HYBRID NUMBERS WITH FIBONACCI AND LUCAS HYBRID NUMBER COEFFICIENTS

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Abstract. In this paper, we introduce hybrid numbers with Fibonacci and Lucas hybrid number coefficients. We give the Binet formulas, generating functions, exponential generating functions for these numbers. Then we define an associate matrix for these numbers. In addition, using this matrix, we present two different versions of Cassini identitiy of these numbers.

1. Introduction

Recently, in [9], Özdemir defined the set of hybrid numbers which contains complex, dual and hyberbolic numbers as

$$\mathbb{K} = \left\{a + b\mathbf{i} + c\boldsymbol{\epsilon} + d\mathbf{h} : a, b, c, d \in \mathbb{R}, \mathbf{i}^2 = -1, \boldsymbol{\epsilon}^2 = 0, \mathbf{h}^2 = 1, \mathbf{ih} = \mathbf{hi} = \boldsymbol{\epsilon} + \mathbf{i}\right\}.$$

This number system is a generalization of complex $\left(\mathbf{i}^2 = -1\right)$, hyperbolic $\left(\mathbf{h}^2 = 1\right)$ and dual number $\left(\boldsymbol{\epsilon}^2 = 0\right)$ systems. Here, $\mathbf{i}$ is complex unit, $\boldsymbol{\epsilon}$ is dual unit and $\mathbf{h}$ is hyperbolic unit. We call these units as hybrid units.

The conjugate of a hybrid number $K = a + b\mathbf{i} + c\boldsymbol{\epsilon} + d\mathbf{h}$ is defined by

$$\overline{K} = a - b\mathbf{i} - c\boldsymbol{\epsilon} - d\mathbf{h}.$$

From the definition of hybrid numbers, the multiplication table of the hybrid units is given by the following table:

$\bullet$	1	$\mathbf{i}$	$\boldsymbol{\epsilon}$	$\mathbf{h}$
1	1	$\mathbf{i}$	$\boldsymbol{\epsilon}$	$\mathbf{h}$
$\mathbf{i}$	$\mathbf{i}$	-1	$1-\mathbf{h}$	$\boldsymbol{\epsilon}+\mathbf{i}$
$\boldsymbol{\epsilon}$	$\boldsymbol{\epsilon}$	$\mathbf{h}+\mathbf{1}$	0	$-\boldsymbol{\epsilon}$
$\mathbf{h}$	$\mathbf{h}$	$-\boldsymbol{\epsilon}-\mathbf{i}$	$\boldsymbol{\epsilon}$	1

This table shows that the multiplication of hybrid numbers is not commutative. Using the above datas, Özdemir [9] investigated various algebraic and geometric properties of hybrid numbers. For instance, he defined a ring isomorphism between the hybrid number ring $\mathbb{K}$ and the ring of real 2×2 matrices $\mathbb{M}_{2\times 2}$. This map is $\varphi : \mathbb{K} \to \mathbb{M}_{2\times 2}$ where

$$\varphi\left(a + b\mathbf{i} + c\boldsymbol{\epsilon} + d\mathbf{h}\right) = \begin{bmatrix} a + c & b - c + d \\ c - b + d & a - c \end{bmatrix}. \quad (1.1)$$

We refer the reader to [9] for more details and properties about hybrid numbers.

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The well-known Fibonacci and Lucas sequences are defined as (for $n \geq 0$)

$$F_{n+2} = F_{n+1} + F_n$$

and

$$L_{n+2} = L_{n+1} + L_n$$

where $F_0 = 0$, $F_1 = 1$, $L_0 = 2$ and $L_1 = 1$. Note that for $n \geq 1$, $F_{n-1}F_{n+1} - F_n^2 = (-1)^n$ and $L_{n-1}L_{n+1} - L_n^2 = 5\,(-1)^{n+1}$.

