Modified Sumudu Transform and Its Properties

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Abstract

Saif et al. (J. Math. Comput. Sci. 21 (2020) 127-135) considered modified Laplace transform and developed some of their certain properties and relations. Motivated by this work, in this paper, we define modified Sumudu transform and investigate many properties and relations including modified Sumudu transforms of the power function, sine, cosine, hyperbolic sine, hyperbolic cosine, exponential function, and function derivatives. Moreover, we attain two shifting properties and a scale preserving theorem for the modified Sumudu transform. We give modified inverse Sumudu transform and investigate some relations and examples. Furthermore, we show that the modified Sumudu transform is the theoretical dual transform to the modified Laplace transform.

2010 Mathematics Subject Classification- Primary 44A10; Secondary 44A35, 44A30.
Keywords and Phrases- Gamma function, Sumudu transform, Laplace transform, convolution.

1. Introduction

Throughout this paper, the symbols $\mathbb{C}$, $\mathbb{R}$, $\mathbb{Z}$, $\mathbb{N}$ and $\mathbb{N}_0$ are referred to the set of all complex numbers, the set of all real numbers, the set of all integers, the set of all-natural numbers, and the set of all non-negative integers, respectively.

Integral transforms have been played a key role to solve the differential or integrodifferential equations *cf.* [1-12]. One of the most useful integral transforms is the Laplace transform, for f being a function defined for $t \geq 0$, defined by

$$F(s) = (f(t)) = \int_0^\infty e^{-st} f(t)\,dt, \tag{1.1}$$

provided that the integral converges. It has powerful applications, not only in applied mathematics but also in other branches of science such as astronomy, engineering, physics, etc., *cf* [5-9]. Also, diverse integral transforms such as Sumudu, Fourier, Elzaki, and M-transforms have been considered, and their properties and applications have been examined in detail by many scientists, *cf.* [1-12] and see also the references cited therein. The Laplace transform is the theoretical dual transform of the Sumudu transform which is introduced by Watugula [10] given by

$$G(u) = \mathbf{S}[f(t)] = \int_0^\infty e^{-t} f(ut)\,dt = \frac{1}{u}\int_0^\infty e^{-\frac{t}{u}} f(t)\,dt, \quad u \in (-\tau_1, \tau_2), \tag{1.2}$$

over the set of functions

$$A = \left\{ f(t) | \exists M, \tau_1, \tau_2 > 0, |f(t)| < Me^{\frac{|t|}{\tau_j}}, \text{ if } t \in (-1)^j \times [0,\infty) \right\}.$$

Several applications of Sumudu transform have been investigated and studied by many physicists and mathematicians, *cf.* [1-4, 6, 10-12]. For instance, Watagula [11] defined two variables Sumudu transform and provided an example solving partial differential equations with known initial conditions. Weerakoon [12] attained the Sumudu transform of partial derivatives and proved its applicability demonstrated utilizing three different partial differential equations. Kilicman et al. [6] studied some properties of the Sumudu transform and relationship between Sumudu and Laplace transforms, and then gave an application of the double Sumudu transform to solve the wave equation in one dimension having singularity at initial conditions. Asiru [1] provided Sumudu transform of several special functions and derived some applications with Abel's integral equation, an integrodifferential equation, a dynamic system with delayed time signals and a differential dynamic system. Belgacem et al. [2] developed fundamental properties including scale and unit-preserving properties of Sumudu transform and proved a solution to an integral production-depreciation problem. Belgacem [3] analyzed deeper Sumudu properties and connections. Belgacem et al. [4] generalized all existing Sumudu integration, differentiation, and Sumudu shifting theorems and convolution theorems. In this study, we introduce modified Sumudu transform and investigated many properties and relations including modified Sumudu transforms of the power function, sine, cosine, hyperbolic sine, hyperbolic cosine, exponential function, and function derivatives. Moreover, we obtain two shifting properties and a scale preserving theorem for the modified Sumudu transform. We provide modified inverse Sumudu transform and derive some relations and examples. Furthermore, we show that modified Sumudu transform is the theoretical dual transform to modified Laplace transform. Lastly, we give duality between the modified Laplace transform and the modified Sumudu transform.

