Forecasting COVID-19 with importance-sampling and path-integrals

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Abstract

Background: Forecasting nonlinear stochastic systems most often is quite difficult, without giving in to temptations to simply simplify models for the sake of permitting simple computations. **Objective:** Here, two basic algorithms, Adaptive Simulated Annealing (ASA) and path-integral codes PATHINT/PATHTREE (and their quantum generalizations qPATHINT/qPATHTREE) are suggested as being useful to fit COVID-19 data and to help predict spread or control of this pandemic. Multiple variables are considered, e.g., potentially including ethnicity, population density, obesity, deprivation, pollution, race, environmental temperature. **Method:** ASA and PATHINT/PATHTREE have been demonstrated as being effective to forecast properties in three disparate disciplines in neuroscience, financial markets, and combat analysis. **Results:** Not only can selected systems in these three disciplines be aptly modeled, but results of detailed calculations have led to new results and insights not previously obtained. **Conclusion:** While optimization and path-integral algorithms are now quite well-known (at least to many scientists), these applications give strong support to a quite generic application of these tools to stochastic nonlinear systems.

Keywords: path integral, importance sampling, financial options, combat analysis

1. Introduction

It is generally recognized that the spread of COVID-19 is affected by multiple variables, e.g., potentially including ethnicity, population density, obesity, deprivation, pollution, race, environmental temperature (Anastassopoulou *et al*, 2020; Bray *et al*, 2020; Li *et al*, 2020). Also, the Centre for Evidence-Based Medicine (CEBM) regularly cites papers on the dynamics of COVID-19 at https://www.cebm.net/evidence-synthesis/transmission-dynamics-of-covid-19/ .

This proposal offers the application of two basic multivariate algorithms to fairly generic issues in forecasting. As such, they may be useful to fit COVID-19 data and to help predict upcoming spread and control of this pandemic.

> (a) Adaptive Simulated Annealing (ASA) developed by the author (Ingber, 1993a) is an importance-sampling optimization code usually used for nonlinear, nonequilibrium, non-stationary, multivariate systems.
>
> (b) PATHINT is a numerical path-integral PATHINT code developed by the author (Ingber, 1993b) used for propagation of nonlinear probability distributions, including discontinuities.

These codes were developed by the author and applied across multiple disciplines.

There is not “one size fits all” in forecasting different systems. This was demonstrated for three systems (Ingber, 2020b), where the author has addressed multiple projects across multiple disciplines using these tools: 72 papers/reports/lectures in neuroscience, e.g. (Ingber, 2018; Ingber, 2021), 31 papers/reports/lectures in finance, e.g. (Ingber & Mondescu, 2003; Ingber, 2020a), 24 papers/reports/lectures in combat analyses, e.g. (Ingber, 1993b; Ingber, 2015), and 11 papers/reports/lectures in optimization, e.g. (Atiya *et al*, 2003; Ingber, 2012), It is reasonable to expect that this approach can be applied to many other projects.

For example, the path-integral representation of multivariate nonlinear stochastic differential equations permits derivation of canonical momenta indicators (CMI) which are faithful to intuitive concepts like Force, Momenta, Mass, etc (Ingber, 1996; Ingber, 2015; Ingber & Mondescu, 2001). Correlations among variables are explicitly included in the CMI.

2. Data

A large and updated database for COVID-19 is maintained by the John Hopkins University (JHU) at https://github.com/CSSEGISandData/COVID-19/blob/master/archived_data/archived_daily_case_updates/01-21-2020_2200.csv . This database was used for a pilot study.

2.1. 50+ Locations

The data being used contains 3340 cities throughout the US and some territories. The locations have been broken into 57 States and Territories ready for production runs.

3. Technical considerations

If there is not time to process large data sets, then the data can be randomly sampled, e.g., as described in another paper, “Developing bid-ask probabilities for high-frequency trading” (Ingber, 2020a).

If the required forecast is longer than the conditional distribution can sustain, PATHINT/PATHTREE can be used to propagate the distribution.

