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Article

Relativistic Ermakov-Milne-Pinney Systems and First Integrals

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Abstract: The Eliezer and Gray physical interpretation of the Ermakov-Lewis invariant is applied as a guiding principle for the derivation of the special relativistic analog of the Ermakov-Milne-Pinney equation and associated first integral. The special relativistic extension of the Ray-Reid system and invariant is obtained. General properties of the relativistic Ermakov-Milne-Pinney are analyzed. The conservative case of the relativistic Ermakov-Milne-Pinney equation is described in terms of a pseudo-potential, reducing the problem to an effective Newtonian form. The non-relativistic limit is considered as well. A relativistic nonlinear superposition law for relativistic Ermakov systems is identified. The generalized Ermakov-Milne-Pinney equation has additional nonlinearities, due to the relativistic effects.

Keywords: Ermakov system; Ermakov-Milne-Pinney equation; relativistic Ermakov-Lewis invariant; relativistic Ray-Reid system; nonlinear superposition law.

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1. Introduction

The Ermakov-Milne-Pinney (EMP) equation [1–3]

$$\ddot{x} + \kappa^2(t)x = C/x^3\,, \tag{1}$$

where C is a real constant usually taken as positive, is a notorious nonlinear non-autonomous ordinary differential equation with many applications, in particular in problems related to the time-dependent harmonic oscillator or in connection with exact solutions of the one-dimensional time-independent Schrödinger equation. In more generality, applications of the EMP equation appear in cosmological models [4–6], Bose-Einstein condensates [7,8], photonic lattices [9], accelerator dynamics [10,11], gravitational wave propagation [12], higher order spin models [13], quantum plasmas [14], limit cycles [15], dynamical symmetries [16], magneto-gasdynamics [17] etc. Historical notes can be found e.g. in [18,19]. We note that the name of the EMP equation is not yet a consensus in the literature. For instance, sometimes it can be referred just as Pinney equation.

Diverse generalizations of the EMP equation have been proposed, as for instance allowing $C(t)$ to have a dependence on time [8,19], the inclusion of dissipation [20,21], unbalanced systems of EMP equations with different frequency functions [22], modified nonlinearities [23], stochastic differential equations with additive noise [24]. Following the generalization trend, it would be relevant to extend the EMP equation into the special relativity domain, with potential applications for the dynamics of charged particles in high energy density fields [25]. To date only few results weakly related to special relativistic EMP equations are available, in connection with the Dirac equation [26] and the relativistic isotropic two-dimensional time-dependent harmonic oscillator [27]. Naturally, the relativistic extension is not straightforward due to the intrinsic extra nonlinearity imposed by the presence of the Lorentz factor. Moreover, the intimate connection between the EMP equation and the time-dependent harmonic oscillator raises the question of what defines the relativistic harmonic oscillator. In this regard, the

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relativistic motion immediately induces anharmonicities even in the case of a quadratic potential with a frequency-amplitude dependence [28,29]. Here we follow the approach of many authors [30–39], adopting the spinless Salpeter equation [40] with a quadratic external potential as our definition of relativistic harmonic oscillator. The choice is due to its simplicity, where just the Newtonian kinetic energy is replaced by its special relativistic counterpart. This version of the relativistic harmonic oscillator model has been recently experimentally probed [41].

The present work proposes a systematic approach towards the relativistic EMP equation and beyond, using as a simple guiding principle the Eliezer and Gray [42] physical interpretation of the Ermakov-Lewis associated first integral for the isotropic two-dimensional time-dependent harmonic oscillator. The reason is that the Eliezer and Gray method based on the conservation of the angular momentum of an auxiliary planar motion provides the standard physical reasoning behind the conservation of the Ermakov invariant [43,44]. Moreover, the extension of the Ray-Reid Ermakov systems [43] to the relativistic domain will be also obtained. In brief, a relativistic extension of the celebrated EMP equation and extensions is proposed (which obviously has no direct relation to special solutions of Klein-Gordon, Dirac or similar relativistic partial differential equations).

This work is organized as follows. In Section II, we review the Eliezer and Gray interpretation. In Section III, we consider the relativistic isotropic two-dimensional time-dependent harmonic oscillator and follow the Eliezer and Gray method, in order to identify a relativistic EMP equation together with the associated conservation law. From the structure of the dynamical equations, the appropriate relativistic Ray-Reid system will be also derived, as well as with the corresponding first integral. Section IV provides an alternative derivation based on a dynamical rescaling of time parameter. Sections V and VI deal with basic properties of the relativistic EMP equation, mainly in the autonomous case. Section VII is dedicated to a nonlinear superposition law relating the solutions of the relativistic EMP system. Finally, Section VIII is reserved to the conclusions and extra remarks.