The Sumudu transformation satisfies the following operational properties, *cf.* [2,4]:

$\mathbf{S}[1]=1$	$\mathbf{S}[\sin(at)]=\frac{au}{1+u^2a^2}$
$\mathbf{S}[t]=u$	$\mathbf{S}[\cos(at)]=\frac{1}{1+u^2a^2}$
$\mathbf{S}[t^n]=n!u^n$	$\mathbf{S}[\sinh(at)]=\frac{au}{1+u^2a^2}$
$\mathbf{S}[e^{at}]=\frac{1}{1-ua}$	$\mathbf{S}[\cosh(at)]=\frac{1}{1+u^2a^2}$
$\mathbf{S}[f(at)]=G(au)$	$\mathbf{S}[e^{at}f(t)]=\frac{1}{1-au}G\left(\frac{u}{1-au}\right)$

(1.3)

Let $f(t)$,$g(t)\in A$ be Sumudu transforms $M(u)$ and $N(u)$, respectively. Then the Sumudu transform of the convolution of f and g is given by

$$\mathbf{S}[(f*g)(t)]=uM(u)N(u), \tag{1.4}$$

where the convolution integral is given by (*cf.* [2,4])

$$(f*g)(t)=\int_0^t g(x)f(t-x)\,dx \tag{1.5}$$

for $f(t)$ and $g(t)$ are piece-wise continuous and of exponential order.

The gamma function is defined by the following improper integral (*cf.* [5-9]):

$$\Gamma(s)=\int_0^\infty e^{-t}t^{s-1}dt, \tag{1.6}$$

where s is a complex number with $Re(s)>0$. The gamma function satisfies the following relations

$$\Gamma(s+1)=s\Gamma(s) \text{ and } \Gamma(n+1)=n!$$

for n being a non-negative integer.

2. Modified Sumudu Transform

In [9], the modified Laplace transform of a function $f(t)$ which is peace-wise continuous and of exponential order is considered as follows

$$_a(f(t))=F(s;a)=\int_0^\infty a^{-st}f(t)\,dt, \tag{2.1}$$

where $Re(s)>0$ and $a\in(0,\infty)\setminus\{1\}$. Note that upon setting $a=e$, modified Laplace transform reduces to usual Laplace transform in (1.1). Then the authors gave several basic properties of modified Laplace transform and provided connections with different functions in [9].

Motivated by the above, we define modified Sumudu transform as follows.

Definition 1. *Let $a\in(0,\infty)\setminus\{1\}$ and*

$$A_a=\left\{f(t)\,|\,\exists M,\tau_1,\tau_2>0 \text{ such that } |f(t)|<Ma^{1/\tau_j}(|t|),\ \text{if } t\in(-1)^j\times[0,\infty)\right\}. \tag{2.2}$$

Then, for $f(t) \in A_a$, we define modified Sumudu transform by the following improper integral:

$$\mathcal{G}_a(u) = \mathcal{S}_a[f(t)] = \frac{1}{u}\int_0^\infty a^{\frac{-s}{u}} f(t)\,dt, \quad u \in (-\tau_1, \tau_2). \tag{2.3}$$

We note that

$$\mathcal{S}_e[f(t)] := \mathbf{S}[f(t)].$$

Let $f(t), g(t) \in A_a$ and $\gamma, \omega \in \mathbb{R}$. The modified Sumudu transformation is a linear transform, namely

$$\begin{aligned}\mathcal{S}_a[\gamma f(t) + \omega g(t)] &= \frac{1}{u}\int_0^\infty a^{\frac{-t}{u}}[\gamma f(t) + \omega g(t)]\,dt \\ &= \frac{\gamma}{u}\int_0^\infty a^{\frac{-t}{u}} f(t)\,dt + \frac{\omega}{u}\int_0^\infty a^{\frac{-t}{u}} g(t)\,dt \\ &= \gamma\mathcal{S}_a[f(t)] + \omega\mathcal{S}_a[g(t)].\end{aligned}$$