The dataset should be broken into independent Training and Testing subsets, to test the trained distribution. If this is not possible, e.g., because of data or time limitations, at the least experts can be used to judge if the model is ready for real-time applications, e.g., the Delphi method (Okoli & Pawlowski, 2004).

If an algorithm like ASA is to be used across a large class of problems, then it must be tunable to different classes. Over the 30+ years of ASA development, the author has worked with many volunteers who have contributed valuable ideas, modifications and corrections to this code. This has resulted in over 150 ASA options that can be used for additional timing additional tuning making it useful across many classes of problems.

The path integral algorithm includes its mathematical equivalents, a large class of stochastic differential equations and a large class of partial differential equations. The advantages of the path integral algorithm are:

> (a) intuitive description in terms of classical forces, inertia, momentum, etc., leading to new indicators.
>
> (b) delivering a cost function derived from a Lagrangian, or its Action (Lagrangian x dt). Sometimes constraints need to be added as Lagrange multipliers, as was required for normalization requirements in financial risk projects (Ingber, 2010).

4. Pilot Study

The shape of the spread of this virus is clearly nonlinear. A simple model was used for a pilot study to at least capture some nonlinearity. For example, just using the daily number of total cases reported, C, the short-time conditional Probability $P(t+1|t)$ is given in terms of its effective Lagrangian L, $P \propto \exp(-L_{eff}\, dt)$ (including the logarithm of the prefactor normalization as it may contain nonlinearities as modeled here):

$$L_{eff} = [(x_{t+1} - x_t - g_x dt) g_{xx} (x_{t+1} - x_t - g_x dt) - 1/2 \log(2\pi dt g^2)$$

$$g_x = a \exp(x^b)$$

$$g_{xx} = c \exp(x^d)$$

$$g = \det(g_{xx}) \qquad (1)$$

with parameters to be fit to data $\{a, b, c, d\}$. This is a simple one-factor model. In more than one dimension, g_{xx} is the metric of this space, the inverse of the covariance matrix.

For the full data set, 100,000 generated-state iterations of this cost/objective function’s states over the JHU data gave

$$a \quad 0.077\,,\ b \quad 0.874\,,\ c \quad 2.79\,,\ d \quad 0.845 \tag{2}$$

4.1. Comet Profile

These codes were run on XSEDE Comet, for 100000 generated states.

"Comet is a dedicated XSEDE cluster designed by Dell and SDSC delivering 2.0 petaflops, featuring Intel next-gen processors with AVX2, Mellanox FDR InfiniBand interconnects and Aeon storage. The standard compute nodes consist of Intel Xeon E5-2680v3 (formerly codenamed Haswell) processors, 128 GB DDR4 DRAM (64 GB per socket), and 320 GB of SSD local scratch memory. The GPU nodes contain four NVIDIA GPUs each. The large memory nodes contain 1.5 TB of DRAM and four Haswell processors each. The network topology is 56 Gbps FDR InfiniBand with rack-level full bisection bandwidth and 4:1 oversubscription cross-rack bandwidth. Comet has 7 petabytes of 200 GB/second performance storage and 6 petabytes of 100 GB/second durable storage. It also has dedicated gateway hosting nodes and a Virtual Machine repository. External connectivity to Internet2 and ESNet is 100 Gbps."

Comet is being phased out and users will soon be using the new Expanse platform.

4.2. Parallel Processing

"Parallel Processing for this project basically is similar to many projects developed by the author as Principal Investigator at the Extreme Science and Engineering Discovery Environment (XSEDE.org) since February 2013. That is "trivial MPI" is used, wherein many simultaneous runs are achieved by simply reading in different data files to ASA, using the "array" feature offered by some XSEDE platforms. As offered in a previous XSEDE Extended Collaborative Support Service (ECSS) ticket:

> Parallelization efficiency is 1 for jobs running on a single core that is max one could get. For multi-threaded apps one can get some to decent bump in speed using multiple cores up to some point before plateauing. However, speed bump with multiple cores often leads drop in parallelization efficiency.
>
> Drawback of using single core is too long run time. Though in this case, you are running array jobs with single core and getting maximum efficiency. This is the ideal situation on 'Comet' because nodes on this machine can be shared. You should explain on Scaling and parallelization efficiency section that your application is not multi-threaded and you use single core on comet to run your jobs. This gives efficiency of 1, which is maximum value achievable. However, you run array of jobs in one submission and each job uses a single core. This is most efficient use of resources because node sharing is allowed on Comet. It won't hurt to write that you have consulted XSEDE staff on this matter."