2. The Eliezer and Gray physical interpretation

We briefly reproduce the Eliezer and Gray physical interpretation [42] of the Ermakov-Lewis invariant, which will serve us as a guidance for a relativistic generalization of the EMP system. Consider the Lagrangian

$$L = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) - V(x,y,t)\,, \quad V(x,y,t) = \frac{\kappa^2(t)(x^2+y^2)}{2} \tag{2}$$

the corresponding auxiliary plane motion

$$\ddot{\mathbf{r}} + \kappa^2(t)\mathbf{r} = 0\,, \tag{3}$$

where $\mathbf{r}$ has the Cartesian components (x,y), and the EMP equation

$$\ddot{\rho} + \kappa^2(t)\rho = \frac{J^2}{\rho^3}\,, \tag{4}$$

where J is a real constant. In terms of polar coordinates (ρ,θ), where $\rho = |\mathbf{r}|, x = \rho\cos\theta, y = \rho\sin\theta$, the equations of motion become

$$\ddot{\rho} - \rho\dot{\theta}^2 + \kappa^2\rho = 0\,, \tag{5}$$

$$\frac{1}{\rho}\frac{d}{dt}(\rho^2\dot{\theta}) = 0\,. \tag{6}$$

Equation (6) implies the constancy of the angular momentum

$$J = \rho^2\dot{\theta} = \text{const.} \tag{7}$$

assuming for simplicity an unit mass. Combining Eqs. (5) and (7) we derive Eq. (4). Considering the equation for the $x-$component of the auxiliary motion together with Eq. (4), the constancy of the Ermakov-Lewis invariant I given by

$$I = \frac{1}{2}\left[(\rho\dot{x} - x\dot{\rho})^2 + \frac{J^2x^2}{\rho^2}\right] \tag{8}$$

is directly verified, $dI/dt = 0$ along trajectories. Expressing in terms of polar coordinates, one has

$$I = \frac{1}{2}(J^2\sin^2\theta + J^2\cos^2\theta) = \frac{J^2}{2}\,. \tag{9}$$

Therefore, the invariance of I is equivalent to the invariance of the angular momentum of the auxiliary plane motion.

For reference, it is worth to consider the Ray-Reid (RR) generalization [43] of the EMP system and invariant, namely

$$\ddot{x} + \kappa^2 x = \frac{f(y/x)}{yx^2}\,, \tag{10}$$

$$\ddot{y} + \kappa^2 y = \frac{g(x/y)}{xy^2}\,, \tag{11}$$

where f, g are arbitrary functions of the indicated arguments. The RR first integral for Eqs. (10) and (11) is

$$I_{RR} = \frac{1}{2}(x\dot{y} - y\dot{x})^2 + \int^{y/x} f(s)ds + \int^{x/y} g(s)ds\,. \tag{12}$$

One can directly verify that $dI_{RR}/dt = 0$ along trajectories.

3. A relativistic Ermakov-Milne-Pinney system

Proceeding in strict analogy with the NR case, consider the equations of motion for the 2D relativistic unit rest mass time-dependent harmonic oscillator, which can be derived [45] from the Lagrangian

$$L = -\frac{c^2}{\gamma} - V(x,y,t)\,, \quad V(x,y,t) = \frac{\kappa^2(t)(x^2+y^2)}{2}\,, \tag{13}$$

where c is the speed of light and $\gamma = [1 - (\dot{x}^2 + \dot{y}^2)/c^2]^{-1/2}$. The Euler-Lagrange equations are

$$\left(1 - \frac{\dot{y}^2}{c^2}\right)\ddot{x} + \frac{\dot{x}\dot{y}}{c^2}\ddot{y} = -\frac{\kappa^2 x}{\gamma^3}\,, \tag{14}$$

$$\left(1 - \frac{\dot{x}^2}{c^2}\right)\ddot{y} + \frac{\dot{x}\dot{y}}{c^2}\ddot{x} = -\frac{\kappa^2 y}{\gamma^3}\,, \tag{15}$$

which can be disentangled as

$$\ddot{x} + \frac{\kappa^2}{\gamma}\frac{x}{\gamma_x^2} = \frac{\kappa^2}{\gamma}\frac{\dot{x}\dot{y}}{c^2}y\,, \tag{16}$$

$$\ddot{y} + \frac{\kappa^2}{\gamma}\frac{y}{\gamma_y^2} = \frac{\kappa^2}{\gamma}\frac{\dot{x}\dot{y}}{c^2}x\,, \tag{17}$$

where $\gamma_x = (1 - \dot{x}^2/c^2)^{-1/2}$, $\gamma_y = (1 - \dot{y}^2/c^2)^{-1/2}$. The equations of motion are in agreement with [27]. Similarly to the NR case, employing cylindrical coordinates $x = \rho\cos\theta, y = \rho\sin\theta$, we have the conserved angular momentum expressed as

$$J = \frac{\partial L}{\partial\dot{\theta}} = \gamma\rho^2\dot{\theta}\,. \tag{18}$$