By Definition 1, for $f(t) = 1$, we observe that

$$\begin{aligned}\mathcal{S}_a[1] &= \frac{1}{u}\int_0^\infty a^{\frac{-t}{u}}\,dt = \lim_{R\to\infty}\frac{1}{u}\int_0^R e^{-\frac{t}{u}\log a}\,dt \\ &= \frac{1}{u}\lim_{R\to\infty}\left.\frac{e^{-\frac{t}{u}\log a}}{-\frac{1}{u}\log a}\right|_0^R = \lim_{R\to\infty}\left(\frac{e^{-\frac{R}{u}\log a}}{-\frac{1}{u}\log a} + \frac{1}{\log a}\right) \\ &= \frac{1}{\log a}, \qquad \frac{\log a}{u} > 0,\end{aligned}$$

and for $f(t) = t$ with $\frac{\log a}{u} > 0$,

$$\begin{aligned}\mathcal{S}_a[t] &= \frac{1}{u}\int_0^\infty t a^{\frac{-t}{u}}\,dt = \lim_{R\to\infty}\left(-\left.\frac{ta^{\frac{-t}{u}}}{\log a}\right|_0^R + \frac{1}{\log a}\int_0^R a^{\frac{-t}{u}}\right)dt \\ &= -\lim_{R\to\infty}\frac{u}{(\log a)^2}\left.a^{\frac{-t}{u}}\right|_0^R dt = \frac{u}{(\log a)^2},\end{aligned}$$

which gives the following theorem.

Theorem 1. *We have*

$$\mathcal{S}_a[1] = \frac{1}{\log a}, \qquad \frac{\log a}{u} > 0 \tag{2.4}$$

and

$$\mathcal{S}_a[t] = \frac{u}{(\log a)^2}, \qquad \frac{\log a}{u} > 0. \tag{2.5}$$

By Definition 1, for $f(t) = t^n$ with $n \in \mathbb{N}$ and $\frac{\log a}{u} > 0$, we observe that

$$\begin{aligned}\mathcal{S}_a[t^n] &= \frac{1}{u}\int_0^\infty t^n a^{\frac{-t}{u}}\,dt = \frac{n}{\log a}\int_0^\infty t^{n-1} a^{\frac{-t}{u}}\,dt \\ &= \frac{n(n-1)}{(\log a)^2} u\int_0^\infty t^{n-2} a^{\frac{-t}{u}}\,dt = \ldots \\ &= \frac{n!u^{n-1}}{(\log a)^n}\int_0^\infty a^{\frac{-t}{u}}\,dt = \frac{n!u^n}{(\log a)^{n+1}},\end{aligned}$$

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and for $f(t) = e^{bt}$,

$$\begin{aligned}\mathcal{S}_a\left[e^{bt}\right] &= \frac{1}{u}\int_0^\infty e^{bt}a^{\frac{-t}{u}}dt = \frac{1}{u}\int_0^\infty e^{t\left(b-\frac{\log a}{u}\right)}dt = \frac{1}{u}\lim_{R\to\infty}\left.\frac{e^{t\left(b-\frac{\log a}{u}\right)}}{b-\frac{\log a}{u}}\right|_0^R \\ &= \frac{1}{u}\lim_{R\to\infty}\left(\frac{e^{t\left(b-\frac{\log a}{u}\right)}}{b-\frac{\log a}{u}} - \frac{1}{b-\frac{\log a}{u}}\right) \\ &= \frac{1}{\log a - bu}, \quad b < \frac{\log a}{u},\end{aligned}$$

which provides the following theorem.

