ON THE HERMITE-HADAMARD-MERCER TYPE INEQUALITIES FOR GENERALIZED PROPORTIONAL FRACTIONAL INTEGRALS

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ABSTRACT. Our aim in this paper is to establish some new Hermite-Hadamard-Mercer type integral inequalities by utilizing the fractional proportional-integral operators. For this purpose, Hermite-Hadamard-Mercer inequalities for differentiable mappings whose derivatives in absolute value are convex.

1. INTRODUCTION AND PRELIMINARES

Over the past few years, various researchers studied the so-called conformable integrals and derivatives. According to this idea, some authors used modified proportional derivatives to create nonlocal fractional integrals and derivatives, called fractional proportional integrals and derivatives, including see exponential functions in their kernels ([5], [7], [31], [32], [33]). Our purpose here new Hermite-Hadamard-Mercer integral inequalities in the article some convex functions using fractional proportional integral operators.

Let $0 < x_1 \leq x_2 \leq \dots \leq x_n$, $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$ nonnegative and $\sum_{j=1}^n \lambda_j = 1$. The Jensen inequality [13] in literature states that if f is a convex function then,

$$f\left(\sum_{j=1}^n \lambda_j x_j\right) \leq \sum_{j=1}^n \lambda_j f(x_j).$$

The $f : J \subset \mathbb{R} \rightarrow \mathbb{R}$ be a convex function defined on an interval J of real numbers $\theta, \beta \in J$ and $\theta < \beta$, if the following inequalities

$$(1.1) \quad f\left(\frac{\theta+\beta}{2}\right) \leq \frac{1}{\beta-\theta} \int_{\theta}^{\beta} f(x) dx \leq \frac{f(\theta)+f(\beta)}{2},$$

holds, it is called Hermite-Hadamard type inequality [5].

Theorem 1. [21] *Let f is a convex function on $[\theta, \beta]$, then*

$$(1.2) \quad f\left(\theta + \beta - \sum_{j=1}^n \lambda_j x_j\right) \leq f(\theta) + f(\beta) - \sum_{j=1}^n \lambda_j f(x_j)$$

for each $x_j \in [\theta, \beta]$ and $\lambda_j \in [0, 1]$ ($j = 1, 2, \dots, n$) with $\sum_{j=1}^n \lambda_j = 1$.

Jensen-Mercer inequality, see ([1], [18], [20], [23]).

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Definition 1. [7] Let $f \in L[\theta, \beta]$. The Generalized proportional integral operators $J_{\theta+}^{\beta, \xi} f$ and $J_{\beta-}^{\beta, \xi} f$ of order β are defined by

$$(1.3) \quad \left(J_{\theta+}^{\beta, \xi} f\right)(x) = \frac{1}{\xi^{\beta} \Gamma(\beta)} \int_{\theta}^x \exp\left[\frac{\xi-1}{\xi}(x-\tau)\right] (x-\tau)^{\beta-1} f(\tau) d\tau, \quad x > \theta$$

and

$$(1.4) \quad \left(J_{\beta-}^{\beta, \xi} f\right)(x) = \frac{1}{\xi^{\beta} \Gamma(\beta)} \int_x^{\beta} \exp\left[\frac{\xi-1}{\xi}(\tau-x)\right] (\tau-x)^{\beta-1} f(\tau) d\tau, \quad \beta > x.$$

where $\xi \in (0, 1]$ and $\beta \in \mathbb{C}$ and $\Re(\beta) > 0$ and Γ is known gamma function. For more details may consult see ([7], [29], [32].)

If we consider $\xi = 1$ in (1.3) and (1.4) in classical Riemann-Liouville fractional integrals are obtained.

2. MAIN RESULTS

Using the Jensen-Mercer inequality, Hermite Hadamard inequalities can be represented in generalized proportional fractional integral forms as follows.

Theorem 2. Let f be a positive continuous, decreasing and convex function on the interval $[\theta, \beta]$, then the following inequality for, generalized proportional fractional integrals holds;

$$(2.1) \quad \begin{aligned} & \frac{1}{\xi^{\sigma}} \left[\frac{1}{\sigma} + \sum_{n=1}^{\infty} \left(\frac{\xi-1}{\xi} \right)^n \frac{1}{\sigma+n} \right] f\left(\theta + \beta - \frac{x+y}{2}\right) \\ & \leq \frac{1}{\xi^{\sigma}} \left[\frac{1}{\sigma} + \sum_{n=1}^{\infty} \left(\frac{\xi-1}{\xi} \right)^n \frac{1}{\sigma+n} \right] [f(\theta) + f(\beta)] \\ & \quad - \frac{\Gamma(\sigma)}{2(y-x)^{\sigma}} \left[J_{x+}^{\sigma, \xi} f(y) + J_{y-}^{\sigma, \xi} f(x) \right] \\ & \leq \frac{1}{\xi^{\sigma}} \left[\frac{1}{\sigma} + \sum_{n=1}^{\infty} \left(\frac{\xi-1}{\xi} \right)^n \frac{1}{\sigma+n} \right] [f(\theta) + f(\beta) - f\left(\frac{x+y}{2}\right)] \end{aligned}$$

and

$$(2.2) \quad \begin{aligned} & \frac{1}{\xi^{\sigma}} \left[\frac{1}{\sigma} + \sum_{n=1}^{\infty} \left(\frac{\xi-1}{\xi} \right)^n \frac{1}{\sigma+n} \right] f\left(\theta + \beta - \frac{x+y}{2}\right) \\ & \leq \frac{\Gamma(\sigma)}{2(y-x)^{\sigma}} \left[J_{(\theta+\beta-y)+}^{\sigma, \xi} f(\theta + \beta - x) + J_{(\theta+\beta-x)-}^{\sigma, \xi} f(\theta + \beta - y) \right] \\ & \leq \frac{1}{\xi^{\sigma}} \left[\frac{1}{\sigma} + \sum_{n=1}^{\infty} \left(\frac{\xi-1}{\xi} \right)^n \frac{1}{\sigma+n} \right] \left[\frac{f(\theta+\beta-x) + f(\theta+\beta-y)}{2} \right] \\ & \leq \frac{1}{\xi^{\sigma}} \left[\frac{1}{\sigma} + \sum_{n=1}^{\infty} \left(\frac{\xi-1}{\xi} \right)^n \frac{1}{\sigma+n} \right] \left[f(\theta) + f(\beta) - \frac{f(x) + f(y)}{2} \right] \end{aligned}$$

Then where $\xi \in (0, 1]$ for all $x, y \in [\theta, \beta]$, $\sigma \in \mathbb{C}$ and $\Re(\sigma) > 0$.

Proof. Using the Jensen -Mercer inequality, we have

$$(2.3) \quad f\left(\theta + \beta - \frac{x_1 + y_1}{2}\right) \leq f(\theta) + f(\beta) - \frac{f(x_1) + f(y_1)}{2}$$

for all $x_1, y_1 \in [\theta, \beta]$. By changing of the variable $x_1 = \tau x + (1-\tau)y$ and $y_1 = (1-\tau)x + \tau y$, $x, y \in [\theta, \beta]$ and $\tau \in [0, 1]$ in (2.3), we obtain

$$(2.4) \quad f\left(\theta + \beta - \frac{x+y}{2}\right) \leq f(\theta) + f(\beta) - \frac{f(\tau x + (1-\tau)y) + f((1-\tau)x + \tau y)}{2}$$