Moreover, we get the Lorentz factor expressible in terms of $\rho, \dot{\rho}$ as

$$\gamma = \left(\frac{1 + J^2/c^2\rho^2}{1 - \dot{\rho}^2/c^2}\right)^{1/2}\,. \tag{19}$$

The NR formal limit $c \to \infty$ yields $\gamma = 1$ also from Eq. (19). Using the angular momentum to eliminate the angular velocity, we obtain

$$\ddot{x} + \frac{\kappa^2}{\gamma}\left(x - \frac{\rho\dot{\rho}\dot{x}}{c^2}\right) = 0\,, \tag{20}$$

$$\ddot{\rho} + \frac{\kappa^2}{\gamma}\left(1 - \frac{\dot{\rho}^2}{c^2}\right)\rho = \frac{J^2}{\gamma^2\rho^3}\,. \tag{21}$$

From the identity $\gamma^2(\rho\dot{x} - x\dot{\rho})^2 = J^2\sin^2\theta$, it becomes self-evident that the quantity

$$I_R = \frac{1}{2}\left[\gamma^2(\rho\dot{x} - x\dot{\rho})^2 + \frac{J^2x^2}{\rho^2}\right] \tag{22}$$

is a first integral, since

$$I_R = \frac{1}{2}(J^2\sin^2\theta + J^2\cos^2\theta) = \frac{J^2}{2}\,, \tag{23}$$

in complete analogy with the Ermakov-Lewis invariant for the NR problem. It can be also directly verified that $dI_R/dt = 0$ along the trajectories of the system (20)-(21). The invariant has the same physical interpretation of the NR Ermakov invariant in terms of the conservation of the angular motion of the associated auxiliary 2D motion. In this context, it is justified to interpret Eqs. (20) and (21) as the (special) relativistic EMP system. Equation (21) is a relativistic EMP equation (REMP), and the first integral in Eq. (22) is the relativistic Ermakov-Lewis invariant of the problem. Obviously, in the formal limit $c \to \infty$ one recovers the NR case. Understanding the Lorentz factor in the sense of Eq. (19), the relativistic EMP system is a pair of nonlinear second-order ordinary differential equations for x, ρ. Unlike the NR case, the equation for x is not uncoupled.

From the structure of Eqs. (20) and (21) and after some trial and error, it is possible to identify a relativistic Ray-Reid (RRR) system, namely

$$\ddot{x} + \frac{\kappa^2}{\gamma}\left(x - \frac{\rho\dot{\rho}\dot{x}}{c^2}\right) = \left(1 - \frac{\dot{x}^2}{c^2}\right)\frac{f(y/x)}{\gamma^2yx^2} - \frac{\dot{x}\dot{y}}{\gamma^2c^2}\frac{g(x/y)}{xy^2}\,, \tag{24}$$

$$\ddot{y} + \frac{\kappa^2}{\gamma}\left(y - \frac{\rho\dot{\rho}\dot{y}}{c^2}\right) = -\frac{\dot{x}\dot{y}}{\gamma^2c^2}\frac{f(y/x)}{yx^2} + \left(1 - \frac{\dot{y}^2}{c^2}\right)\frac{g(x/y)}{\gamma^2xy^2}\,, \tag{25}$$

where $\rho\dot{\rho} = x\dot{x} + y\dot{y}$ and the Lorentz factor is understood as a function of $\dot{x}, \dot{y}$. The invariant for Eqs. (24) and (25) is

$$I_{RRR} = \frac{\gamma^2}{2}(x\dot{y} - y\dot{x})^2 + \int^{y/x} f(s)ds + \int^{x/y} g(s)ds\,. \tag{26}$$

It can be verified that $dI_{RRR}/dt = 0$ along trajectories. It is apparent that Eqs. (24), (25) and (26) define a RRR system and its invariant, showing a complete symmetry between the x and y variables and recovering the RR system and invariant in the formal NR limit $c \to \infty$, as shown by comparison with

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Eqs. (10)-(12). A derivation provided by a dynamical rescaling of time will be described in the next Section.

Although our treatment has as motivation the relativistic time-dependent harmonic oscillator, it is clear that the invariants shown in Eqs. (22) and (26) do not depend on κ. Therefore, one is authorized to allow for more general functional dependencies of κ, e.g. $\kappa = \kappa(t, x, y, \dot{x}, \dot{y}, \dots)$ in Eqs. (24) and (25), maintaining the constancy of the RRR invariant. Similar remarks apply to the NR case [46–48].