Theorem 2. *Let $n\in\mathbb{N}$. We have*

$$\mathcal{S}_a\left[t^n\right] = \frac{n!u^n}{(\log a)^{n+1}}, \quad 0 < \frac{\log a}{u} \tag{2.6}$$

and

$$\mathcal{S}_a\left[e^{bt}\right] = \frac{1}{\log a - bu}, \quad b < \frac{\log a}{u}. \tag{2.7}$$

From Definition 1 and using formula (2.7), we have

$$\begin{aligned}\mathcal{S}_a\left[\sin(bt)\right] &= \mathcal{S}_a\left[\frac{e^{ibt}-e^{-ibt}}{2i}\right] \\ &= \frac{1}{2i}\left(\mathcal{S}_a\left[e^{ibt}\right] - \mathcal{S}_a\left[e^{-ibt}\right]\right) \\ &= \frac{1}{2i}\left(\frac{1}{\log a - ibu} - \frac{1}{\log a + ibu}\right) \\ &= \frac{bu}{(\log a)^2 + b^2u^2}\end{aligned}$$

and

$$\begin{aligned}\mathcal{S}_a\left[\cos(bt)\right] &= \mathcal{S}_a\left[\frac{e^{ibt}+e^{-ibt}}{2}\right] \\ &= \frac{1}{2}\left(\mathcal{S}_a\left[e^{-ibt}\right] + \mathcal{S}_a\left[e^{-ibt}\right]\right) \\ &= \frac{1}{2}\left(\frac{1}{\log a - ibu} + \frac{1}{\log a - ibu}\right) \\ &= \frac{\log a}{(\log a)^2 + b^2u^2},\end{aligned}$$

where $i = \sqrt{-1}$. Thus we give the following theorem.

Theorem 3. *We have*

$$\mathcal{S}_a\left[\sin(bt)\right] = \frac{bu}{(\log a)^2 + b^2u^2} \text{ and } \mathcal{S}_a\left[\cos(bt)\right] = \frac{\log a}{(\log a)^2 + b^2u^2}. \tag{2.8}$$

By Definition 1, and utilizing formula (2.7), we derive

$$\mathcal{S}_a\left[\sinh(bt)\right] = \mathcal{S}_a\left[\frac{e^{bt}-e{-bt}}{2}\right] = \frac{bu}{(\log a)^2 - b^2u^2}$$

and

$$\mathcal{S}_a\left[\cosh(bt)\right] = \mathcal{S}_a\left[\frac{e^{bt}+e{-bt}}{2}\right] = \frac{\log a}{(\log a)^2 - b^2u^2}.$$

Therefore, we give the following theorem.

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Theorem 4. *We have*

$$\mathcal{S}_a\left[\sinh\left(bt\right)\right]=\frac{bu}{\left(\log a\right)^2-b^2u^2}\text{ and }\mathcal{S}_a\left[\cosh\left(bt\right)\right]=\frac{\log a}{\left(\log a\right)^2-b^2u^2}. \tag{2.9}$$

By Definition 1 and (1.6), for $b\in\mathbb{R}$ with $b>-1$, we derive

$$\begin{aligned}\mathcal{S}_a\left[t^b\right] &= \frac{1}{u}\int_0^\infty t^b a^{\frac{-t}{u}}dt=\frac{u^{b-1}}{\left(\log a\right)^b}\int_0^\infty\left(\frac{t\log a}{u}\right)^b e^{\frac{-t}{u}\log a}dt\\ &= \frac{u^b}{\left(\log a\right)^{b+1}}\int_0^\infty t^b e^{-t}dt=\frac{u^b}{\left(\log a\right)^{b+1}}\Gamma\left(b+1\right).\end{aligned}$$

Thus we give the following theorem.

Theorem 5. *The following*

$$\mathcal{S}_a\left[t^b\right]=\frac{u^b}{\left(\log a\right)^{b+1}}\Gamma\left(b+1\right)$$

is valid for $b\in\mathbb{R}$ with $b>-1$.