In [18], the authors introduced the Fibonacci hybrid numbers and derived some combinatorial properties of these numbers. For $n \geq 0$, they defined the nth Fibonacci hybrid and nth Lucas hybrid numbers as

$$FH_n = F_n + F_{n+1}\mathbf{i} + F_{n+2}\boldsymbol{\epsilon} + F_{n+3}\mathbf{h}$$

and

$$LH_n = L_n + L_{n+1}\mathbf{i} + L_{n+2}\boldsymbol{\epsilon} + L_{n+3}\mathbf{h}$$

where $FH_0 = \mathbf{i}+\boldsymbol{\epsilon}+2\mathbf{h}$, $FH_1 = 1+\mathbf{i}+2\boldsymbol{\epsilon}+3\mathbf{h}$, $LH_0 = 2+\mathbf{i}+3\boldsymbol{\epsilon}+4\mathbf{h}$ and $LH_1 = 1+3\mathbf{i}+4\boldsymbol{\epsilon}+7\mathbf{h}$. They also gave the Binet formulas of these hybrid numbers as

$$FH_n = \frac{\underline{\alpha}\alpha^n - \underline{\beta}\beta^n}{\alpha - \beta}$$

and

$$LH_n = \underline{\alpha}\alpha^n + \underline{\beta}\beta^n,$$

respectively, where $\underline{\alpha} = 1 + \alpha\mathbf{i} + \alpha^2\boldsymbol{\epsilon} + \alpha^3\mathbf{h}$, $\underline{\beta} = 1 + \beta\mathbf{i} + \beta^2\boldsymbol{\epsilon} + \beta^3\mathbf{h}$, $\alpha = \left(1 + \sqrt{5}\right)/2$ and $\beta = \left(1 - \sqrt{5}\right)/2$.

Hybrid number sequences have been studied by many researchers. For instance, in [3], Cerda-Morales studied generalized hybrid Fibonacci numbers and their properties. In [1], Catarino introduced a new sequence of numbers called k-Pell hybrid numbers and gave some algebraic properties involving this sequence. Catarino and Bilgici defined modified k-Pell hybrid sequence in [2]. In [5], Kızılateş investigated the q-Fibonacci and the q-Lucas hybrid numbers and gave some algebraic properties of these numbers. In [6], the same author introduced the Horadam hybrid polynomials called Horadam hybrinomials. Liana et al. studied Pell hybrinomials in [8]. In [11], Soykan and Taşdemir introduced the generalized Tetranacci hybrid numbers and presented some combinatorial properties of these hybrid numbers. In [12–18], Liana and Wloch introduced several hybrid number sequences and polynomials and gave various properties of them. In [20], Tan and Ait-Amrane examined the bi-periodic Horadam hybrid numbers. Taşyurdu [21] defined the Tribonacci and Tribonacci-Lucas hybrid numbers and presented some matrix applications related to these numbers. In [22], Yağmur examined the generalized hybrid Tribonacci numbers. In [19], Şentürk et al. studied Horadam hybrid numbers and obtained various properties.

Recently, in [4], using an associate matrix, Irmak gave various identities about Fibonacci and Lucas quaternions by matrix methods.

In this paper, motivated by the above papers, we introduce hybrid numbers with Fibonacci and Lucas hybrid number coefficients. We give the Binet formulas, generating functions, exponential generating functions for these numbers. Then we define an associate matrix for these numbers. Finally, using this matrix, we present two different versions of Cassini identitiy of these numbers.

2. Main Results

In this section, we firstly define hybrid numbers with Fibonacci and Lucas hybrid number coefficients, respectively. Note that we consider a more general form of hybrid numbers. Then we give Binet formulas, generating functions, exponential generating functions, and some summation formulas for these numbers.