Multiplying both sides of (2.4) by $\frac{\tau^{\sigma-1}e^{\frac{\xi-1}{\xi}\tau}}{\xi^\sigma}$ and integrate with respect to τ over $[0, 1]$, we get

$$(2.5) \quad \int_0^1 \frac{\tau^{\sigma-1}e^{\frac{\xi-1}{\xi}\tau}}{\xi^\sigma} f\left(\theta + \beta - \frac{x+y}{2}\right) d\tau \leq [f(\theta) + f(\beta)] \int_0^1 \frac{\tau^{\sigma-1}e^{\frac{\xi-1}{\xi}\tau}}{\xi^\sigma} d\tau \\ - \int_0^1 \frac{\tau^{\sigma-1}e^{\frac{\xi-1}{\xi}\tau}}{\xi^\sigma} \left[\frac{f(\tau x + (1-\tau)y) + f((1-\tau)x + \tau y)}{2} \right] d\tau$$

$$(2.6) \quad \frac{1}{\xi^\sigma} \left[\frac{1}{\sigma} + \sum_{n=1}^{\infty} \left(\frac{\xi-1}{\xi} \right)^n \frac{1}{\sigma+n} \right] f\left(\theta + \beta - \frac{x+y}{2}\right) \\ \leq \left[\frac{1}{\sigma} + \sum_{n=1}^{\infty} \left(\frac{\xi-1}{\xi} \right)^n \frac{1}{\sigma+n} \right] [f(\theta) + f(\beta)] \\ - \frac{1}{2} \int_0^1 \frac{\tau^{\sigma-1}e^{\frac{\xi-1}{\xi}\tau}}{\xi^\sigma} f(\tau x + (1-\tau)y) d\tau \\ - \frac{1}{2} \int_0^1 \frac{\tau^{\sigma-1}e^{\frac{\xi-1}{\xi}\tau}}{\xi^\sigma} f((1-\tau)x + \tau y) d\tau$$

i.e

$$(2.7) \quad \frac{1}{\xi^\sigma} \left[\frac{1}{\sigma} + \sum_{n=1}^{\infty} \left(\frac{\xi-1}{\xi} \right)^n \frac{1}{\sigma+n} \right] f\left(\theta + \beta - \frac{x+y}{2}\right) \\ \leq \left[\frac{1}{\sigma} + \sum_{n=1}^{\infty} \left(\frac{\xi-1}{\xi} \right)^n \frac{1}{\sigma+n} \right] [f(\theta) + f(\beta)] \\ - \frac{\Gamma(\sigma)}{2(y-x)^\sigma} \left[J_{x^+}^{\sigma,\xi} f(y) + J_{y^-}^{\sigma,\xi} f(x) \right].$$

and so the first inequality of (2.1) proved. To be able to prove the second inequality in (2.1), first we have to pay attention to that if f is a convex function, in case that, for $\tau \in [0, 1]$, it gives

$$(2.8) \quad f\left(\frac{x+y}{2}\right) = f\left(\frac{\tau x + (1-\tau)y + (1-\tau)x + \tau y}{2}\right) \\ \leq \frac{f(\tau x + (1-\tau)y) + f((1-\tau)x + \tau y)}{2}$$

Multiplying both sides of (2.8) by $\frac{\tau^{\sigma-1}e^{\frac{\xi-1}{\xi}\tau}}{\xi^\sigma}$ and integrate with respect to τ over $[0, 1]$, we have

$$(2.9) \quad \frac{1}{\xi^\sigma} \left[\frac{1}{\sigma} + \sum_{n=1}^{\infty} \left(\frac{\xi-1}{\xi} \right)^n \frac{1}{\sigma+n} \right] f\left(\frac{x+y}{2}\right) \\ \leq \frac{1}{2} \int_0^1 \frac{\tau^{\sigma-1}e^{\frac{\xi-1}{\xi}\tau}}{\xi^\sigma} f(\tau x + (1-\tau)y) d\tau \\ + \frac{1}{2} \int_0^1 \frac{\tau^{\sigma-1}e^{\frac{\xi-1}{\xi}\tau}}{\xi^\sigma} f((1-\tau)x + \tau y) d\tau \\ = \frac{\Gamma(\sigma)}{2(y-x)^\sigma} \left[J_{x^+}^{\sigma,\xi} f(y) + J_{y^-}^{\sigma,\xi} f(x) \right].$$

Adding $[f(\theta) + f(\beta)] \int_0^1 \frac{\tau^{\sigma-1}e^{\frac{\xi-1}{\xi}\tau}}{\xi^\sigma}$ to both sides of (2.7) we find the second inequality of (2.1).

now we prove the inequality (2.2). From the convexity of f we have

$$(2.10) \quad f\left(\theta + \beta - \frac{x_1+y_1}{2}\right) = f\left(\frac{\theta+\beta-x_1+\theta+\beta-y_1}{2}\right) \\ \leq \frac{1}{2} [f(\theta + \beta - x_1) + f(\theta + \beta - y_1)]$$

for all $x_1, y_1 \in [\theta, \beta]$. With variable replacement $\theta + \beta - x_1 = \tau(\theta + \beta - x) + (1 - \tau)(\theta + \beta - y)$ and $\theta + \beta - y_1 = (1 - \tau)(\theta + \beta - x) + \tau(\theta + \beta - y)$, $x, y \in [\theta, \beta]$ and $\tau \in [0, 1]$ in (2.10), we obtain

$$(2.11) \quad \begin{aligned} & f\left(\theta + \beta - \frac{x+y}{2}\right) \\ & \leq \frac{1}{2} [f(\tau(\theta + \beta - x) + (1 - \tau)(\theta + \beta - y)) + f((1 - \tau)(\theta + \beta - x) + \tau(\theta + \beta - y))] . \end{aligned}$$

Multiplying both sides of (2.11) by $\frac{\tau^{\sigma-1} e^{\frac{\xi-1}{\xi}\tau}}{\xi^\sigma}$ and integrate with respect to τ over $[0, 1]$, we have,

$$(2.12) \quad \begin{aligned} & \frac{1}{\xi^\sigma} \left[\frac{1}{\sigma} + \sum_{n=1}^{\infty} \left(\frac{\xi-1}{\xi} \right)^n \frac{1}{\sigma+n} \right] f\left(\theta + \beta - \frac{x+y}{2}\right) \\ & \leq \frac{1}{2} \int_0^1 \frac{\tau^{\sigma-1} e^{\frac{\xi-1}{\xi}\tau}}{\xi^\sigma} f(\tau(\theta + \beta - x) + (1 - \tau)(\theta + \beta - y)) d\tau \\ & \quad + \frac{1}{2} \int_0^1 \frac{\tau^{\sigma-1} e^{\frac{\xi-1}{\xi}\tau}}{\xi^\sigma} f((1 - \tau)(\theta + \beta - x) + \tau(\theta + \beta - y)) d\tau \\ & = \frac{1}{2\xi^\sigma} \left[\int_{\theta+\beta-y}^{\theta+\beta-x} e^{\frac{\xi-1}{\xi}(u - \frac{\theta+\beta-y}{y-x})} \left(u - \frac{\theta+\beta-y}{y-x}\right)^{\sigma-1} f(u) du \right] \\ & \quad + \frac{1}{2\xi^\sigma} \left[\int_{\theta+\beta-y}^{\theta+\beta-x} e^{\frac{\xi-1}{\xi}(u - \frac{\theta+\beta-x}{y-x})} \left(u - \frac{\theta+\beta-x}{y-x}\right)^{\sigma-1} f(u) du \right] \\ & = \frac{\Gamma(\sigma)}{2(y-x)^\sigma} \left[J_{(\theta+\beta-y)^+}^{\sigma, \xi} f(\theta + \beta - x) + J_{(\theta+\beta-x)^-}^{\sigma, \xi} f(\theta + \beta - y) \right] \end{aligned}$$

The proof of first inequality of (2.2) is completed. On the other hand, using the convexity of f we can write

$$(2.13) \quad f(\tau(\theta + \beta - x) + (1 - \tau)(\theta + \beta - y)) \leq \tau f((\theta + \beta - x)) + (1 - \tau) f(\theta + \beta - y)$$

and

$$(2.14) \quad f((1 - \tau)(\theta + \beta - x) + \tau(\theta + \beta - y)) \leq (1 - \tau) f((\theta + \beta - x)) + \tau f(\theta + \beta - y)$$

by adding these inequalities and using the Jensen-Mercer inequality, we have

$$(2.15) \quad \begin{aligned} & f(\tau(\theta + \beta - x) + (1 - \tau)(\theta + \beta - y)) + f((1 - \tau)(\theta + \beta - x) + \tau(\theta + \beta - y)) \\ & \leq f((\theta + \beta - x)) + f(\theta + \beta - y) \\ & \leq 2[f(\theta) + f(\beta)] - [f(x) + f(y)] . \end{aligned}$$

Multiplying both sides of (2.15) by $\frac{\tau^{\sigma-1} e^{\frac{\xi-1}{\xi}\tau}}{\xi^\sigma}$ and then integrating the resulting inequality with respect to τ over $[0, 1]$, we have second and third inequalities of (2.2). $\square$

Remark 1. If we consider $\xi = 1$, $\sigma = 1$ in Theorem, the following inequality is obtain

$$\begin{aligned} f\left(\theta + \beta - \frac{x+y}{2}\right) & \leq f(\theta) + f(\beta) - \int_0^1 f(\tau x + (1 - \tau)y) d\tau \\ & \leq f(\theta) + f(\beta) - f\left(\frac{x+y}{2}\right) \end{aligned}$$

and

$$\begin{aligned} f\left(\theta + \beta - \frac{x+y}{2}\right) & \leq \frac{1}{y-x} \int_x^y f(\theta + \beta - \tau) d\tau \\ & \leq f(\theta) + f(\beta) - \frac{f(x) + f(y)}{2} \end{aligned}$$

which is proved by Kian and Moslehian in [18].