We now investigate some formulas for modified Sumudu transform of derivatives of functions. By Definition 1, for $\frac{\log a}{u}>0$, we see that

$$\mathcal{S}_a\left[f'\left(t\right)\right]=\frac{1}{u}\int_0^\infty f'\left(t\right)a^{\frac{-t}{u}}dt=\frac{1}{u}\left(\log a\mathcal{G}_a\left(u\right)-f\left(0\right)\right). \tag{2.10}$$

By means of (2.10), we acquire

$$\begin{aligned}\mathcal{S}_a\left[f^{(2)}\left(t\right)\right] &= \frac{1}{u}\int_0^\infty f^{(2)}\left(t\right)a^{\frac{-t}{u}}dt\\ &= \frac{1}{u^2}\left(\left(\log a\right)^2\mathcal{G}_a\left(u\right)-\log af\left(0\right)-uf'\left(0\right)\right).\end{aligned}$$

Continuing this process, we get

$$\mathcal{S}_a\left[f^{(n)}\left(t\right)\right]=\frac{\left(\log a\right)^n}{u^n}\mathcal{G}_a\left(u\right)-\frac{1}{u^n}\sum_{i=0}^{n-1}\left(\log a\right)^{n-i-1}u^i f^{(i)}\left(0\right). \tag{2.11}$$

By (2.11), we provide the following theorem.

Theorem 6. *The following modified Sumudu transform*

$$\mathcal{S}_a\left[f^{(n)}\left(t\right)\right]=\frac{\left(\log a\right)^n}{u^n}\mathcal{G}_a\left(u\right)-\frac{1}{u^n}\sum_{i=0}^{n-1}\left(\log a\right)^{n-i-1}u^i f^{(i)}\left(0\right). \tag{2.12}$$

is valid for $n\in\mathbb{N}$ and $\frac{\log a}{u}>0$.

By Definition 1, for $u\in\left(-\tau_1,\tau_2\right)$, we observe that

$$\begin{aligned}\frac{d}{du}\mathcal{G}_a\left(u\right) &= -\frac{1}{u^2}\int_0^\infty a^{\frac{-t}{u}}f\left(t\right)dt+\frac{\log a}{u^3}\int_0^\infty a^{\frac{-t}{u}}tf\left(t\right)dt\\ &= -\frac{1}{u}\mathcal{G}_a\left(u\right)+\frac{\log a}{u^2}\mathcal{S}_a\left[tf\left(t\right)\right]\end{aligned}$$

and

$$\begin{aligned}\frac{d^2}{du^2}\mathcal{G}_a(u) &= \frac{d}{du}\left(-\frac{1}{u}\mathcal{G}_a(u)+\frac{\log a}{u^2}\mathcal{S}_a[tf(t)]\right)\\ &= \frac{1}{u^2}\mathcal{G}_a(u)-\frac{1}{u}\left(-\frac{1}{u}\mathcal{G}_a(u)+\frac{\log a}{u^2}\mathcal{S}_a[tf(t)]\right)-\frac{2}{u^3}\mathcal{S}_a[tf(t)]\\ &\quad+\frac{1}{u^2}\left(-\frac{1}{u}\mathcal{S}_a[tf(t)]+\frac{\log a}{u^2}\mathcal{S}_a\left[t^2f(t)\right]\right)\\ &= \frac{2}{u^2}\mathcal{G}_a(u)-\frac{3+\log a}{u^3}\mathcal{S}_a[tf(t)]+\frac{\log a}{u^4}\mathcal{S}_a\left[t^2f(t)\right].\\ \frac{d^3}{du^3}\mathcal{G}_a(u) &= \frac{d}{du}\left(\frac{2}{u^2}\mathcal{G}_a(u)-\frac{3+\log a}{u^3}\mathcal{S}_a[tf(t)]+\frac{\log a}{u^4}\mathcal{S}_a\left[t^2f(t)\right]\right)\\ &= -\frac{6}{u^3}\mathcal{G}_a(u)+\frac{12+6\log a}{u^4}\mathcal{S}_a[tf(t)]\\ &\quad-\frac{(\log a)^2+4\log a}{u^5}\mathcal{S}_a\left[t^2f(t)\right]+\frac{(\log a)^2}{u^6}\mathcal{S}_a\left[t^3f(t)\right].\end{aligned}$$

Therefore, we give the following theorem.