Definition 2.1. *For $n \geq 0$, the nth term of hybrid number with Fibonacci hybrid number coefficients is given by*

$$\mathbb{F}_n = FH_n + FH_{n+1}\mathbf{i} + FH_{n+2}\boldsymbol{\epsilon} + FH_{n+3}\mathbf{h}. \tag{2.1}$$

Definition 2.2. *For $n \geq 0$, the nth term of hybrid numbers with Lucas hybrid number coefficients is given by*

$$\mathbb{L}_n = LH_n + LH_{n+1}\mathbf{i} + LH_{n+2}\boldsymbol{\epsilon} + LH_{n+3}\mathbf{h}. \tag{2.2}$$

Remark 2.3. *If we expand the definitions of $\mathbb{F}_n$ and $\mathbb{L}_n$, we get*

$$\mathbb{F}_n = F_n - F_{n+2} + 2F_{n+3} + F_{n+6} + 2F_{n+1}\mathbf{i} + 2F_{n+2}\boldsymbol{\epsilon} + 2F_{n+3}\mathbf{h}$$

and

$$\mathbb{L}_n = L_n - L_{n+2} + 2L_{n+3} + L_{n+6} + 2L_{n+1}\mathbf{i} + 2L_{n+2}\boldsymbol{\epsilon} + 2L_{n+3}\mathbf{h},$$

respectively.

Theorem 2.4. *For $n \geq 0$, the Binet formulas of hybrid numbers with Fibonacci and Lucas hybrid number coefficients are given by*

$$\mathbb{F}_n = \frac{(\underline{\alpha})^2 \alpha^n - (\underline{\beta})^2 \beta^n}{\alpha - \beta} \tag{2.3}$$

and

$$\mathbb{L}_n = (\underline{\alpha})^2 \alpha^n + (\underline{\beta})^2 \beta^n, \tag{2.4}$$

respectively, where $\underline{\alpha} = 1 + \alpha\mathbf{i} + \alpha^2\boldsymbol{\epsilon} + \alpha^3\mathbf{h}$, $\underline{\beta} = 1 + \beta\mathbf{i} + \beta^2\boldsymbol{\epsilon} + \beta^3\mathbf{h}$, $\alpha = \left(1+\sqrt{5}\right)/2$ and $\beta = \left(1-\sqrt{5}\right)/2$.

Proof. Using the Binet formula of hybrid Fibonacci numbers, we have

$$\begin{aligned}
\mathbb{F}_n &= \frac{\underline{\alpha}\alpha^n - \underline{\beta}\beta^n}{\alpha - \beta} + \frac{\underline{\alpha}\alpha^{n+1} - \underline{\beta}\beta^{n+1}}{\alpha - \beta}\mathbf{i} + \frac{\underline{\alpha}\alpha^{n+2} - \underline{\beta}\beta^{n+2}}{\alpha - \beta}\boldsymbol{\epsilon} + \frac{\underline{\alpha}\alpha^{n+3} - \underline{\beta}\beta^{n+3}}{\alpha - \beta}\mathbf{h} \\
&= \frac{\underline{\alpha}\alpha^n\left(1 + \alpha\mathbf{i} + \alpha^2\boldsymbol{\epsilon} + \alpha^3\mathbf{h}\right) - \underline{\beta}\beta^n\left(1 + \beta\mathbf{i} + \beta^2\boldsymbol{\epsilon} + \beta^3\mathbf{h}\right)}{\alpha - \beta} \\
&= \frac{(\underline{\alpha})^2 \alpha^n - (\underline{\beta})^2 \beta^n}{\alpha - \beta}.
\end{aligned}$$

By a similar calculation, we obtain

$$\mathbb{L}_n = (\underline{\alpha})^2 \alpha^n + (\underline{\beta})^2 \beta^n$$

□

Theorem 2.5. *The generating functions of hybrid numbers with Fibonacci and Lucas hybrid number coefficients are*

$$F(x) = \sum_{n\geq 0} \mathbb{F}_n x^n = \frac{11 + 7x + 2\mathbf{i} + 2(1+x)\boldsymbol{\epsilon} + (4+2x)\mathbf{h}}{1 - x - x^2}$$

and

$$L(x)=\sum_{n\geq 0}\mathbb{L}_n x^n=\frac{25+15x+2(1+2x)\,\mathbf{i}+2(3+x)\,\boldsymbol{\epsilon}+2(4+3x)\,\mathbf{h}}{1-x-x^2},$$

recpectively.