4. Derivation from a dynamical rescaling of time

Start from the dynamical system

$$x'' + \omega^2 x = \frac{f(y/x)}{yx^2}\,, \quad y'' + \omega^2 y = \frac{g(x/y)}{xy^2}\,, \tag{27}$$

where a prime denotes derivative with respect to the independent variable τ, or $x' = dx/d\tau, y' = dy/d\tau$, and where ω can be anything in the same spirit of the last Section. In the same way as for the RR system, it is obvious that (27) possess the invariant

$$I = \frac{1}{2}(xy' - yx')^2 + \int^{y/x} f(s)ds + \int^{x/y} g(s)ds\,, \quad \frac{dI}{d\tau} = 0\,. \tag{28}$$

If we now move to a new independent variable t defined by

$$\frac{dt}{d\tau} = \gamma\,, \quad \gamma = \left(1 - \frac{\dot{x}^2 + \dot{y}^2}{c^2}\right)^{-1/2}\,, \quad \dot{x} = \frac{dx}{dt}\,, \quad \dot{y} = \frac{dy}{dt}\,, \tag{29}$$

then it can be checked after some algebra that the RRR system (24) and (25) is recovered, with $\kappa^2 = \omega^2/\gamma$, and that the invariant (28) transforms into (26). The procedure gives a more transparent derivation of the RRR system from a dynamical rescaling of time starting from the RR system.

5. The $J = 0$ case

For a vanishing angular momentum, one has $\dot{\theta} = 0$ so that it can be chosen $\theta = 0$, without loss of generality. In this case, $y = 0$ and Eq. (20) becomes

$$\frac{d}{dt}(\gamma \dot{x}) = -\kappa^2 x\,, \quad \gamma = \left(1 - \frac{\dot{x}^2}{c^2}\right)^{-1/2}\,, \tag{30}$$

which is [49] the equation for an one-dimensional (1D) relativistic time-dependent harmonic oscillator.

In the case of a constant frequency κ, obviously the energy is conserved. In terms of rescaled variables $\bar{x} = \kappa x/c$, $\bar{t} = \kappa t$, $\bar{v} = d\bar{x}/d\bar{t}$, the corresponding first integral $H_{1D} \geq 1$ is

$$H_{1D} = (1 - \bar{v}^2)^{-1/2} + \frac{\bar{x}^2}{2}\,, \tag{31}$$

with phase-space contour plots shown in Fig. 1. Clearly, only bounded trajectories are admissible, as expected. The return points are located at $\bar{x} = \pm\sqrt{2}(H_{1D} - 1)^{1/2}$. In rescaled coordinates, the NR limit $H_{1D} \simeq 1$ corresponds to circular trajectories in phase space. For larger values of H_{1D}, the return points are also larger, implying more relativistic effects and increasing anharmonicity, as seen in Fig. 1.

It is instructive to rewrite the conservation law in a Newtonian form,

$$\frac{\bar{v}^2}{2} + V_{1D}(\bar{x}) = 0\,, \tag{32}$$

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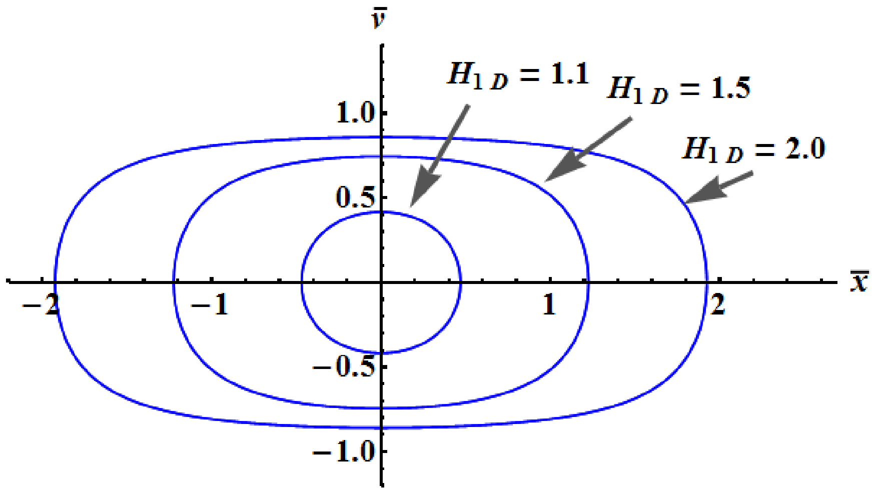

Figure 1. Phase-space contour plots of the energy first integral (31) for the 1D conservative relativistic harmonic oscillator described by Eq. (30) with constant κ, for $\bar{x} = \kappa x/c$, $\bar{v} = \dot{x}/c$ and different values of H_{1D}, as indicated.

where the pseudo-potential $V_{1D}(\bar{x})$ is defined by

$$V_{1D}(\bar{x}) = -\frac{1}{2} + \frac{1}{2(H_{1D} - \bar{x}^2/2)^2}, \tag{33}$$

shown in Fig. 2. In the NR limit, the variable $\bar{x}$ is limited to small values, so that $V_{1D} = \text{cte.} + \bar{x}^2/(2H_{1D}^3) + \ldots$ with $H_{1D} \simeq 1$, while larger values of H_{1D} correspond to enhanced relativistic effects and anharmonicity.