Theorem 7. *For $u\in(-\tau_1,\tau_2)$ and $\frac{\log a}{u}>0$, we have*

$$\frac{d}{du}\mathcal{G}_a(u)=-\frac{1}{u}\mathcal{G}_a(u)+\frac{\log a}{u^2}\mathcal{S}_a[tf(t)]$$

$$\frac{d^2}{du^2}\mathcal{G}_a(u)=\frac{2}{u^2}\mathcal{G}_a(u)-\frac{3+\log a}{u^3}\mathcal{S}_a[tf(t)]+\frac{\log a}{u^4}\mathcal{S}_a\left[t^2f(t)\right]$$

and

$$\begin{aligned}\frac{d^3}{du^3}\mathcal{G}_a(u) &= -\frac{6}{u^3}\mathcal{G}_a(u)+\frac{12+6\log a}{u^4}\mathcal{S}_a[tf(t)]\\ &\quad-\frac{(\log a)^2+4\log a}{u^5}\mathcal{S}_a\left[t^2f(t)\right]+\frac{(\log a)^2}{u^6}\mathcal{S}_a\left[t^3f(t)\right].\end{aligned}$$

From Definition 1, we observe that

$$\mathcal{S}_a[f(bt)]=\frac{1}{u}\int_0^\infty f(bt)\,a^{\frac{-t}{u}}dt=\frac{1}{u}\int_0^\infty f(bt)\,a^{\frac{-bt}{bu}}dt$$

and setting $\omega=bt$, then

$$\mathcal{S}_a[f(bt)]=\frac{1}{bu}\int_0^\infty f(\omega)\,a^{\frac{-\omega}{bu}}d\omega=\mathcal{G}_a(bu).$$

Thereby, we give the following theorem.

Theorem 8. *The following*

$$\mathcal{S}_a[f(bt)]=G_a(bu) \tag{2.13}$$

holds.

Let $\delta(t-c)$ be unit step function given the below:

$$\delta(t-c)=\{.0,\quad x<c,1,\quad x\geq c.$$

Then, we get

$$\mathcal{S}_a[\delta(t-c)]=\frac{1}{u}\int_0^c 0a^{\frac{-t}{u}}dt+\frac{1}{u}\int_c^\infty a^{\frac{-t}{u}}dt=\frac{a^{\frac{-c}{u}}}{\log a},\quad \frac{\log a}{u}>0.$$

By Definition 1, for $\mathcal{S}_a[f(t)]=G_a(u)$, we observe

$$\mathcal{S}_a\left[a^{bt}f(t)\right]=\frac{1}{u}\int_0^c a^{-\left(\frac{1-bu}{u}\right)t}f(t)\,dt=G_a\left(\frac{u}{1-bu}\right)$$

and

$$\begin{aligned}\mathcal{S}_a\left[f(t-c)\,\delta(t-c)\right] &= \frac{1}{u}\int_c^\infty a^{\frac{-t}{u}} f(t-c)\,dt \\ &= \frac{1}{u}\int_0^\infty a^{\frac{-t+c}{u}} f(t)\,dt = a^{-\frac{c}{u}}\mathcal{S}_a\left[f(t)\right].\end{aligned}$$

Hence, two shifting properties of modified Sumudu transform are given by the following theorem.