Proof. By taking the generating function of both sides of eq. (2.1), we directly have

$$\begin{aligned}\sum_{n\geq 0}\mathbb{F}_n x^n &= \sum_{n\geq 0}FH_n x^n+\left(\sum_{n\geq 0}FH_{n+1}x^n\right)\mathbf{i}+\left(\sum_{n\geq 0}FH_{n+2}x^n\right)\boldsymbol{\epsilon}+\left(\sum_{n\geq 0}FH_{n+3}x^n\right)\mathbf{h}\\ &= \frac{11+7x+2\mathbf{i}+2(1+x)\,\boldsymbol{\epsilon}+(4+2x)\,\mathbf{h}}{1-x-x^2}.\end{aligned}$$

Similarly, we get

$$\begin{aligned}\sum_{n\geq 0}\mathbb{L}_n x^n &= \sum_{n\geq 0}LH_n x^n+\left(\sum_{n\geq 0}LH_{n+1}x^n\right)\mathbf{i}+\left(\sum_{n\geq 0}LH_{n+2}x^n\right)\boldsymbol{\epsilon}+\left(\sum_{n\geq 0}LH_{n+3}x^n\right)\mathbf{h}\\ &= \frac{25+15x+2(1+2x)\,\mathbf{i}+2(3+x)\,\boldsymbol{\epsilon}+2(4+3x)\,\mathbf{h}}{1-x-x^2}.\end{aligned}$$

□

Theorem 2.6. *For $m,n\in\mathbb{Z}$, the generating functions of $\mathbb{F}_{n+m}$ and $\mathbb{L}_{n+m}$ are*

$$\begin{aligned}\sum_{n\geq 0}\mathbb{F}_{n+m}x^n &= \frac{FH_m+FH_{m-1}x}{1-x-x^2}+\left(\frac{FH_{m+1}+FH_m x}{1-x-x^2}\right)\mathbf{i}\\ &\quad+\left(\frac{FH_{m+2}+FH_{m+1}x}{1-x-x^2}\right)\boldsymbol{\epsilon}+\left(\frac{FH_{m+3}+FH_{m+2}x}{1-x-x^2}\right)\mathbf{h}\end{aligned}$$

and

$$\begin{aligned}\sum_{n\geq 0}\mathbb{L}_{n+m}x^n &= \frac{LH_m+LH_{m-1}x}{1-x-x^2}+\left(\frac{LH_{m+1}+LH_m x}{1-x-x^2}\right)\mathbf{i}\\ &\quad+\left(\frac{LH_{m+2}+LH_{m+1}x}{1-x-x^2}\right)\boldsymbol{\epsilon}+\left(\frac{LH_{m+3}+LH_{m+2}x}{1-x-x^2}\right)\mathbf{h}\end{aligned}$$

respectively.

Proof. The proof is similar to the proof of above theorem. □

Theorem 2.7. *Exponential generating functions of $\mathbb{F}_n$ and $\mathbb{L}_n$ are given by*

$$\sum_{n\geq 0}\mathbb{F}_n\frac{x^n}{n!}=\frac{(\underline{\alpha})^2e^{\alpha x}-(\underline{\beta})^2e^{\beta x}}{\alpha-\beta}$$

and

$$\sum_{n\geq 0}\mathbb{L}_n\frac{x^n}{n!}=(\underline{\alpha})^2e^{\alpha x}+(\underline{\beta})^2e^{\beta x},$$

respectively.