Theorem 3. Let f be a positive continuous, decreasing and convex function on the interval $[\theta, \beta]$ then the following inequality for, generalized proportional fractional integrals holds;

$$\begin{aligned}
 & \frac{2}{\xi^\sigma} \left[\frac{1}{\sigma} + \sum_{n=1}^{\infty} \left(\frac{\xi-1}{\xi} \right)^n \frac{1}{\sigma+n} \right] f \left(\theta + \beta - \frac{x+y}{2} \right) \\
 (2.16) \quad & \leq \frac{2^\sigma \Gamma(\sigma)}{(y-x)^\sigma} \left[J_{(\theta+\beta-\frac{x+y}{2})-}^{\sigma, \xi} f(\theta + \beta - y) + J_{(\theta+\beta-\frac{x+y}{2})+}^{\sigma, \xi} f(\theta + \beta - x) \right] \\
 & \leq \frac{2}{\xi^\sigma} \left[\frac{1}{\sigma} + \sum_{n=1}^{\infty} \left(\frac{\xi-1}{\xi} \right)^n \frac{1}{\sigma+n} \right] \left[[f(\theta) + f(\beta)] - \frac{f(x)+f(y)}{2} \right].
 \end{aligned}$$

Then where $\xi \in (0, 1]$ for all $x, y \in [\theta, \beta]$, $\sigma \in \mathbb{C}$ and $\Re(\sigma) > 0$.

Proof. To prove the first (2.16), by writting $x_1 = \frac{\tau}{2}x + \frac{2-\tau}{2}y$ and $y_1 = \frac{2-\tau}{2}x + \frac{\tau}{2}y$ for $x, y \in [\theta, \beta]$ and $\tau \in [0, 1]$ in the inequality (2.10) we get

$$(2.17) \quad \leq \left[f \left(\theta + \beta - \left(\frac{\tau}{2}x + \frac{2-\tau}{2}y \right) \right) + f \left(\theta + \beta - \left(\frac{2-\tau}{2}x + \frac{\tau}{2}y \right) \right) \right]$$

and then, Multiplying both sides of (2.17) by $\frac{\tau^{\sigma-1} e^{\frac{\xi-1}{\xi}\tau}}{\xi^\sigma}$ and integrate with respect to τ over $[0, 1]$, we get

$$\begin{aligned}
 & \frac{2}{\xi^\sigma} \left[\frac{1}{\sigma} + \sum_{n=1}^{\infty} \left(\frac{\xi-1}{\xi} \right)^n \frac{1}{\sigma+n} \right] f \left(\theta + \beta - \frac{x+y}{2} \right) \\
 & \leq \int_0^1 \frac{\tau^{\sigma-1} e^{\frac{\xi-1}{\xi}\tau}}{\xi^\sigma} f \left(\theta + \beta - \left(\frac{\tau}{2}x + \frac{2-\tau}{2}y \right) \right) d\tau \\
 & + \int_0^1 \frac{\tau^{\sigma-1} e^{\frac{\xi-1}{\xi}\tau}}{\xi^\sigma} f \left(\theta + \beta - \left(\frac{2-\tau}{2}x + \frac{\tau}{2}y \right) \right) d\tau \\
 (2.18) \quad & = \int_{\theta+\beta-\frac{x+y}{2}}^{\theta+\beta-y} \left(\frac{2(u-(\theta+\beta-y))}{y-x} \right)^{\sigma-1} e^{\frac{\xi-1}{\xi} \frac{2(u-(\theta+\beta-y))}{y-x}} f(u) \frac{2du}{y-x} \\
 & + \int_{\theta+\beta-x}^{\theta+\beta-\frac{x+y}{2}} \left(\frac{2((\theta+\beta-x)-u)}{y-x} \right)^{\sigma-1} e^{\frac{\xi-1}{\xi} \frac{2((\theta+\beta-x)-u)}{y-x}} f(u) \frac{2du}{y-x} \\
 & = \frac{2^\sigma \Gamma(\sigma)}{(y-x)^\sigma} \left[J_{(\theta+\beta-\frac{x+y}{2})-}^{\sigma, \xi} f(\theta + \beta - y) + J_{(\theta+\beta-\frac{x+y}{2})+}^{\sigma, \xi} f(\theta + \beta - x) \right]
 \end{aligned}$$

the first inequality (2.16) is proved. To be able to prove the second inequality of (2.16) by using Jensen Mercer inequality, we get

$$f \left(\theta + \beta - \left(\frac{\tau}{2}x + \frac{2-\tau}{2}y \right) \right) \leq f(\theta) + f(\beta) - \left[\frac{\tau}{2}f(x) + \frac{2-\tau}{2}f(y) \right]$$

and

$$(2.19) \quad f \left(\theta + \beta - \left(\frac{2-\tau}{2}x + \frac{\tau}{2}y \right) \right) \leq f(\theta) + f(\beta) - \left[\frac{2-\tau}{2}f(x) + \frac{\tau}{2}f(y) \right]$$

by adding these inequalities, we have

$$\begin{aligned}
 (2.20) \quad & f \left(\theta + \beta - \left(\frac{\tau}{2}x + \frac{2-\tau}{2}y \right) \right) + f \left(\theta + \beta - \left(\frac{2-\tau}{2}x + \frac{\tau}{2}y \right) \right) \\
 & \leq 2[f(\theta) + f(\beta)] - \frac{f(x)+f(y)}{2}
 \end{aligned}$$

Multiplying both sides of (2.21) by $\frac{\tau^{\sigma-1} e^{\frac{\xi-1}{\xi}\tau}}{\xi^\sigma}$ and integrate with respect to τ over $[0, 1]$, we find second inequality of (2.16). $\square$

3. HERMITE-HADAMARD MERCER TYPE INEQUALITIES FOR FRACTIONAL INTEGRALS

Now, we give the new following lemmas for our results.

Lemma 1. *Let f be a positive continuous, decreasing and convex function a differentiable mapping on (θ, β) with $\theta < \beta$. If $f' \in L[\theta, \beta]$, then the following equality for, generalized proportional fractional integrals holds;*

$$\begin{aligned}
 (3.1) \quad & \frac{e^{\frac{\xi-1}{\xi}}}{2} [f(\theta + \beta - x) + f(\theta + \beta - y)] \\
 & - \frac{\xi^{\sigma-1}(\xi-1)\Gamma(\sigma)}{2(y-x)^{\sigma+1}} \left[J_{(\theta+\beta-x)^-}^{\sigma-1,\xi} f(\theta + \beta - y) + J_{(\theta+\beta-y)^+}^{\sigma-1,\xi} f(\theta + \beta - x) \right] \\
 & - \frac{\xi^{\sigma}\Gamma(\sigma)}{2(y-x)^{\sigma}} \left[J_{(\theta+\beta-x)^-}^{\sigma,\xi} f(\theta + \beta - y) + J_{(\theta+\beta-y)^+}^{\sigma,\xi} f(\theta + \beta - x) \right] \\
 & = \frac{y-x}{2} \int_0^1 e^{\frac{\xi-1}{\xi}\tau} \tau^{\sigma} f'(\theta + \beta - (\tau x + (1-\tau)y)) d\tau \\
 & - \frac{y-x}{2} \int_0^1 e^{\frac{\xi-1}{\xi}(1-\tau)} (1-\tau)^{\sigma} f'(\theta + \beta - (\tau x + (1-\tau)y)) d\tau
 \end{aligned}$$

Then where $\tau \in [0, 1]$, $\xi \in (0, 1]$ for all $x, y \in [\theta, \beta]$, $\sigma \in \mathbb{C}$ and $\Re(\sigma) > 0$.

