, the equality holds if and only if $a_{2i-1} + a_{2i} = a_j$, $1 \leq i \leq k$, $2k+1 \leq j \leq n$.

Proof. We shall prove the above Proposition by mathematical induction.

Let

$$P(n) : \sum_{1 \leq i < j \leq n} a_i a_j - \sum_{i=1}^{k} a_{2i-1} a_{2i} \leq \frac{n-k-1}{2(n-k)} \left(\sum_{i=1}^{n} a_i \right)^2,$$

with equality holding if and only if $a_{2i-1} + a_{2i} = a_j$, $1 \leq i \leq k$, $2k+1 \leq j \leq n$.

First we show that $P(2k)$ is true. Indeed

$$\sum_{1 \leq i < j \leq 2k} a_i a_j - \sum_{i=1}^{k} a_{2i-1} a_{2i} \leq \frac{k-1}{2k} \left(\sum_{i=1}^{2k} a_i \right)^2 \Leftrightarrow$$

$$(k-1)(a_1^2+a_2^2+...+a_{2k}^2)-2\sum_{1\le i<j\le 2k} a_ia_j+2k(a_1a_2+a_3a_4+...+a_{2k-1}a_{2k})\ge 0 \Leftrightarrow$$

$$(a_1+a_2-a_3-a_4)^2+...+(a_1+a_2-a_{2k-1}-a_{2k})^2+...+(a_{2k-3}+a_{2k-2}-a_{2k-1}-a_{2k})^2\ge 0.$$

Clearly the equality holds if and only if $a_1+a_2=..=a_{2k-1}+a_{2k}$.

Next, assume $P(n)$ and prove $P(n+1)$.

$$P(n+1):\sum_{1\le i<j\le n+1} a_ia_j-\sum_{i=1}^{k} a_{2i-1}a_{2i}\le \frac{n-k}{2(n-k+1)}\left(\sum_{i=1}^{n+1} a_i\right)^2,$$

with equality holding if and only if $a_{2i-1}+a_{2i}=a_j$, $1\le i\le k$, $2k+1\le j\le n+1$.

By using $P(n)$, one has

$$\sum_{1\le i<j\le n+1} a_ia_j-\sum_{i=1}^{k} a_{2i-1}a_{2i}=$$

$$=\sum_{1\le i<j\le n} a_ia_j+a_{n+1}\left(\sum_{i=1}^{n} a_i\right)-\sum_{i=1}^{k} a_{2i-1}a_{2i}\le$$

$$\le\frac{n-k-1}{2(n-k)}\left(\sum_{i=1}^{n} a_i\right)^2+a_{n+1}\left(\sum_{i=1}^{n} a_i\right)\le$$

$$\le\frac{n-k}{2(n-k+1)}\left(\sum_{i=1}^{n+1} a_i\right)^2.$$

The last inequality is equivalent to

$$\left[(n-k)a_{n+1}-\left(\sum_{i=1}^{n} a_i\right)\right]^2\ge 0.$$

The equality holds if and only if

$$a_{2i-1}+a_{2i}=a_j,\ 1\le i\le k,\ 2k+1\le j\le n+1.$$

□

4. A Chen inequality

As an application of Proposition 1, we give a simple proof of the Chen inequality for the Chen invariant $\delta^k(2,...,2)$ of submanifolds in Riemannian space forms.

Let $\tilde{M}(c)$ be an m-dimensional Riemannian space form of constant sectional curvature c. The standard examples are the Euclidean space $\mathbb{E}^m$, the sphere S^m and the hyperbolic space H^m.

Let M be an n-dimensional submanifold of $\tilde{M}(c)$ and denote by h the second fundamental form of M in $\tilde{M}(c)$. Recall that the mean curvature vector $H(p)$ at $p \in M$ is given by

$$H(p) = \frac{1}{n}\sum_{i=1}^{n} h(e_i, e_i),$$

where $\{e_1, ..., e_n\}$ is an orthonormal basis of T_pM.

The submanifold M is said to be *minimal* if $H(p) = 0, \forall p \in M$.

The Gauss equation is (see [4])

$$R(X, Y, Z, W) = c + g(h(X, Z), h(Y, W)) - g(h(X, W), h(Y, Z)),$$

for any vector fields X, Y, Z, W tangent to M.

Theorem 4.1. *Let $\tilde{M}(c)$ be an m-dimensional Riemannian space form of constant sectional curvature c and M an n-dimensional submanifold of $\tilde{M}(c)$. Then one has the following Chen inequality:*

$$\delta^k(2, ..., 2) \leq \frac{n^2(n-k-1)}{2(n-k)}||H||^2 + \left[\frac{n(n-1)}{2} - k\right] c.$$

Moreover, the equality holds at a point $p \in M$ if and only if there exist suitable orthonormal bases $\{e_1, ..., e_n\} \subset T_pM$ and $\{e_{n+1}, ..., e_m\} \subset T_p^{\perp}M$ such that the shape operators take the forms

$$A_{e_{n+1}} = \begin{pmatrix} a_1 & 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & a_2 & 0 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & a_{2k-1}\; 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \;\; a_{2k} & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 & \mu & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 0 & \dots & \mu \end{pmatrix}, \quad a_{2i-1} + a_{2i} = \mu,\ 1 \leq i \leq k,$$

$$A_{e_r} = \begin{pmatrix} A_1^r & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & A_2^r & 0 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & A_k^r & 0 & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 \end{pmatrix}, \quad r = n+2, ..., m,$$

where A_j^r are symmetric 2×2 matrices with trace $A_j^r = 0$, $\forall j = 1, ..., k$.