4.3. Xeon Processor

The full US run was done on the author's P1 Gen 3 Thinkpad with a Xeon processor. Previous runs show full agreement between the Comet and the Thinkpad runs when "-ffloat-store" is added to the compile parameters. A full US run of 100,000 generated states with 3239 non-zero locations took 1 hr 47 min 17 sec. (All runs including subsets of the full US therefore took about twice that long.)

5. All Results

All locations were processed to exclude those with all "0" for all days, 99 of them.

Note that a few locations, those with just sub-location as it turned out, gave parameter values that hit boundaries of assigned parameter maximums or minimums. Since these were few exceptions, the decision was made to keep the default ranges given in Table 1.

Table 1

Par	Min	Max
0	-2	2

1	-2	2
2	0.1	2
3	-2	2

Final Results for all 58 Locations are given in Table 2.

Table 2

RUNS_COVID/asa_usr_out_01-Alabama
final cost value = 0.0006165903

parameter	value
0	0.07526909
1	0.7867917
2	0.1
3	1.036661

RUNS_COVID/asa_usr_out_02-Alaska
final cost value = 0.0008660421

parameter	value
0	0.03041555
1	0.9221085
2	0.1
3	0.9276368

RUNS_COVID/asa_usr_out_03-Arizona
final cost value = 0.003912767

parameter	value
0	0.08377208
1	0.818453
2	0.1
3	1.287453

RUNS_COVID/asa_usr_out_04-Arkansas
final cost value = 0.0004816542

parameter	value
0	0.07941101
1	0.7750893
2	0.1
3	1.183597

RUNS_COVID/asa_usr_out_05-California
final cost value = 0.0009490655

parameter	value
0	0.06696538
1	0.8527078
2	0.1
3	1.292374

RUNS_COVID/asa_usr_out_06-Colorado
final cost value = 0.000503892

parameter	value
0	0.02576714
1	0.875757
2	0.1076587
3	1.033743

RUNS_COVID/asa_usr_out_07-Connecticut
final cost value = 0.006819112

parameter	value
0	0.03883795
1	0.7877583
2	0.1499112
3	1.133882

RUNS_COVID/asa_usr_out_08-Delaware
final cost value = 0.004949477

parameter	value
0	0.1227538
1	0.6899159
2	0.261152
3	0.9695861

RUNS_COVID/asa_usr_out_09-Diamond_Princess
final cost value = -0.05391078

parameter	value
0	-4.98784e-07
1	-1.992295
2	0.1
3	-2

RUNS_COVID/asa_usr_out_10-District_of_Columbia
final cost value = 0.06929941

parameter	value
0	2
1	0.4060976
2	2
3	0.7794162

RUNS_COVID/asa_usr_out_11-Florida
final cost value = 0.0008101027

parameter	value
0	0.0844608
1	0.8210241
2	0.1
3	1.270596

RUNS_COVID/asa_usr_out_12-Georgia
final cost value = 0.0002643592

parameter	value
0	0.04424673
1	0.8548552
2	0.1
3	1.162738

RUNS_COVID/asa_usr_out_13-Grand_Princess
final cost value = -0.063622

parameter	value
0	-6.538724e-08
1	-1.806268
2	0.1

3 -2

RUNS_COVID/asa_usr_out_14-Guam
final cost value = 0.0465182
parameter value
0 0.008877227
1 1.154289
2 0.1
3 1.147461

RUNS_COVID/asa_usr_out_15-Hawaii
final cost value = 0.006862005
parameter value
0 0.01611866
1 1.050401
2 0.1
3 1.102763

RUNS_COVID/asa_usr_out_16-Idaho
final cost value = 0.0007488098
parameter value
0 0.05115676
1 0.8504985
2 0.1
3 1.084733