Theorem 9. *Let* $\mathcal{S}_a\left[f(t)\right] = G_a(u)$*. Each of the following properties*

$$\mathcal{S}_a\left[a^{bt} f(t)\right] = G_a\left(\frac{u}{1-bu}\right) \qquad \textit{(The first shifting property)}$$

and

$$\mathcal{S}_a\left[f(t-c)\,\delta(t-c)\right] = a^{-\frac{c}{u}}\mathcal{S}_a\left[f(t)\right] \qquad \textit{(The second shifting property)}$$

holds for $\frac{\log a}{u} > 0$.

By (1.5), it can be readily shown that the set of all modified Sumudu transformable functions form a commutative semigroup with respect to the convolution operator $*$.

By Definition 1 and (1.5), we attain

$$\mathcal{S}_a\left[(f*g)(t)\right] = \frac{1}{u}\int_0^\infty a^{\frac{-t}{u}}(f*g)(t)\,dt = \frac{1}{u}\int_0^\infty a^{\frac{-t}{u}}\left(\int_0^t g(\omega)\,f(t-\omega)\,d\omega\right)dt.$$

Setting $t-\omega=\gamma$ yields $dt = d\gamma$, and we have

$$\begin{aligned}\mathcal{S}_a\left[(f*g)(t)\right] &= \frac{1}{u}\int_0^\infty\int_0^\infty a^{\frac{-\gamma+\omega}{u}} g(\omega)\,f(\gamma)\,d\omega d\gamma \\ &= u\left(\frac{1}{u}\int_0^\infty a^{\frac{-\omega}{u}} f(\omega)\,d\omega\right)\left(\frac{1}{u}\int_0^\infty a^{\frac{-\gamma}{u}} g(\gamma)\,d\gamma\right) \\ &= u\mathcal{S}_a\left[f(t)\right]\mathcal{S}_a\left[g(t)\right].\end{aligned}$$

Thus, we give the following theorem.

Theorem 10. *For* $f(t)$ *and* $g(t)$ *being piece-wise continuous and of exponential order functions on* $[0,\infty]$*, let* $\mathcal{S}_a\left[f(t)\right] = G_a(u)$ *and* $\mathcal{S}_a\left[g(t)\right] = H_a(u)$*. Modified Sumudu transform of the convolution is as follows:*

$$\mathcal{S}_a\left[(f*g)(t)\right] = uG_a(u)\,H_a(u).$$

3. Further Remarks

By (1.1) and (1.2), the Sumudu transform is the theoretical dual transform to the Laplace transform given below (*cf.* [2,4])

$$G(u) = \frac{F(1/u)}{u} \text{ and } F(s) = \frac{G(1/s)}{s}. \tag{3.1}$$

Using (2.1) and (2.3), for $f(t) \in A_a$ and $-\tau_1 < u < \tau_2$, we observe that

$$G_a(u) = \mathcal{S}_a\left[f(t)\right] = \frac{1}{u}\int_0^\infty a^{\frac{-t}{u}} f(t)\,dt = \frac{1}{u}F\left(\frac{1}{u};a\right) \tag{3.2}$$

and

$$F(s;a) =_a (f(t)) = \frac{1}{s}\frac{1}{\frac{1}{s}}\int_0^\infty a^{-st} f(t)\,dt = \frac{1}{s}G_a\left(\frac{1}{s}\right) \tag{3.3}$$

which are the modified version of the duality in (3.1). Therefore, the relations (3.2) and (3.3) between the modified Sumudu transform and the modified Laplace transform means to acquire one from the other when needed. For example, since ${}_a(\sinh(bt)) = \frac{b}{s^2(\log a)^2 - b^2}$, recall from Theorem 4, we have

$$F(s;a) = \frac{1}{s}\frac{b(1/s)}{(\log a)^2 - (1/s)^2 b^2} = \frac{1}{s}G_a\left(\frac{1}{s}\right).$$

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Hence, we can say from (3.2) and (3.3) that the modified Sumudu transform is the theoretical dual transform to the modified Laplace transform.