Proof. Let $p \in M$, $\pi_1, ..., \pi_k \subset T_pM$ mutually orthogonal plane sections and $\{e_1, e_2\} \subset \pi_1, ..., \{e_{2k-1}, e_{2k}\} \subset \pi_k$ orthonormal bases. We construct $\{e_1, ..., e_{2k}, e_{2k+1}, ..., e_n\} \subset T_pM$ and $\{e_{n+1}, ..., e_m\} \subset T_p^{\perp}M$ orthonormal bases, respectively.

Denote by $h_{ij}^r = g(h(e_i, e_j), e_r)$, $i, j = 1, ..., n$, $r \in \{n+1, ..., m\}$, the components of the second fundamental form.

By the Gauss equation, we have

$$\tau = \sum_{1 \le i < j \le n} K(e_i \wedge e_j) = \sum_{1 \le i < j \le n} R(e_i, e_j, e_i, e_j) =$$

$$= \frac{n(n-1)}{2} c + \sum_{r=n+1}^{m} \sum_{1 \le i < j \le n} [h_{ii}^r h_{jj}^r - (h_{ij}^r)^2].$$

Also the Gauss equation implies

$$K(\pi_i) = K(e_{2i-1} \wedge e_{2i}) = R(e_{2i-1}, e_{2i}, e_{2i-1}, e_{2i}) =$$

$$= c + \sum_{r=n+1}^{m} [h_{2i-1,2i-1}^r h_{2i,2i}^r - (h_{2i-1,2i}^r)^2], \ \forall i = 1, ..., k.$$

Then we get

$$\tau - \sum_{i=1}^{k} K(\pi_i) =$$

$$= \left[\frac{n(n-1)}{2} - k\right] c + \sum_{r=n+1}^{m} \left[\sum_{1 \le i < j \le n} h_{ii}^r h_{jj}^r - \sum_{i=1}^{k} h_{2i-1,2i-1}^r h_{2i,2i}^r\right] -$$

$$- \sum_{r=n+1}^{m} \sum_{\substack{1 \le i < j \le n, \\ (i,j) \neq (1,2), ..., (2k-1,2k)}} (h_{ij}^r)^2.$$

By using the algebraic inequality from the previous section, we obtain

$$\tau - \sum_{i=1}^{k} K(\pi_i) \le \frac{n-k-1}{2(n-k)} \sum_{r=n+1}^{m} \left(\sum_{i=1}^{n} h_{ii}^r\right)^2 + \left[\frac{n(n-1)}{2} - k\right] c =$$

$$= \frac{n^2(n-k-1)}{2(n-k)} ||H||^2 + \left[\frac{n(n-1)}{2} - k\right] c,$$

which implies the desired inequality.

If the equality case holds at a point $p \in M$, then we have equalities in all the inequalities in the proof, i.e.,

$$\begin{cases} h_{2i-1,2i-1}^r + h_{2i,2i}^r = h_{jj}^r, \ 1 \le i \le k, \ 2k+1 \le j \le n, \\ h_{ij}^r = 0, \ \forall 1 \le i < j \le n, \ (i,j) \neq (1,2), ..., (2k-1,2k), \end{cases}$$

for any $r \in \{n+1, ..., m\}$.

We choose e_{n+1} parallel to $H(p)$. Then the shape operators take the above forms. □

Corollary 4.1. *Let $\tilde{M}(c)$ be an m-dimensional Riemannian space form of constant sectional curvature c and M an n-dimensional submanifold of $\tilde{M}(c)$. If there exists a point $p \in M$ such that $\delta^k(2, ..., 2)(p) > \left[\frac{n(n-1)}{2} - k\right] c$, then M is not minimal.*

For $k = 1$, one derives Chen first inequality (see [1]).

Corollary 4.2. *Let $\tilde{M}(c)$ be an m-dimensional Riemannian space form of constant sectional curvature c and M an n-dimensional submanifold of $\tilde{M}(c)$. Then one has*

$$\inf K \geq \tau - \frac{n-2}{2}\left[\frac{n^2}{n-1}||H||^2 + (n+1)c\right].$$

Equality holds if and only if, with respect to suitable frame fields $\{e_1, ..., e_n, e_{n+1}, ..., e_m\}$, the shape operators take the following forms:

$$A_{e_{n+1}} = \begin{pmatrix} a & 0 & 0 & ... & 0 \\ 0 & \mu - a & 0 & ... & 0 \\ 0 & 0 & \mu & ... & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & ... & \mu \end{pmatrix},$$

$$A_{e_r} = \begin{pmatrix} h^r_{11} & h^r_{12} & 0 & ... & 0 \\ h^r_{12} & -h^r_{11} & 0 & ... & 0 \\ 0 & 0 & 0 & ... & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & ... & 0 \end{pmatrix}, \quad r = n+2, ..., m.$$

Example. The generalized Clifford torus.

Let $T = S^k(\frac{1}{\sqrt{2}}) \times S^k(\frac{1}{\sqrt{2}}) \subset S^{2k+1} \subset \mathbb{E}^{2k+2}$.

It is known that T is a minimal hypersurface of S^{2k+1}, but a non-minimal submanifold of $\mathbb{E}^{2k+2}$.

Obviously $\delta^k(2, ..., 2) = \tau = 2k(k-1)$.

Then $T \subset S^{2k+1}$ does not satisfy the equality case of Theorem 4.1.

If we consider $T \subset \mathbb{E}^{2k+2}$, then it satisfies the equality case of Theorem 4.1.

Remark. By using the inequality from Proposition 1, we can obtain Chen inequalities for the invariant $\delta^k(2, ..., 2)$ on submanifolds in other ambient spaces, for instance, complex space forms, Sasakian space forms, Hessian manifolds of constant Hessian curvature, etc.

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