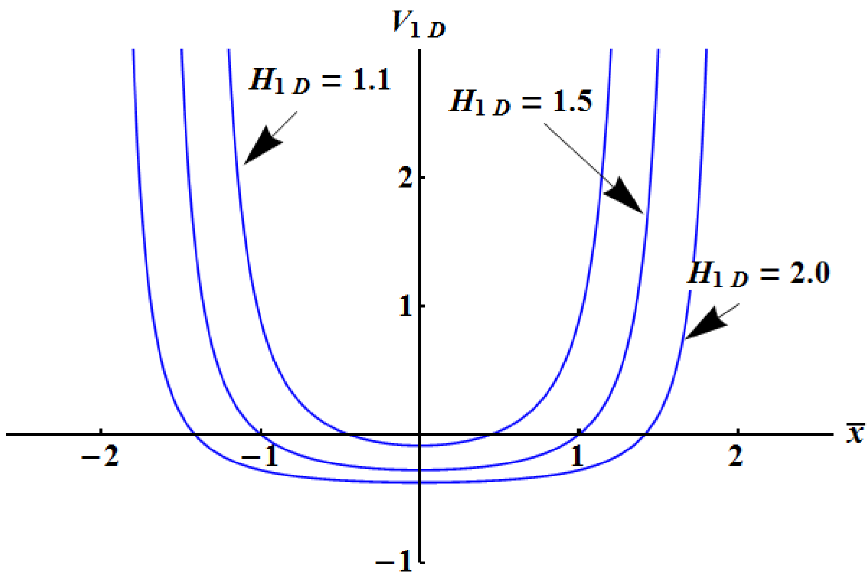

Figure 2. Pseudo-potential $V_{1D}(\bar{x})$ from Eq. (33) and different values of H_{1D}, as indicated.

The conserved energy can be used for the quadrature of the motion in terms of elliptic functions, but the analytic solution is too cumbersome to be of much interest. Some exact periodic solutions for the 1D conservative relativistic harmonic oscillator can be found in [50].

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6. Relativistic conservative Ermakov-Milne-Pinney equation

The REMP defined in Eq. (21) with $J \neq 0$ does not have collapsing ($\rho \to 0$) solutions due to the inverse cubic term. For instance, suppose $\kappa =$ cte. and the rescaling $\bar{\rho} = \kappa\rho/c, \bar{t} = \kappa t, \bar{v} = d\bar{\rho}/d\bar{t}, \bar{J} = \kappa J/c^2$, so that

$$\frac{d\bar{v}}{d\bar{t}} + \frac{1}{\gamma}(1 - \bar{v}^2)\bar{\rho} = \frac{\bar{J}^2}{\gamma^2\bar{\rho}^3} \,. \tag{34}$$

The energy first integral is

$$H = \gamma + \frac{\bar{\rho}^2}{2} \geq 1 \,, \quad \gamma = \left(\frac{1 + \bar{J}^2/\bar{\rho}^2}{1 - \bar{v}^2}\right)^{1/2} \,. \tag{35}$$

It is immediate to conclude that $0 < \bar{\rho} < \sqrt{2H}$, otherwise H would be not conserved.

Rewriting in a Newtonian form yields

$$\bar{v}^2/2 + V(\bar{\rho}) = 0 \,, \tag{36}$$

with a pseudo-potential defined by

$$V(\bar{\rho}) = -\frac{1}{2} + \frac{1 + \bar{J}^2/\bar{\rho}^2}{2(H - \bar{\rho}^2/2)^2} \,, \tag{37}$$

shown in Fig. 3. It is possible to show that $V(\bar{\rho}_*) < 0$, where $0 < \bar{\rho}_* < 2H$ is the equilibrium point of V, viz.

$$\bar{\rho}_* = \frac{1}{2}\left(-3\bar{J}^2 + \sqrt{9\bar{J}^4 + 16\bar{J}^2 H}\right)^{1/2} \,. \tag{38}$$

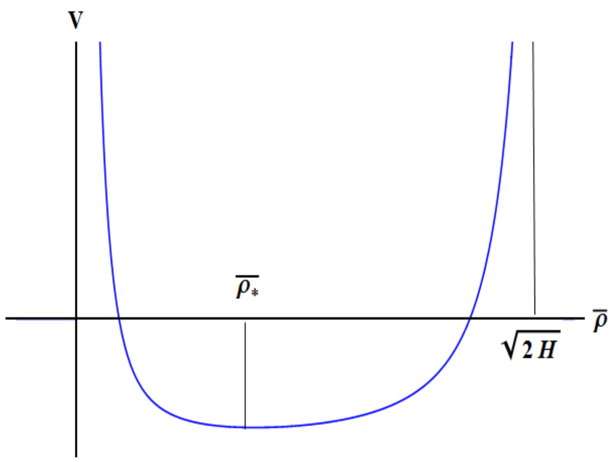

Figure 3. Pseudo-potential V from Eq. (37).