Proof. It is necessary to note,

$$\begin{aligned}
 (3.2) \quad & \int_0^1 e^{\frac{\xi-1}{\xi}\tau} \tau^{\sigma} f'(\theta + \beta - (\tau x + (1-\tau)y)) d\tau \\
 & - \int_0^1 e^{\frac{\xi-1}{\xi}(1-\tau)} (1-\tau)^{\sigma} f'(\theta + \beta - (\tau x + (1-\tau)y)) d\tau \\
 & = I_1 - I_2
 \end{aligned}$$

integrating by parts, we get

$$\begin{aligned}
 (3.3) \quad & I_1 = \int_0^1 e^{\frac{\xi-1}{\xi}\tau} \tau^{\sigma} f'(\theta + \beta - (\tau x + (1-\tau)y)) d\tau \\
 & = \frac{e^{\frac{\xi-1}{\xi}\tau} \tau^{\sigma} f(\theta + \beta - (\tau x + (1-\tau)y))}{(y-x)} \Big|_0^1 \\
 & - \frac{1}{(y-x)} \int_0^1 e^{\frac{\xi-1}{\xi}\tau} \left(\frac{\xi-1}{\xi} \tau^{\sigma} + \sigma \tau^{\sigma-1} \right) f(\theta + \beta - (\tau x + (1-\tau)y)) d\tau \\
 & = \frac{e^{\frac{\xi-1}{\xi}\tau} \tau^{\sigma} f(\theta + \beta - (\tau x + (1-\tau)y))}{(y-x)} \Big|_0^1 \\
 & - \frac{(\xi-1)}{\xi(y-x)} \int_0^1 e^{\frac{\xi-1}{\xi}\tau} \tau^{\sigma} f(\theta + \beta - (\tau x + (1-\tau)y)) d\tau \\
 & - \frac{\sigma}{(y-x)} \int_0^1 e^{\frac{\xi-1}{\xi}\tau} \tau^{\sigma-1} f(\theta + \beta - (\tau x + (1-\tau)y)) d\tau \\
 & = \frac{e^{\frac{\xi-1}{\xi}} f(\theta + \beta - x)}{(y-x)} - \frac{\xi^{\sigma-1}(\xi-1)}{\xi^{\sigma}(y-x)} \int_{\theta+\beta-y}^{\theta+\beta-x} e^{\frac{\xi-1}{\xi} \left(\frac{u-(\theta+\beta-y)}{y-x} \right)} \left(\frac{u-(\theta+\beta-y)}{y-x} \right)^{\sigma} f(u) \frac{du}{y-x} \\
 & - \frac{\xi^{\sigma}\sigma}{\xi^{\sigma}(y-x)} \int_{\theta+\beta-y}^{\theta+\beta-x} e^{\frac{\xi-1}{\xi} \left(\frac{u-(\theta+\beta-y)}{y-x} \right)} \left(\frac{u-(\theta+\beta-y)}{y-x} \right)^{\sigma-1} f(u) \frac{du}{y-x} \\
 & = \frac{e^{\frac{\xi-1}{\xi}} f(\theta + \beta - x)}{(y-x)} - \frac{\xi^{\sigma-1}(\xi-1)\Gamma(\sigma)}{(y-x)^{\sigma+2}} J_{(\theta+\beta-x)^-}^{\sigma-1,\xi} f(\theta + \beta - y) - \frac{\xi^{\sigma}\sigma\Gamma(\sigma)}{(y-x)^{\sigma+1}} J_{(\theta+\beta-x)^-}^{\sigma,\xi} f(\theta + \beta - y)
 \end{aligned}$$

Similary we get

$$\begin{aligned}
 (3.4) \quad I_2 &= \int_0^1 e^{\frac{\xi-1}{\xi}(1-\tau)} (1-\tau)^\sigma f'(\theta + \beta - (\tau x + (1-\tau)y)) d\tau \\
 &= \frac{e^{\frac{\xi-1}{\xi}(1-\tau)} (1-\tau)^\sigma f(\theta + \beta - (\tau x + (1-\tau)y))}{(y-x)} \Big|_0^1 \\
 &\quad - \frac{1}{(y-x)} \int_0^1 e^{\frac{\xi-1}{\xi}(1-\tau)} \left(\frac{\xi-1}{\xi} (1-\tau)^\sigma + \sigma (1-\tau)^{\sigma-1} \right) f(\theta + \beta - (\tau x + (1-\tau)y)) d\tau \\
 &= \frac{e^{\frac{\xi-1}{\xi}(1-\tau)} (1-\tau)^\sigma f(\theta + \beta - (\tau x + (1-\tau)y))}{(y-x)} \Big|_0^1 \\
 &\quad - \frac{\xi-1}{\xi(y-x)} \int_0^1 e^{\frac{\xi-1}{\xi}(1-\tau)} (1-\tau)^\sigma f(\theta + \beta - (\tau x + (1-\tau)y)) d\tau \\
 &\quad - \frac{\sigma}{(y-x)} \int_0^1 e^{\frac{\xi-1}{\xi}(1-\tau)} (1-\tau)^{\sigma-1} f(\theta + \beta - (\tau x + (1-\tau)y)) d\tau \\
 &= -\frac{e^{\frac{\xi-1}{\xi}} f(\theta + \beta - y)}{(y-x)} + \frac{\xi^{\sigma-1}(\xi-1)}{\xi^\sigma(y-x)} \int_{\theta+\beta-y}^{\theta+\beta-x} e^{\frac{\xi-1}{\xi}(1-\frac{u-(\theta+\beta-y)}{y-x})} \left(1 - \frac{u-(\theta+\beta-y)}{y-x}\right)^\sigma f(u) \frac{du}{y-x} \\
 &\quad + \frac{\sigma \xi^\sigma}{\xi^\sigma(y-x)} \int_{\theta+\beta-y}^{\theta+\beta-x} e^{\frac{\xi-1}{\xi}(1-\frac{u-(\theta+\beta-y)}{y-x})} \left(1 - \frac{u-(\theta+\beta-y)}{y-x}\right)^{\sigma-1} f(u) \frac{du}{y-x} \\
 &= -\frac{e^{\frac{\xi-1}{\xi}} f(\theta + \beta - y)}{(y-x)} + \frac{\xi^{\sigma-1}(\xi-1)\Gamma(\sigma)}{(y-x)^{\sigma+2}} J_{(\theta+\beta-y)^-}^{\sigma-1,\xi} f(\theta + \beta - x) + \frac{\xi^\sigma \sigma \Gamma(\sigma)}{(y-x)^{\sigma+1}} J_{(\theta+\beta-y)^+}^{\sigma,\xi} f(\theta + \beta - x)
 \end{aligned}$$

We can write

$$\begin{aligned}
 (3.5) \quad I &= I_1 - I_2 \\
 &= \frac{e^{\frac{\xi-1}{\xi}}}{\xi^\sigma(y-x)} [f(\theta + \beta - x) + f(\theta + \beta - y)] - \frac{(\xi-1)\Gamma(\sigma)}{\xi^{\sigma+1}(y-x)^{\sigma+2}} \left[J_{(\theta+\beta-x)^-}^{\sigma-1,\xi} f(\theta + \beta - y) + J_{(\theta+\beta-y)^+}^{\sigma-1,\xi} f(\theta + \beta - x) \right] \\
 &\quad - \frac{\sigma \Gamma(\sigma)}{\xi^\sigma(y-x)^{\sigma+1}} \left[J_{(\theta+\beta-x)^-}^{\sigma,\xi} f(\theta + \beta - y) + J_{(\theta+\beta-y)^+}^{\sigma,\xi} f(\theta + \beta - x) \right]
 \end{aligned}$$

Multiplying the both sides by $\frac{y-x}{2}$, we proof is obtained (3.1). $\square$

Corollary 1. If we consider $\xi = 1$, $\sigma = 1$ in Lemma 1, then we have the following equality

$$\begin{aligned}
 &\frac{[f(\theta+\beta-x)+f(\theta+\beta-y)]}{2} - \\
 &\quad - \frac{1}{y-x} \left[\int_{\theta+\beta-y}^{\theta+\beta-x} f(u) du \right] \\
 &= \frac{y-x}{2} \int_0^1 (2\tau-1) f'(\theta + \beta - (\tau x + (1-\tau)y)) d\tau.
 \end{aligned}$$

which is proved by Sarikaya and Ogülmüş in [27].