RUNS_COVID/asa_usr_out_17-Illinois
final cost value = 0.0004481785
parameter value
0 0.06157631
1 0.8171975
2 0.8193241
3 1.021197

RUNS_COVID/asa_usr_out_18-Indiana
final cost value = 0.0003787652
parameter value
0 0.0412226
1 0.8332504
2 0.1
3 0.9823153

RUNS_COVID/asa_usr_out_19-Iowa
final cost value = 0.0003525547
parameter value
0 0.07068677
1 0.7683947
2 0.1387974
3 1.049687

RUNS_COVID/asa_usr_out_20-Kansas
final cost value = 0.0002747757
parameter value
0 0.0456688

1 0.8592813
2 0.1
3 1.161988

RUNS_COVID/asa_usr_out_21-Kentucky
final cost value = 0.0002246308
parameter value
0 0.03505446
1 0.8823249
2 0.1
3 0.9808715

RUNS_COVID/asa_usr_out_22-Louisiana
final cost value = 0.0008015797
parameter value
0 0.1070208
1 0.7072564
2 2
3 0.7402889

RUNS_COVID/asa_usr_out_23-Maine
final cost value = 0.001441506
parameter value
0 0.03198315
1 0.7940144
2 0.1823495
3 0.6823531

RUNS_COVID/asa_usr_out_24-Maryland
final cost value = 0.002061062
parameter value
0 0.0638636
1 0.7898237
2 0.1
3 1.089192

RUNS_COVID/asa_usr_out_25-Massachusetts
final cost value = 0.004352416
parameter value
0 0.06403747
1 0.7364045
2 0.1
3 1.128749

RUNS_COVID/asa_usr_out_26-Michigan
final cost value = 0.0004323011
parameter value
0 0.04372185
1 0.7974153
2 0.311704
3 0.8720471