The return points correspond to $V = 0$, similarly to Fig. 2, with stable oscillations around $\bar{\rho}_*$. Elementary algebra shows that the condition for periodic motions ($V(\bar{\rho}_*) < 0$) is equivalent to

$$\bar{J}^2 < F(H) = \frac{4}{27}\left(-9H + H^3 + (3 + H^2)^{3/2}\right) \,, \tag{39}$$

or, sufficiently small angular momentum. In the NR limit one has $\bar{J}^2 < (H-1)^2$ disregarding $\mathcal{O}(H-1)^3$ terms, so that the periodicity condition can be shown to be automatically satisfied. The characteristic function F is shown in Fig. 4. A numerical investigation shows that the periodicity

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condition is always meet, for meaningful initial conditions $0 < \bar{\rho}(0) < \sqrt{2H}$, $-1 < \bar{v}(0) < 1$, as expected.

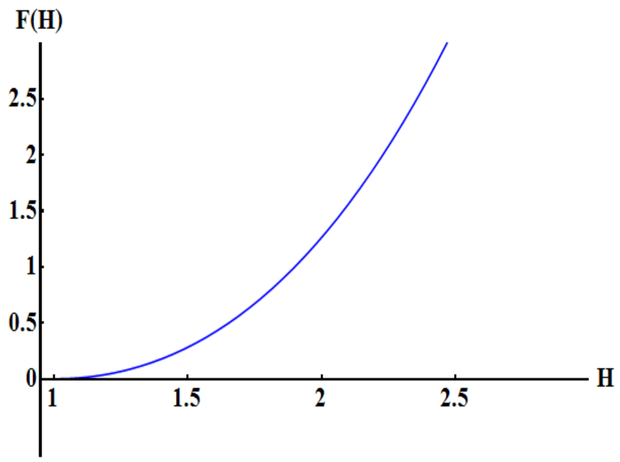

Figure 4. Characteristic function $F(H)$ from Eq. (39) for $H \geq 1$.

The quadrature of Eq. (36) can be made in terms of elliptic functions, after determining the return points corresponding to the potential well from Eq. (37). However, the result is exceedingly complicated.

7. Nonlinear superposition law

Suppose $\rho = \rho(t)$ a particular solution of the REMP equation (21). Introducing the new variables

$$Q = \frac{x}{\rho}, \quad T = \int \frac{dt}{\gamma \rho^2}, \tag{40}$$

where $\gamma = \gamma(t)$ is given by Eq. (19) converts the relativistic Ermakov-Lewis invariant (22) into

$$I_R = \frac{1}{2}\left(\frac{dQ}{dT}\right)^2 + \frac{J^2 Q^2}{2}, \tag{41}$$

formally the same as the energy first integral for a 1D conservative NR harmonic oscillator. A quadrature yields

$$Q = \frac{\sqrt{2I_R}}{|J|} \sin(JT + \delta), \tag{42}$$

or

$$x = \rho \sin\left(J \int \frac{dt}{\gamma \rho^2} + \delta\right), \tag{43}$$

since $I_R = J^2/2$, adopting $J > 0$ and where δ is a constant phase. The nonlinear superposition law (43) generalizes the Newtonian result [46] to the relativistic context. In concrete applications, typically the particular solution ρ should be numerically found.

8. Conclusion

In this work, considerable progress was achieved, in the generalization of Ermakov systems towards the special relativity domain. The Eliezer and Gray physical interpretation of the Ermakov-Lewis invariant, was used as a guide for the derivation of the relativistic analog of the EMP equation, together with the corresponding first integral for the relativistic planar time-dependent harmonic oscillator. General aspects of the relativistic EMP equation have been addressed, and a nonlinear superposition law was derived. In spite of the successful results, it is still possible to derive

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other classes of relativistic Ermakov systems, not arising from the correspondence with the Eliezer and Gray physical interpretation. For instance, symmetry principles can be a guiding principle, although the $SL(2,\Re)$ group structure of non-relativistic Ermakov systems obviously tends to be broken in the relativistic domain. In the same footing, to carry to the relativistic domain the linearization of standard Ermakov systems would be a probably unfeasible task. Finally, an intriguing question remains about the RRR system defined by Eqs. (24) and (25). If one starts from the Lagrangian (13) adding an Ermakov term $U(y/x)/(x^2+y^2)$ to the potential V, we find precisely the system (24)-(25), where f, g are suitable related to U, but with a factor $1/\gamma$ instead of $1/\gamma^2$ on all terms of the right-hand side. In this case, the quantity I_{RRR} in Eq. (26) would be not conserved, and no other first integral in direct correspondence with the non-relativistic Ray and Reid invariant would be available. In other words, the Hamiltonian structure from (non-relativistic) RR systems [47] is almost immediately translated to the relativistic context, except for an *ad hoc* modification of the $\gamma-$factor power, as long as one prefers to keep the existence of a Ray and Reid invariant.

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Conflicts of Interest: The author declares no conflict of interest.