Proof. Using the eq. (2.3) and eq. (2.4), we obtain

$$\begin{aligned}\sum_{n\geq 0}\mathbb{F}_n\frac{x^n}{n!} &= \sum_{n\geq 0}\left(\frac{(\underline{\alpha})^2\alpha^n-(\underline{\beta})^2\beta^n}{\alpha-\beta}\right)\frac{x^n}{n!}\\ &= \frac{(\underline{\alpha})^2}{\alpha-\beta}\sum_{n\geq 0}\frac{(\alpha x)^n}{n!}-\frac{(\underline{\beta})^2}{\alpha-\beta}\sum_{n\geq 0}\frac{(\beta x)^n}{n!}\\ &= \frac{(\underline{\alpha})^2}{\alpha-\beta}e^{\alpha x}-\frac{(\underline{\beta})^2}{\alpha-\beta}e^{\beta x}\\ &= \frac{(\underline{\alpha})^2e^{\alpha x}-(\underline{\beta})^2e^{\beta x}}{\alpha-\beta}\end{aligned}$$

and

$$\begin{aligned}\sum_{n\geq 0}\mathbb{L}_n\frac{x^n}{n!} &= \sum_{n\geq 0}\left((\underline{\alpha})^2\alpha^n+(\underline{\beta})^2\beta^n\right)\frac{x^n}{n!}\\ &= (\underline{\alpha})^2\sum_{n\geq 0}\frac{(\alpha x)^n}{n!}+(\underline{\beta})^2\sum_{n\geq 0}\frac{(\beta x)^n}{n!}\\ &= (\underline{\alpha})^2e^{\alpha x}+(\underline{\beta})^2e^{\beta x}.\end{aligned}$$

□

Now we give some summation formulas for the numbers $\mathbb{F}_n$ and $\mathbb{L}_n$ without proof.

Proposition 2.8. *For the $\mathbb{F}_n$ and $\mathbb{L}_n$, we have*

$$\sum_{k=0}^{n}\mathbb{F}_k=\mathbb{F}_{n+2}-(18+2\mathbf{i}+4\boldsymbol{\epsilon}+6\mathbf{h}),$$

$$\sum_{k=0}^{n}\mathbb{L}_k=\mathbb{L}_{n+2}-(40+6\mathbf{i}+8\boldsymbol{\epsilon}+14\mathbf{h}),$$

$$\sum_{k=0}^{n}\binom{n}{k}\mathbb{F}_k=\mathbb{F}_{2n},$$

$$\sum_{k=0}^{n}\binom{n}{k}\mathbb{L}_k=\mathbb{L}_{2n}.$$

3. A Matrix Approach For Hybrid Numbers with Fibonacci and Lucas Hybrid Number Coefficients

Firstly, let us consider the following matrix:

$$Q=\begin{pmatrix}1 & 1\\ 1 & 0\end{pmatrix}.$$

This Q-matrix was studied by Charles H. King [7] in 1960 for his Master's thesis. It is well-known that

$$Q^n=\begin{pmatrix}F_{n+1} & F_n\\ F_n & F_{n-1}\end{pmatrix}.$$

In 1963, Hoggatt and Ruggles [10] was introduced the following R-matrix:

$$R = \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}.$$

It is easily seen that

$$RQ^n = \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix} = \begin{pmatrix} L_{n+1} & L_n \\ L_n & L_{n-1} \end{pmatrix}.$$

Now, motivated by the [4], we define an associate matrix as

$$A = \begin{pmatrix} 18 + 2\mathbf{i} + 4\boldsymbol{\epsilon} + 6\mathbf{h} & 11 + 2\mathbf{i} + 2\boldsymbol{\epsilon} + 4\mathbf{h} \\ 11 + 2\mathbf{i} + 2\boldsymbol{\epsilon} + 4\mathbf{h} & 7 + 2\boldsymbol{\epsilon} + 2\mathbf{h} \end{pmatrix}.$$