Remark 2. If we consider $\xi = 1$, $\sigma = 1$, $x = \theta$, $y = \beta$ in Lemma 1, in that case the following equality we get,

$$\begin{aligned}
 (3.6) \quad &\frac{f(\theta)+f(\beta)}{2} - \\
 &\quad - \frac{1}{\beta-\theta} \left[\int_{\theta}^{\beta} f(u) du \right] \\
 &= \frac{\beta-\theta}{2} \int_0^1 (2\tau-1) f'(\theta + \beta - (\tau\theta + (1-\tau)\beta)) d\tau.
 \end{aligned}$$

which is proved by Dragomir and Agarwal in [9].

Remark 3. If we consider $\xi = 1$, in Lemma 1, in that case the following equality we get,

$$\begin{aligned} & f(\theta + \beta - x) + f(\theta + \beta - y) \\ & - \frac{\sigma \Gamma(\sigma)}{2(y-x)^\sigma} \left[J_{(\theta+\beta-x)^-}^\sigma f(\theta + \beta - y) + J_{(\theta+\beta-y)^+}^\sigma f(\theta + \beta - x) \right] \\ & = \frac{y-x}{2} \int_0^1 \tau^\sigma f'(\theta + \beta - (\tau x + (1-\tau)y)) d\tau \\ & - \frac{y-x}{2} \int_0^1 (1-\tau)^\sigma f'(\theta + \beta - (\tau x + (1-\tau)y)) d\tau \end{aligned}$$

which is proved by Sarikaya and Ogülmüş in [27].

Lemma 2. Let f be a positive continuous, decreasing and convex function a differentiable mapping on (θ, β) with $\theta < \beta$. If $f' \in L[\theta, \beta]$, then the following equality for fractional integrals holds;

$$\begin{aligned} (3.7) \quad & \frac{\xi^{\sigma-1} 2^\sigma (\xi-1)}{(y-x)^\sigma} \left[J_{(\theta+\beta-\frac{x+y}{2})^+}^{\sigma-1, \xi} f(\theta + \beta - x) + J_{(\theta+\beta-\frac{x+y}{2})^-}^{\sigma-1, \xi} f(\theta + \beta - y) \right] \\ & + \frac{\xi^{\sigma} 2^{\sigma-1}}{(y-x)^\sigma} \left[J_{(\theta+\beta-\frac{x+y}{2})^+}^{\sigma, \xi} f(\theta + \beta - x) + J_{(\theta+\beta-\frac{x+y}{2})^-}^{\sigma, \xi} f(\theta + \beta - y) \right] \\ & - e^{\frac{\xi-1}{\xi}} f\left(\theta + \beta - \frac{x+y}{2}\right) \\ & = \frac{y-x}{4} \int_0^1 e^{\frac{\xi-1}{\xi} \tau} \tau^\sigma \left[f'(\theta + \beta - (\frac{2-\tau}{2}x + \frac{\tau}{2}y)) - f'(\theta + \beta - (\frac{\tau}{2}x + \frac{2-\tau}{2}y)) \right] d\tau. \end{aligned}$$

Then where $\tau \in [0, 1]$, $\xi \in (0, 1]$ for all $x, y \in [\theta, \beta]$, $\sigma \in \mathbb{C}$ and $\Re(\sigma) > 0$.

Proof. It suffices to note that

$$\begin{aligned} (3.8) \quad & \int_0^1 e^{\frac{\xi-1}{\xi} \tau} \tau^\sigma \left[f'(\theta + \beta - (\frac{2-\tau}{2}x + \frac{\tau}{2}y)) - f'(\theta + \beta - (\frac{\tau}{2}x + \frac{2-\tau}{2}y)) \right] d\tau \\ & = \int_0^1 e^{\frac{\xi-1}{\xi} \tau} \tau^\sigma f'(\theta + \beta - (\frac{2-\tau}{2}x + \frac{\tau}{2}y)) d\tau \\ & - \int_0^1 e^{\frac{\xi-1}{\xi} \tau} \tau^\sigma f'(\theta + \beta - (\frac{\tau}{2}x + \frac{2-\tau}{2}y)) d\tau \\ & = I_1 - I_2 \end{aligned}$$

Integrating by parts, we get

$$\begin{aligned}
 I_1 &= \int_0^1 e^{\frac{\xi-1}{\xi}\tau} \tau^\sigma f' \left(\theta + \beta - \left(\frac{2-\tau}{2}x + \frac{\tau}{2}y \right) \right) d\tau \\
 &= e^{\frac{\xi-1}{\xi}\tau} \tau^\sigma \frac{2}{y-x} f \left(\theta + \beta - \left(\frac{2-\tau}{2}x + \frac{\tau}{2}y \right) \right) \Big|_0^1 \\
 &\quad - \int_0^1 \frac{2}{y-x} e^{\frac{\xi-1}{\xi}\tau} \left[\frac{\xi-1}{\xi} \tau^\sigma + \sigma \tau^{\sigma-1} \right] f \left(\theta + \beta - \left(\frac{2-\tau}{2}x + \frac{\tau}{2}y \right) \right) d\tau \\
 &= e^{\frac{\xi-1}{\xi}} \frac{2}{y-x} f \left(\theta + \beta - \frac{x+y}{2} \right) \\
 &\quad - \frac{2(\xi-1)}{\xi(y-x)} \int_0^1 e^{\frac{\xi-1}{\xi}\tau} \tau^\sigma f \left(\theta + \beta - \left(\frac{2-\tau}{2}x + \frac{\tau}{2}y \right) \right) d\tau \\
 &\quad - \frac{2\sigma}{(y-x)} \int_0^1 e^{\frac{\xi-1}{\xi}\tau} \tau^{\sigma-1} f \left(\theta + \beta - \left(\frac{2-\tau}{2}x + \frac{\tau}{2}y \right) \right) d\tau \\
 (3.9) \quad &= e^{\frac{\xi-1}{\xi}} \frac{2}{y-x} f \left(\theta + \beta - \frac{x+y}{2} \right) \\
 &\quad - \frac{2(\xi-1)}{\xi(y-x)} \int_{\theta+\beta-x}^{\theta+\beta-\frac{x+y}{2}} e^{\frac{\xi-1}{\xi} \left[\frac{2((\theta+\beta-x)-u)}{y-x} \right]} \left(\frac{2((\theta+\beta-x)-u)}{y-x} \right)^\sigma f(u) \left(\frac{-2du}{y-x} \right) \\
 &\quad - \frac{2\sigma}{(y-x)} \int_{\theta+\beta-x}^{\theta+\beta-\frac{x+y}{2}} e^{\frac{\xi-1}{\xi} \left[\frac{2((\theta+\beta-x)-u)}{y-x} \right]} \left(\frac{2((\theta+\beta-x)-u)}{y-x} \right)^{\sigma-1} f(u) \left(\frac{-2du}{y-x} \right) \\
 &= -e^{\frac{\xi-1}{\xi}} \frac{2}{y-x} f \left(\theta + \beta - \frac{x+y}{2} \right) \\
 &\quad + \frac{\xi^{\sigma-1} 2^{\sigma+2} (\xi-1)}{(y-x)^{\sigma+1}} J_{(\theta+\beta-\frac{x+y}{2})^+, \xi}^{\sigma-1, \xi} f(\theta + \beta - x) \\
 &\quad + \frac{\xi^\sigma 2^{\sigma+1}}{(y-x)^{\sigma+1}} J_{(\theta+\beta-\frac{x+y}{2})^+, \xi}^{\sigma, \xi} f(\theta + \beta - x).
 \end{aligned}$$