References

1. Ermakov, V.P. Differentzial'nyya uravneniya vtorago poryadka. Usloviya integriruemosti v konechnom vide. *Univ. Isz. Kiev Series III* **1880**, *9*, 1–25 (English translation: Harin, A.O. Second-order differential equations. Conditions of complete integrability. *Appl. Anal. Discrete Math.* **2008**, *2*, 123–145).
2. Milne, W.E. The numerical determination of characteristic numbers. *Phys. Rev.* **1930**, *35*, 863–867.
3. Pinney, E. The nonlinear equation $y''(x)+p(x)y(x)+cy^{-3}(x)=0$. *Proc. Amer. Math. Soc.* **1950**, *1*, 681.
4. Hawkins, R.M.; Lidsey, J.E. Ermakov-Pinney equation in scalar field cosmologies. *Phys. Rev. D* **2002**, *66*, 023523–023530.
5. Rosu, H.C.; Mancas, S.C.; Chen, P. Barotropic FRW Cosmologies with Chiellini damping. *Phys. Lett. A* **2015**, *379*, 882–887.
6. Tsamparlis, M.; Paliathanasis, A.; Basilakos, S.; Capozziello, S. Scalar-tensor gravity cosmology: Noether symmetries and analytical solutions. *Phys. Rev. D* **2014**, *89*, 063532–063540.
7. Haas, F. Anisotropic Bose-Einstein condensates and completely integrable dynamical systems. *Phys. Rev. A* **2002**, *65*, 033603–033608.
8. Herring, G.; Kevrekidis, P.G.; Williams, F.; Christodoulakis, T.; Frantzeskakis, D.F. From Feshbach-resonance managed Bose-Einstein condensates to anisotropic universes: applications of the Ermakov-Pinney equation with time-dependent nonlinearity. *Phys. Lett. A* **2007**, *367*, 140–148.
9. Rodríguez-Lara, B.M.; Aleahmad, P.; Moya-Cessa, H.M.; Christodoulides, D.N. Ermakov–Lewis symmetry in photonic lattices. *Opt. Lett.* **2014**, *39*, 2083–2085.
10. Courant, E.D.; Snyder, H.D. Theory of the alternating-gradient synchrotron. *Ann. Phys.* **1958**, *3*, 1–48.
11. Qin, H.; Davidson, R.C. Symmetries and invariants of the oscillator and envelope equations with time-dependent frequency. *Phys. Rev. ST Accel. Beams* **2006**, *9*, 054001–054005.
12. Geralico, A.; Landolfi, G.; Ruggeri, C.; Soliani, G. Novel approach to the study of quantum effects in the early Universe. *Phys. Rev. D* **2004**, *69*, 043504–043513.
13. Fring, A.; Frith, T. Metric versus observable operator representation, higher spin models. *Eur. Phys. J. Plus* **2018**, *133*, 57–65.
14. Haas, F.; Bret, A. Nonlinear low-frequency collisional quantum Buneman instability. *Europhys. Lett.* **2012**, *97*, 26001, 7 pages.
15. Llibre, J.; Pérez-Chavela, E. Limit cycles for a class of second order differential equations. *Phys. Lett. A* **2011**, *375*, 1080–1083.
16. Haas, F.; Goedert, J. On the linearization of the generalized Ermakov systems. *J. Phys. A: Math. Gen.* **1999**, *32*, 2835–2844.
17. Rogers, C.; Schief, W. K. The pulsrodon in 2+1-dimensional magneto-gasdynamics: Hamiltonian structure and integrability. *J. Math. Phys.* **2011**, *52*, 083701–083721.