Then we can easily see that

$$Q^n A = \begin{pmatrix} \mathbb{F}_{n+1} & \mathbb{F}_n \\ \mathbb{F}_n & \mathbb{F}_{n-1} \end{pmatrix} \tag{3.1}$$

and

$$RQ^n A = \begin{pmatrix} \mathbb{L}_{n+1} & \mathbb{L}_n \\ \mathbb{L}_n & \mathbb{L}_{n-1} \end{pmatrix}. \tag{3.2}$$

Theorem 3.1 (First type of Cassini Identity)**.** *For $n \geq 1$, we have*

$$\mathbb{F}_{n-1}\mathbb{F}_{n+1} - \mathbb{F}_n^2 = (-1)^n (1 - 34\mathbf{i} + 12\boldsymbol{\epsilon} - 6\mathbf{h})$$

and

$$\mathbb{L}_{n-1}\mathbb{L}_{n+1} - \mathbb{L}_n^2 = 5(-1)^{n+1} (1 - 34\mathbf{i} + 12\boldsymbol{\epsilon} - 6\mathbf{h})$$

respectively.

Proof. By using the matrices (3.1) and (3.2), we have

$$\begin{aligned} \mathbb{F}_{n-1}\mathbb{F}_{n+1} - \mathbb{F}_n^2 &= \begin{vmatrix} \mathbb{F}_{n+1} & \mathbb{F}_n \\ \mathbb{F}_n & \mathbb{F}_{n-1} \end{vmatrix} \\ &= |Q^n A| \\ &= (-1)^n \left[(7 + 2\boldsymbol{\epsilon} + 2\mathbf{h})(18 + 2\mathbf{i} + 4\boldsymbol{\epsilon} + 6\mathbf{h}) - (11 + 2\mathbf{i} + 2\boldsymbol{\epsilon} + 4\mathbf{h})^2 \right] \\ &= (-1)^n (1 - 34\mathbf{i} + 12\boldsymbol{\epsilon} - 6\mathbf{h}) \end{aligned}$$

and

$$\begin{aligned} \mathbb{L}_{n-1}\mathbb{L}_{n+1} - \mathbb{L}_n^2 &= \begin{vmatrix} \mathbb{L}_{n+1} & \mathbb{L}_n \\ \mathbb{L}_n & \mathbb{L}_{n-1} \end{vmatrix} \\ &= |RQ^n A| \\ &= 5(-1)^{n+1} \left[(7 + 2\boldsymbol{\epsilon} + 2\mathbf{h})(18 + 2\mathbf{i} + 4\boldsymbol{\epsilon} + 6\mathbf{h}) - (11 + 2\mathbf{i} + 2\boldsymbol{\epsilon} + 4\mathbf{h})^2 \right] \\ &= 5(-1)^{n+1} (1 - 34\mathbf{i} + 12\boldsymbol{\epsilon} - 6\mathbf{h}). \end{aligned}$$

□

Theorem 3.2 (Second type of Cassini Identity)**.** *For $n \geq 1$, we have*

$$\mathbb{F}_{n+1}\mathbb{F}_{n-1} - \mathbb{F}_n^2 = (-1)^n (1 - 26\mathbf{i} + 28\boldsymbol{\epsilon} - 14\mathbf{h})$$

and

$$\mathbb{L}_{n+1}\mathbb{L}_{n-1} - \mathbb{L}_n^2 = 5(-1)^{n+1} (1 - 26\mathbf{i} + 28\boldsymbol{\epsilon} - 14\mathbf{h}).$$