Similary we get

$$\begin{aligned}
 I_2 &= \int_0^1 e^{\frac{\xi-1}{\xi}\tau} \tau^\sigma f' \left(\theta + \beta - \left(\frac{\tau}{2}x + \frac{2-\tau}{2}y \right) \right) d\tau \\
 &= e^{\frac{\xi-1}{\xi}\tau} \tau^\sigma \frac{2}{y-x} f \left(\theta + \beta - \left(\frac{\tau}{2}x + \frac{2-\tau}{2}y \right) \right) \Big|_0^1 \\
 &\quad - \int_0^1 \frac{2}{y-x} e^{\frac{\xi-1}{\xi}\tau} \left[\frac{\xi-1}{\xi} \tau^\sigma + \sigma \tau^{\sigma-1} \right] f \left(\theta + \beta - \left(\frac{\tau}{2}x + \frac{2-\tau}{2}y \right) \right) d\tau \\
 &= e^{\frac{\xi-1}{\xi}} \frac{2}{y-x} f \left(\theta + \beta - \frac{x+y}{2} \right) \\
 &\quad - \frac{2(\xi-1)}{\xi(y-x)} \int_0^1 e^{\frac{\xi-1}{\xi}\tau} \tau^\sigma f \left(\theta + \beta - \left(\frac{\tau}{2}x + \frac{2-\tau}{2}y \right) \right) d\tau \\
 &\quad - \frac{2\sigma}{(y-x)} \int_0^1 e^{\frac{\xi-1}{\xi}\tau} \tau^{\sigma-1} f \left(\theta + \beta - \left(\frac{\tau}{2}x + \frac{2-\tau}{2}y \right) \right) d\tau \\
 (3.10) \quad &= e^{\frac{\xi-1}{\xi}} \frac{2}{y-x} f \left(\theta + \beta - \frac{x+y}{2} \right) \\
 &\quad - \frac{2(\xi-1)}{\xi(y-x)} \int_{\theta+\beta-y}^{\theta+\beta-\frac{x+y}{2}} e^{\frac{\xi-1}{\xi} \left[\frac{2(u-(\theta+\beta-y))}{y-x} \right]} \left(\frac{2(u-(\theta+\beta-y))}{y-x} \right)^\sigma f(u) \left(\frac{2du}{y-x} \right) \\
 &\quad - \frac{2\sigma}{(y-x)} \int_{\theta+\beta-y}^{\theta+\beta-\frac{x+y}{2}} e^{\frac{\xi-1}{\xi} \left[\frac{2(u-(\theta+\beta-y))}{y-x} \right]} \left(\frac{2(u-(\theta+\beta-y))}{y-x} \right)^{\sigma-1} f(u) \left(\frac{2du}{y-x} \right) \\
 &= e^{\frac{\xi-1}{\xi}} \frac{2}{y-x} f \left(\theta + \beta - \frac{x+y}{2} \right) \\
 &\quad - \frac{\xi^{\sigma-1} 2^{\sigma+2} (\xi-1)}{(y-x)^{\sigma+1}} J_{(\theta+\beta-\frac{x+y}{2})^-, \xi}^{\sigma-1, \xi} f(\theta + \beta - y) \\
 &\quad - \frac{\xi^\sigma 2^{\sigma+1}}{(y-x)^{\sigma+1}} J_{(\theta+\beta-\frac{x+y}{2})^-, \xi}^{\sigma, \xi} f(\theta + \beta - y).
 \end{aligned}$$

We can write

$$(3.11) \quad \begin{aligned} I_1 - I_2 &= -e^{\frac{\xi-1}{\xi}} \frac{4}{y-x} f\left(\theta + \beta - \frac{x+y}{2}\right) \\ &+ \frac{\xi^{\sigma-1} 2^{\sigma+2} (\xi-1)}{(y-x)^{\sigma+1}} \left[J_{\left(\theta+\beta-\frac{x+y}{2}\right)^+}^{\sigma-1, \xi} f(\theta + \beta - x) + J_{\left(\theta+\beta-\frac{x+y}{2}\right)^-}^{\sigma-1, \xi} f(\theta + \beta - y) \right] \\ &+ \frac{\xi^{\sigma} 2^{\sigma+1}}{(y-x)^{\sigma+1}} \left[J_{\left(\theta+\beta-\frac{x+y}{2}\right)^+}^{\sigma, \xi} f(\theta + \beta - x) + J_{\left(\theta+\beta-\frac{x+y}{2}\right)^-}^{\sigma, \xi} f(\theta + \beta - y) \right]. \end{aligned}$$

Multiplying the both sides by $\frac{y-x}{4}$, we obtain the conclusion (3.6). $\square$

Remark 4. If we consider $x = \theta, y = \beta$ and $\xi = 1$ in Theorem 2 becomes Theorem 3 proved by Sarikaya et. al in [25].

Remark 5. If we consider $x = \theta, \sigma = 1, y = \beta$ and $\xi = 1$ in Theorem 2 becomes Theorem 2 gives [9, Theorem2.2].

Remark 6. If we consider $\xi = 1$, in Lemma 2, in that case the following equality we get,

$$\begin{aligned} &\frac{2^{\sigma-1}}{(y-x)^{\sigma}} \left[J_{\left(\theta+\beta-\frac{x+y}{2}\right)^+}^{\sigma} f(\theta + \beta - x) + J_{\left(\theta+\beta-\frac{x+y}{2}\right)^-}^{\sigma} f(\theta + \beta - y) \right] \\ &- f\left(\theta + \beta - \frac{x+y}{2}\right) \\ &= \frac{y-x}{4} \int_0^1 \tau^{\sigma} \left[f'\left(\theta + \beta - \left(\frac{2-\tau}{2}x + \frac{\tau}{2}y\right)\right) - f'\left(\theta + \beta - \left(\frac{\tau}{2}x + \frac{2-\tau}{2}y\right)\right) \right] d\tau \end{aligned}$$

which is proved by Sarikaya and Ogülmüş in [27].

Theorem 4. Let f be a positive continuous, decreasing and convex function a differentiable mapping on (θ, β) with $\theta < \beta$. If $|f'|$ is convex on $[\theta, \beta]$, then the following inequality for fractional integrals holds;

$$(3.12) \quad \begin{aligned} &\left| \frac{e^{\frac{\xi-1}{\xi}}}{2} [f(\theta + \beta - x) + f(\theta + \beta - y)] \right. \\ &\quad - \frac{\xi^{\sigma-1} (\xi-1) \Gamma(\sigma)}{2(y-x)^{\sigma+1}} \left[J_{\left(\theta+\beta-x\right)^-}^{\sigma-1, \xi} f(\theta + \beta - y) + J_{\left(\theta+\beta-y\right)^+}^{\sigma-1, \xi} f(\theta + \beta - x) \right] \\ &\quad \left. - \frac{\xi^{\sigma} \Gamma(\sigma)}{2(y-x)^{\sigma}} \left[J_{\left(\theta+\beta-x\right)^-}^{\sigma, \xi} f(\theta + \beta - y) + J_{\left(\theta+\beta-y\right)^+}^{\sigma, \xi} f(\theta + \beta - x) \right] \right| \\ &\leq \left(|f'(\theta)| + |f'(\beta)| - \frac{|f'(x)| + |f'(y)|}{2} \right) \\ &\quad \times \left(\frac{1}{\sigma+1} - \frac{1}{\sigma+1} + \sum_{n=1}^{\infty} \left(\frac{\xi-1}{\xi} \right)^n \frac{1}{n!} \left(\frac{1}{\sigma+n+1} - \frac{1}{\sigma+1+n} \right) \right) \end{aligned}$$

Then where $\tau \in [0, 1]$, $\xi \in (0, 1]$ for all $x, y \in [\theta, \beta]$, $\sigma \in \mathbb{C}$ and $\Re(\sigma) > 0$.