18. Cariñena, J.F.; Lucas, J.; Rañada, M.F. A nonlinear superposition rule for solutions of the Milne-Pinney equation. *SIGMA* **2008**, *4*, 031, 18 pages.
19. Morris, R.M.; Leach, P.G.L. The effects of symmetry-breaking functions on the Ermakov-Pinney equation. *Appl. Anal. Discrete Math.* **2017**, *11*, 62–73.
20. Haas, F. The damped Pinney equation and its applications to dissipative quantum mechanics. *Physica Scripta* **2010**, *81*, 025004, 8 pages.
21. Guha, P.; Ghose-Choudhury, A. Nonlocal transformations of the generalized Liénard type equations and dissipative Ermakov-Milne-Pinney systems. *Int. J. Geom. Meth. Mod. Phys.* **2019**, *16*, 1950107, 18 pages.
22. Athorne, C. On solving a class of unbalanced Ermakov- Pinney systems. *J. Phys. A: Math. Gen.* **2001**, *34*, L563–L566.
23. Reid, J.L. An exact solution of the nonlinear differential equation $\ddot{y} + p(t)y = qm(t)/y^{2m1}$. *Proc. Am. Math. Soc.* **1971**, *27*, 601–602.
24. Cervantes-López, E.; Espinoza, P.B.; Gallegos, A.; Rosu, H.C. Ermakov–Ray–Reid systems with additive noise. *Physica A* **2015**, *439*, 44–47.
25. Hartemann, F.V. *High-field Electrodynamics*. CRC Press: Boca Raton, USA, 2002.
26. Thylwe, K.-E.; McCabe, P. Bohr-Sommerfeld quantization condition for Dirac states derived from an Ermakov-type invariant. *J. Math. Phys.* **2013**, *54*, 052301–052308.
27. Petrov, S.V. Classical dynamics of the relativistic oscillator. *Eur. J. Phys.* **2016**, *37*, 065605, 11 pages.
28. Guerrero, J.; Aldaya, V. A perturbative approach to the relativistic harmonic oscillator. *Mod. Phys. Lett. A* **1999**, *14*, 1689–1699.
29. Beléndez, A.; Pascual, C.; Méndez, D.I.; Neipp, C. Solution of the relativistic (an)harmonic oscillator using the harmonic balance method. *J. Sound Vib.* **2008**, *311*, 1447–1456.
30. Edery, A.; Laporte, P. First and second-order relativistic corrections to the two and higher-dimensional isotropic harmonic oscillator obeying the spinless Salpeter equation. *J. Phys. Commun.* **2018**, *2*, 025024, 16 pages.
31. Poszwa, A. Relativistic generalizations of the quantum harmonic oscillator *Acta Physica Polonica A* **2014**, *126*, 1226–1234.
32. Znojil, M. Harmonic oscillations in quasi-relativistic regime. *J. Phys. A: Math. Gen* **1996**, *29*, 2905–2918.
33. Balasubramanya, M.K. The action variable and frequency of a relativistic harmonic oscillator. *Theor. Math. Phys.* **2010**, *162*, 341–346.
34. Gold, L. Note on the relativistic harmonic oscillator. *J. Franklin Inst.* **1957**, *264*, 25–27.
35. Harvey, A.L. Relativistic harmonic oscillator. *Phys. Rev. D* **1972**, *6*, 1474–1475.
36. Teychenné, D.; Bonnaud, G.; Bobin, J. Oscillatory relativistic motion of a particle in a power-law or sinusoidal-shaped potential well. *Phys. Rev. E* **1994**, *49*, 3253–3263.
37. Chargui, Y. On an approximation of the two-body spinless Salpeter equation. *Eur. Phys. J. Plus* **2018**, *133*, 543, 6 pages.
38. Llibre, J.; Makhlouf, A. On the periodic solutions of the relativistic driven harmonic oscillator. *J. Math. Phys.* **2020**, *61*, 012501–012507.
39. Li, Z.; Liu, J.; Lucha, W.; Ma, W.; Schöberl, F.F. Relativistic harmonic oscillator. *J. Math. Phys.* **2005**, *46*, 103514–103525.
40. Salpeter, E.E. Mass corrections to the fine structure of hydrogen-like atoms. *Phys. Rev.* **1952**, *87*, 328–343.
41. Fujiwara, K.M.; Geiger, Z.A.; Singh, K.; Senaratne, R.; Rajagopal, S.V.; Lipatov, M.; Shimasaki, T.; Weld, D.M. Experimental realization of a relativistic harmonic oscillator. *New J. Phys.* **2018**, *20*, 063027, 7 pages.
42. Eliezer, C.J.; Gray, A. A note on the time-dependent harmonic oscillator. *SIAM J. Appl. Math.* **1976**, *30*, 463–468.
43. Ray, J.R.; Reid, J.L. More exact invariants for the time-dependent harmonic oscillator. *Phys. Lett. A* **1979**, *71*, 317–318.
44. Mancas, S.C.; Rosu, H.C. Ermakov-Lewis invariants and Reid systems. *Phys. Lett. A* **2014**, *378*, 2113–2117.
45. Goldstein, H.; Poole, C.; Safko, J. *Classical Mechanics*, 3rd ed. Addison Wesley: San Francisco, USA, 2002; pp. 316.
46. Reid, J.L.; Ray, J.R. Ermakov systems, nonlinear superposition, and solutions of nonlinear equations of motion. *J. Math. Phys.* **1980**, *21*, 1583–1587.
47. Haas, F.; Goedert, J. On the Hamiltonian structure of Ermakov systems. *J. Phys. A: Math. Gen.* **1996**, *29*, 4083–4092.

 doi:10.20944/preprints202101.0143.v1

48. Goedert, J.; Haas, F. On the Lie symmetries of a class of generalized Ermakov systems. *Phys. Lett. A* **1998**, *239*, 348–352.
49. Martin, J.; Bouquet, S. Search for first integrals in relativistic time-dependent Hamiltonian systems. *J. Math. Phys.* **1994**, *35*, 181–189.
50. Mickens, R.E. Periodic solutions of the relativistic harmonic oscillator. *J. Sound Vib.* **1998**, *212*, 905–908.