We now introduce modified inverse Sumudu transform of a function $f(t)$ as follows:

$$f(t) = \mathcal{S}_a^{-1}[G_a(u)] = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} a^{ut} G_a(u)\,du, \qquad (\gamma > 0). \tag{3.4}$$

It can be readily seen that modified inverse Sumudu transform is a linear transform, namely, for $\beta, \omega \in \mathbb{R}$,

$$\mathcal{S}_a^{-1}[\beta G_a(u) + \omega H_a(u)] = \beta \mathcal{S}_a^{-1}[G_a(u)] + \omega \mathcal{S}_a^{-1}[H_a(u)] = \beta f(t) + \omega g(t),$$

where $\mathcal{S}_a[f(t)] = G_a(u)$ and $\mathcal{S}_a[g(t)] = H_a(u)$.

From (3.4) and Theorem 10, we get

$$\mathcal{S}_a^{-1}[G_a(u) H_a(u)] = \frac{1}{u}(f * g)(t).$$

Some examples of the modified inverse Sumudu transform are stated below.

$$\mathcal{S}_a^{-1}\left[\frac{1}{\log a}\right] = 1 \quad \text{by (2.4)}$$

$$\mathcal{S}_a^{-1}[u^n] = \frac{(\log a)^{n+1} t^n}{n!} \quad \text{by (2.5)}$$

$$\mathcal{S}_a^{-1}\left[\frac{1}{\log a - bu}\right] = e^{bt} \quad \text{by (2.7)}$$

$$\mathcal{S}_a^{-1}\left[\frac{u}{(\log a)^2 + b^2u^2}\right] = \frac{\sin(bt)}{b} \quad \text{by (2.8)}$$

$$\mathcal{S}_a^{-1}\left[\frac{1}{(\log a)^2 + b^2u^2}\right] = \frac{\cos(bt)}{\log a} \quad \text{by (2.8)}$$

$$\mathcal{S}_a^{-1}\left[\frac{u}{(\log a)^2 - b^2u^2}\right] = \frac{\sinh(bt)}{b} \quad \text{by (2.9)}$$

$$\mathcal{S}_a^{-1}\left[\frac{1}{(\log a)^2 - b^2u^2}\right] = \frac{\cos h(bt)}{\log a} \quad \text{by (2.9).}$$

4. Conclusions

Saif et al. [9] defined the modified Laplace transform as follows:

$$_a(f(t)) = F(s;a) = \int_0^\infty a^{-st} f(t)\,dt,$$

if the integral converges. Several properties and interesting formulas for modified Laplace transform were investigated in [9]. Inspired by this study, in this paper, we have considered modified Sumudu transform by the following improper integral:

$$G_a(u) = \mathcal{S}_a[f(t)] = \frac{1}{u}\int_0^\infty a^{\frac{-t}{u}} f(t)\,dt, \quad u \in (-\tau_1, \tau_2) \text{ and } a \in (0,\infty)/\{0\}$$

for

$$f(t) \in A_\lambda = \left\{ f(t) | \exists M, \tau_1, \tau_2 > 0 \text{ such that } |f(t)| < Ma^{1/\tau_j}(|t|), \text{ if } t \in (-1)^j \times [0,\infty) \right\}.$$

Then, we have given many properties and relations covering modified Sumudu transforms of the power function, sine, cosine, hyperbolic sine, hyperbolic cosine, exponential function, and function derivatives. We also attained two shifting properties and a scale preserving theorem for the modified Sumudu transform. Moreover, we have provided modified inverse Sumudu transform and developed some relations and examples.

Furthermore, we have shown that the modified Sumudu transform is the theoretical dual transform to the modified Laplace transform.

Funding

The author has no received any financial support for the research, authorship or publication of this study.

The Declaration of Ethics Committee Approval

The author declares that this document does not require an ethics committee approval or any special permission.

The Declaration of Conflict of Interest/Common Interest

“No conflict of interest or common interest has been declared by the author”.

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