RUNS_COVID/asa_usr_out_27-Minnesota
final cost value = 0.0004295167

parameter	value
0	0.06178572
1	0.8006544
2	0.1
3	1.253828

RUNS_COVID/asa_usr_out_28-Mississippi
final cost value = 0.000463057

parameter	value
0	0.1097083
1	0.6931913
2	0.1054405
3	0.9913985

RUNS_COVID/asa_usr_out_29-Missouri
final cost value = 0.000257466

parameter	value
0	0.05215969
1	0.8596338
2	0.1
3	1.055173

RUNS_COVID/asa_usr_out_30-Montana
final cost value = 0.0004428111

parameter	value
0	0.03814208
1	0.899361
2	0.1
3	0.9560651

RUNS_COVID/asa_usr_out_31-Nebraska
final cost value = 0.0003267622

parameter	value
0	0.04517647
1	0.8218373
2	0.1
3	1.145402

RUNS_COVID/asa_usr_out_32-Nevada
final cost value = 0.001893444

parameter	value
0	0.03241173
1	0.9219539
2	0.1
3	1.156847

RUNS_COVID/asa_usr_out_33-New_Hampshire
final cost value = 0.003540204

parameter	value
0	0.05990541
1	0.713824
2	1.999386
3	0.5164278

RUNS_COVID/asa_usr_out_34-New_Jersey
final cost value = 0.003764219

parameter	value
0	2
1	0.3257865
2	2
3	1.048204

RUNS_COVID/asa_usr_out_35-New_Mexico
final cost value = 0.001152665

parameter	value
0	0.1004785
1	0.695894
2	0.6817652
3	0.7827343

RUNS_COVID/asa_usr_out_36-New_York
final cost value = 0.0007147068

parameter	value
0	0.04110297
1	0.7541359
2	0.1
3	1.054681

RUNS_COVID/asa_usr_out_37-North_Carolina
final cost value = 0.0003851502

parameter	value
0	0.08513204
1	0.7664615
2	0.1
3	1.003618

RUNS_COVID/asa_usr_out_38-North_Dakota
final cost value = 0.0003929314

parameter	value
0	0.04932907
1	0.8614704
2	0.1
3	0.9475553

RUNS_COVID/asa_usr_out_39-Northern_Mariana_Islands
final cost value = 0.0110592

parameter	value
0	0.03284899
1	0.6745235
2	0.1
3	0.3440228

RUNS_COVID/asa_usr_out_40-Ohio
final cost value = 0.0004090411

parameter	value
0	0.04184926
1	0.8463094
2	0.1

3	1.049081

RUNS_COVID/asa_usr_out_41-Oklahoma
final cost value = 0.0004630219

parameter	value
0	0.04501715
1	0.8799497
2	0.3494903
3	0.9048175

RUNS_COVID/asa_usr_out_42-Oregon
final cost value = 0.0009208029

parameter	value
0	0.05225226
1	0.816799
2	0.2053155
3	0.9100787

RUNS_COVID/asa_usr_out_43-Pennsylvania
final cost value = 0.0005589026

parameter	value
0	0.04241694
1	0.8052484
2	0.1
3	1.015383

RUNS_COVID/asa_usr_out_44-Puerto_Rico
final cost value = 0.000312391

parameter	value
0	0.03449601
1	0.9045291
2	0.1
3	1.088644

RUNS_COVID/asa_usr_out_45-Rhode_Island
final cost value = 0.0111474

parameter	value
0	0.04708741
1	0.7901072
2	0.1
3	1.442058

RUNS_COVID/asa_usr_out_46-South_Carolina
final cost value = 0.0009722008

parameter	value
0	0.09290075
1	0.7718165
2	0.1
3	1.095007

RUNS_COVID/asa_usr_out_47-South_Dakota
final cost value = 0.0003353859

parameter	value
0	0.05975135

1 0.7782754
2 0.1
3 0.9210539

RUNS_COVID/asa_usr_out_48-Tennessee
final cost value = 0.0005178384
parameter value
0 0.09073933
1 0.7754924
2 2
3 0.8461525

RUNS_COVID/asa_usr_out_49-Texas
final cost value = 0.0001681769
parameter value
0 0.05172033
1 0.8703259
2 0.1
3 1.330712

RUNS_COVID/asa_usr_out_50-US
final cost value = 1.27974e-05
parameter value
0 0.05285717
1 0.8271716
2 0.1090954
3 1.204249

RUNS_COVID/asa_usr_out_51-Utah
final cost value = 0.003623466
parameter value
0 0.04933961
1 0.8573352
2 0.1
3 1.086935

RUNS_COVID/asa_usr_out_52-Vermont
final cost value = 0.0008160128
parameter value
0 0.006796208
1 0.9282152
2 0.1
3 0.4539584

RUNS_COVID/asa_usr_out_53-Virgin_Islands
final cost value = 0.03999473
parameter value
0 0.06337426
1 0.8251611
2 0.1
3 1.064258

RUNS_COVID/asa_usr_out_54-Virginia
final cost value = 0.0001637778

parameter	value
0	0.05090517
1	0.8072254
2	0.1
3	0.9941458

RUNS_COVID/asa_usr_out_55-Washington
final cost value = 0.0005114633

parameter	value
0	0.05293824
1	0.8114288
2	0.1
3	1.026359

RUNS_COVID/asa_usr_out_56-West_Virginia
final cost value = 0.0004020269

parameter	value
0	0.02179989
1	0.9542683
2	0.1
3	0.8805843

RUNS_COVID/asa_usr_out_57-Wisconsin
final cost value = 0.0005233912

parameter	value
0	0.05836374
1	0.8373115
2	0.1
3	1.251952

RUNS_COVID/asa_usr_out_58-Wyoming
final cost value = 0.0008048018

parameter	value
0	0.05755666
1	0.7254984
2	0.1
3	0.8178418

6. Conclusion

Two algorithms are suggested for fitting data and forecasting COVID-19, ASA for importance-sampling and fitting parameters to models, and PATHINT/PATHTREE. These algorithms have been applied to several disciplines — neuroscience, financial markets, combat analysis. While optimization and path-integral algorithms are now quite well-known (at least to many scientists), these previous applications give strong support to application of these tools to COVID-19 data.

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