Proof. By means of the *Lemma 1* and Jensen Mercer inequality, we find that

$$\begin{aligned}
 (3.13) \quad & \left| \frac{e^{\frac{\xi-1}{\xi}}}{2} [f(\theta + \beta - x) + f(\theta + \beta - y)] - \frac{\xi^{\sigma-1}(\xi-1)\Gamma(\sigma)}{2(y-x)^{\sigma+1}} \left[J_{(\theta+\beta-x)^-}^{\sigma-1,\xi} f(\theta + \beta - y) + J_{(\theta+\beta-y)^+}^{\sigma-1,\xi} f(\theta + \beta - x) \right] \right. \\
 & \left. - \frac{\xi^{\sigma}\Gamma(\sigma)}{2(y-x)^{\sigma}} \left[J_{(\theta+\beta-x)^-}^{\sigma,\xi} f(\theta + \beta - y) + J_{(\theta+\beta-y)^+}^{\sigma,\xi} f(\theta + \beta - x) \right] \right| \\
 & \leq \frac{y-x}{2} \int_0^1 \left| e^{\frac{\xi-1}{\xi}\tau} \tau^{\sigma} - e^{\frac{\xi-1}{\xi}(1-\tau)} (1-\tau)^{\sigma} \right| \left| f'(\theta + \beta - (\tau x + (1-\tau)y)) \right| d\tau \\
 & = \frac{y-x}{2} \left\{ \int_0^{\frac{1}{2}} \left[e^{\frac{\xi-1}{\xi}(1-\tau)} (1-\tau)^{\sigma} - e^{\frac{\xi-1}{\xi}\tau} \tau^{\sigma} \right] \left[|f'(\theta)| + |f'(\beta)| - (\tau |f'(x)| + (1-\tau) |f'(y)|) \right] d\tau \right. \\
 & \quad \left. + \int_{\frac{1}{2}}^1 \left[e^{\frac{\xi-1}{\xi}\tau} \tau^{\sigma} - e^{\frac{\xi-1}{\xi}(1-\tau)} (1-\tau)^{\sigma} \right] \left[|f'(\theta)| + |f'(\beta)| - (\tau |f'(x)| + (1-\tau) |f'(y)|) \right] d\tau \right\} \\
 & = \frac{y-x}{2} (R_1 + R_2).
 \end{aligned}$$

calculating R_1 and R_2 , we obtain

$$\begin{aligned}
 (3.14) \quad R_1 &= \left(|f'(\theta)| + |f'(\beta)| \right) \left(\int_0^{\frac{1}{2}} \left(e^{\frac{\xi-1}{\xi}(1-\tau)} (1-\tau)^{\sigma} - e^{\frac{\xi-1}{\xi}\tau} \tau^{\sigma} \right) d\tau \right) \\
 &\quad - |f'(x)| \left[\int_0^{\frac{1}{2}} e^{\frac{\xi-1}{\xi}(1-\tau)} (1-\tau)^{\sigma} \tau d\tau - \int_0^{\frac{1}{2}} e^{\frac{\xi-1}{\xi}\tau} \tau^{\sigma+1} d\tau \right] \\
 &\quad + |f'(y)| \left[\int_0^{\frac{1}{2}} e^{\frac{\xi-1}{\xi}(1-\tau)} (1-\tau)^{\sigma+1} d\tau - \int_0^{\frac{1}{2}} e^{\frac{\xi-1}{\xi}\tau} \tau^{\sigma} (1-\tau) d\tau \right] \\
 &= \left(|f'(\theta)| + |f'(\beta)| \right) \left(\frac{1}{\sigma+1} - \frac{\frac{1}{2}}{\sigma+1} + \sum_{n=1}^{\infty} \frac{\left(\frac{\xi-1}{\xi}\right)^n \left(\frac{1}{2}\right)^{\sigma+1+n} 2}{(\sigma+n+1)n!} \right) \\
 &\quad - \left\{ |f'(x)| \left(\frac{1}{(\sigma+1)(\sigma+2)} - \frac{\frac{1}{2}}{\sigma+1} + \sum_{n=1}^{\infty} \left(\frac{\xi-1}{\xi}\right)^n \frac{1}{n!} \left[\frac{1}{(\sigma+1+n)(\sigma+2+n)} - \frac{\frac{1}{2}}{(\sigma+1+n)} \right] \right) \right. \\
 &\quad \left. + |f'(y)| \left(\frac{1}{\sigma+2} - \frac{\frac{1}{2}}{\sigma+1} + \sum_{n=1}^{\infty} \left(\frac{\xi-1}{\xi}\right)^n \frac{1}{n!} \left[\frac{1}{(\sigma+2+n)} - \frac{\frac{1}{2}}{(\sigma+1+n)} \right] \right) \right\}
 \end{aligned}$$

and

$$\begin{aligned}
 (3.15) \quad R_2 &= \left(|f'(\theta)| + |f'(\beta)| \right) \left(\int_{\frac{1}{2}}^1 \left(e^{\frac{\xi-1}{\xi}\tau} \tau^{\sigma} - e^{\frac{\xi-1}{\xi}(1-\tau)} (1-\tau)^{\sigma} \right) d\tau \right) \\
 &\quad - |f'(x)| \left[\int_{\frac{1}{2}}^1 e^{\frac{\xi-1}{\xi}\tau} \tau^{\sigma+1} d\tau - \int_{\frac{1}{2}}^1 e^{\frac{\xi-1}{\xi}(1-\tau)} (1-\tau)^{\sigma} \tau d\tau \right] \\
 &\quad + |f'(y)| \left[\int_{\frac{1}{2}}^1 e^{\frac{\xi-1}{\xi}\tau} \tau^{\sigma} (1-\tau) d\tau - \int_{\frac{1}{2}}^1 e^{\frac{\xi-1}{\xi}(1-\tau)} (1-\tau)^{\sigma+1} d\tau \right] \\
 &= \left(|f'(\theta)| + |f'(\beta)| \right) \left(\frac{1}{\sigma+1} - \frac{\frac{1}{2}}{\sigma+1} + \sum_{n=1}^{\infty} \frac{\left(\frac{\xi-1}{\xi}\right)^n \left(\frac{1}{2}\right)^{\sigma+1+n} 2}{(\sigma+n+1)n!} \right) \\
 &\quad - \left\{ |f'(x)| \left(\frac{1}{\sigma+2} - \frac{\frac{1}{2}}{\sigma+1} + \sum_{n=1}^{\infty} \left(\frac{\xi-1}{\xi}\right)^n \frac{1}{n!} \left[\frac{1}{(\sigma+2+n)} - \frac{\frac{1}{2}}{(\sigma+1+n)} \right] \right) \right. \\
 &\quad \left. + |f'(y)| \left(\frac{1}{(\sigma+1)(\sigma+2)} - \frac{\frac{1}{2}}{\sigma+1} + \sum_{n=1}^{\infty} \left(\frac{\xi-1}{\xi}\right)^n \frac{1}{n!} \left[\frac{1}{(\sigma+1+n)(\sigma+2+n)} - \frac{\frac{1}{2}}{(\sigma+1+n)} \right] \right) \right\}
 \end{aligned}$$

By adding R_1 and R_2 , we get the inequality (3.11). $\square$

Remark 7. If we consider $x = \theta, y = \beta$ and $\xi = 1$ in Theorem 2 becomes Theorem 3 proved by Sarikaya et. al in [25].

Remark 8. If we consider $x = \theta, \sigma = 1, y = \beta$ and $\xi = 1$ in Theorem 2 becomes Theorem 2 gives [9, Theorem2.2].

Remark 9. If we consider $\xi = 1$, in Theorem 4, in that case the following inequality we get,

$$\begin{aligned} & \left| f(\theta + \beta - x) + f(\theta + \beta - y) \right. \\ & \quad \left. - \frac{\sigma \Gamma(\sigma)}{2(y-x)^\sigma} \left[J_{(\theta+\beta-x)-}^\sigma f(\theta + \beta - y) + J_{(\theta+\beta-y)+}^\sigma f(\theta + \beta - x) \right] \right| \\ & \leq \left(\left| f'(\theta) \right| + \left| f'(\beta) \right| - \frac{|f'(x)| + |f'(y)|}{2} \right) \end{aligned}$$

which is proved by Sarikaya and Ogülmüş in [27].

Theorem 5. Let f be a positive continuous, decreasing and convex function a differentiable mapping on (θ, β) with $\theta < \beta$. If $|f'|$ is convex on $[\theta, \beta]$, then the following inequality for fractional integrals holds;

$$\begin{aligned} (3.16) \quad & \left| \frac{\xi^{\sigma-1} 2^\sigma (\xi-1)}{(y-x)^\sigma} \left[J_{(\theta+\beta-\frac{x+y}{2})+}^{\sigma-1, \xi} f(\theta + \beta - x) + J_{(\theta+\beta-\frac{x+y}{2})-}^{\sigma-1, \xi} f(\theta + \beta - y) \right] \right. \\ & \quad \left. + \frac{\xi^\sigma 2^{\sigma-1}}{(y-x)^\sigma} \left[J_{(\theta+\beta-\frac{x+y}{2})+}^{\sigma, \xi} f(\theta + \beta - x) + J_{(\theta+\beta-\frac{x+y}{2})-}^{\sigma, \xi} f(\theta + \beta - y) \right] \right. \\ & \quad \left. - e^{\frac{\xi-1}{\xi}} f\left(\theta + \beta - \frac{x+y}{2}\right) \right| \\ & \leq \frac{y-x}{2} \left[\frac{1}{\sigma+1} + \sum_{n=1}^{\infty} \frac{1}{n!} \left(\frac{\xi-1}{\xi} \right)^n \frac{1}{\sigma+n+1} \right] \left(\left| f'(\theta) \right| + \left| f'(\beta) \right| - \frac{|f'(x)| + |f'(y)|}{2} \right) \end{aligned}$$

Then where $\tau \in [0, 1]$, $\xi \in (0, 1]$ for all $x, y \in [\theta, \beta]$, $\sigma \in \mathbb{C}$ and $\Re(\sigma) > 0$.

Proof. Using the Lemma 2 and Jensen -Mercer inequality, we find

$$\begin{aligned}
 & \left| \frac{\xi^{\sigma-1} 2^\sigma (\xi-1)}{(y-x)^\sigma} \left[J_{(\theta+\beta-\frac{x+y}{2})^+}^{\sigma-1, \xi} f(\theta+\beta-x) + J_{(\theta+\beta-\frac{x+y}{2})^-}^{\sigma-1, \xi} f(\theta+\beta-y) \right] \right. \\
 & \quad \left. + \frac{\xi^{\sigma} 2^{\sigma-1}}{(y-x)^\sigma} \left[J_{(\theta+\beta-\frac{x+y}{2})^+}^{\sigma, \xi} f(\theta+\beta-x) + J_{(\theta+\beta-\frac{x+y}{2})^-}^{\sigma, \xi} f(\theta+\beta-y) \right] \right. \\
 & \quad \left. - e^{\frac{\xi-1}{\xi}} f\left(\theta+\beta-\frac{x+y}{2}\right) \right| \\
 & \leq \frac{y-x}{4} \int_0^1 e^{\frac{\xi-1}{\xi} \tau} \tau^\sigma \left| f'\left(\theta+\beta-\left(\frac{2-\tau}{2}x + \frac{\tau}{2}y\right)\right) \right| d\tau \\
 & \quad + \frac{y-x}{4} \int_0^1 e^{\frac{\xi-1}{\xi} \tau} \tau^\sigma \left| f'\left(\theta+\beta-\left(\frac{\tau}{2}x + \frac{2-\tau}{2}y\right)\right) \right| d\tau \\
 & = \frac{y-x}{4} \int_0^1 e^{\frac{\xi-1}{\xi} \tau} \tau^\sigma \left[\left| f'(\theta) \right| + \left| f'(\beta) \right| - \left(\frac{2-\tau}{2} \left| f'(x) \right| + \frac{\tau}{2} \left| f'(y) \right| \right) \right] d\tau \\
 & \quad + \frac{y-x}{4} \int_0^1 e^{\frac{\xi-1}{\xi} \tau} \tau^\sigma \left[\left| f'(\theta) \right| + \left| f'(\beta) \right| - \left(\frac{\tau}{2} \left| f'(x) \right| + \frac{2-\tau}{2} \left| f'(y) \right| \right) \right] d\tau \\
 & = \frac{y-x}{4} \left(\left| f'(\theta) \right| + \left| f'(\beta) \right| \right) \left[\frac{1}{\sigma+1} + \sum_{n=1}^{\infty} \frac{1}{n!} \left(\frac{\xi-1}{\xi} \right)^n \frac{1}{\sigma+n+1} \right] \\
 & \quad - \frac{y-x}{4} \left| f'(x) \right| \left[\frac{1}{\sigma+1} - \frac{1}{2(\sigma+2)} + \sum_{n=1}^{\infty} \left(\frac{\xi-1}{\xi} \right)^n \frac{1}{n!} \left(\frac{1}{\sigma+n+1} - \frac{1}{2(\sigma+n+2)} \right) \right] \\
 & \quad - \frac{y-x}{4} \left| f'(y) \right| \left[\frac{1}{2(\sigma+2)} + \sum_{n=1}^{\infty} \left(\frac{\xi-1}{\xi} \right)^n \frac{1}{n!} \left(\frac{1}{2(\sigma+n+2)} \right) \right] \\
 & \quad + \frac{y-x}{4} \left(\left| f'(\theta) \right| + \left| f'(\beta) \right| \right) \left[\frac{1}{\sigma+1} + \sum_{n=1}^{\infty} \frac{1}{n!} \left(\frac{\xi-1}{\xi} \right)^n \frac{1}{\sigma+n+1} \right] \\
 & \quad - \frac{y-x}{4} \left| f'(x) \right| \left[\frac{1}{2(\sigma+2)} + \sum_{n=1}^{\infty} \left(\frac{\xi-1}{\xi} \right)^n \frac{1}{n!} \left(\frac{1}{2(\sigma+n+2)} \right) \right] \\
 & \quad - \frac{y-x}{4} \left| f'(y) \right| \left[\frac{1}{\sigma+1} - \frac{1}{2(\sigma+2)} + \sum_{n=1}^{\infty} \left(\frac{\xi-1}{\xi} \right)^n \frac{1}{n!} \left(\frac{1}{\sigma+n+1} - \frac{1}{2(\sigma+n+2)} \right) \right]
 \end{aligned}
 \tag{3.17}$$

□

Remark 10. If we consider $\xi = 1$, in Theorem 5, in that case the following inequality we get,

$$\begin{aligned}
 & \left| \frac{2^{\sigma-1}}{(y-x)^\sigma} \left[J_{(\theta+\beta-\frac{x+y}{2})^+}^{\sigma} f(\theta+\beta-x) + J_{(\theta+\beta-\frac{x+y}{2})^-}^{\sigma} f(\theta+\beta-y) \right] \right. \\
 & \quad \left. - f\left(\theta+\beta-\frac{x+y}{2}\right) \right| \\
 & \leq \frac{y-x}{2} \left(\left| f'(\theta) \right| + \left| f'(\beta) \right| - \frac{\left| f'(x) \right| + \left| f'(y) \right|}{2} \right)
 \end{aligned}$$

which is proved by Sarikaya and Ogülmüş in [27].

Corollary 2. If we consider $\xi = 1$ and $\sigma = 1$, in Theorem 5, in that case the following inequality we get,

$$\begin{aligned}
 & \left| \frac{1}{y-x} \int_{\theta+\beta-y}^{\theta+\beta-x} f(u) du - f\left(\theta+\beta-\frac{x+y}{2}\right) \right| \\
 & \leq \frac{y-x}{4} \left[\left| f'(\theta) \right| + \left| f'(\beta) \right| - \frac{\left| f'(x) \right| + \left| f'(y) \right|}{2} \right]
 \end{aligned}$$

which is proved by Sarikaya and Ogülmüş in [27].

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