Proof. Again, by using the matrices (3.1) and (3.2), we have

$$\begin{aligned}\mathbb{F}_{n+1}\mathbb{F}_{n-1}-\mathbb{F}_n^2 &= (-1)^n\left[(18+2\mathbf{i}+4\boldsymbol{\epsilon}+6\mathbf{h})(7+2\boldsymbol{\epsilon}+2\mathbf{h})-(11+2\mathbf{i}+2\boldsymbol{\epsilon}+4\mathbf{h})^2\right]\\ &= (-1)^n(1-26\mathbf{i}+28\boldsymbol{\epsilon}-14\mathbf{h})\end{aligned}$$

and

$$\begin{aligned}\mathbb{L}_{n+1}\mathbb{L}_{n-1}-\mathbb{L}_n^2 &= 5(-1)^{n+1}\left[(18+2\mathbf{i}+4\boldsymbol{\epsilon}+6\mathbf{h})(7+2\boldsymbol{\epsilon}+2\mathbf{h})-(11+2\mathbf{i}+2\boldsymbol{\epsilon}+4\mathbf{h})^2\right]\\ &= 5(-1)^{n+1}(1-26\mathbf{i}+28\boldsymbol{\epsilon}-14\mathbf{h}).\end{aligned}$$

□

Now, let us define the conjugate matrix of A as

$$\overline{A}=\begin{pmatrix}18-2\mathbf{i}-4\boldsymbol{\epsilon}-6\mathbf{h} & 11-2\mathbf{i}-2\boldsymbol{\epsilon}-4\mathbf{h}\\ 11-2\mathbf{i}-2\boldsymbol{\epsilon}-4\mathbf{h} & 7-2\boldsymbol{\epsilon}-2\mathbf{h}\end{pmatrix}. \tag{3.3}$$

Thus, using the matrix $\overline{A}$, we can give the two types of Cassini identity for the conjugate hybrid numbers with Fibonacci and Lucas hybrid number coefficient respectively. Note that

$$\begin{aligned}\overline{A}Q^n &= \begin{pmatrix}18-2\mathbf{i}-4\boldsymbol{\epsilon}-6\mathbf{h} & 11-2\mathbf{i}-2\boldsymbol{\epsilon}-4\mathbf{h}\\ 11-2\mathbf{i}-2\boldsymbol{\epsilon}-4\mathbf{h} & 7-2\boldsymbol{\epsilon}-2\mathbf{h}\end{pmatrix}\begin{pmatrix}F_{n+1} & F_n\\ F_n & F_{n-1}\end{pmatrix}\\ &= \begin{pmatrix}\overline{\mathbb{F}}_{n+1} & \overline{\mathbb{F}}_n\\ \overline{\mathbb{F}}_n & \overline{\mathbb{F}}_{n-1}\end{pmatrix}\end{aligned}$$

and

$$\begin{aligned}\overline{A}RQ^n &= \begin{pmatrix}18-2\mathbf{i}-4\boldsymbol{\epsilon}-6\mathbf{h} & 11-2\mathbf{i}-2\boldsymbol{\epsilon}-4\mathbf{h}\\ 11-2\mathbf{i}-2\boldsymbol{\epsilon}-4\mathbf{h} & 7-2\boldsymbol{\epsilon}-2\mathbf{h}\end{pmatrix}\begin{pmatrix}1 & 2\\ 2 & -1\end{pmatrix}\begin{pmatrix}F_{n+1} & F_n\\ F_n & F_{n-1}\end{pmatrix}\\ &= \begin{pmatrix}\overline{\mathbb{L}}_{n+1} & \overline{\mathbb{L}}_n\\ \overline{\mathbb{L}}_n & \overline{\mathbb{L}}_{n-1}\end{pmatrix}.\end{aligned}$$

Theorem 3.3. *For $n \geq 1$, we have*

$$\overline{\mathbb{F}}_{n-1}\overline{\mathbb{F}}_{n+1}-\left(\overline{\mathbb{F}}_n\right)^2=(-1)^n(1+26\mathbf{i}-28\boldsymbol{\epsilon}+14\mathbf{h}),$$

$$\overline{\mathbb{F}}_{n+1}\overline{\mathbb{F}}_{n-1}-\left(\overline{\mathbb{F}}_n\right)^2=(-1)^n(1+34\mathbf{i}-12\boldsymbol{\epsilon}+6\mathbf{h}),$$

$$\overline{\mathbb{L}}_{n-1}\overline{\mathbb{L}}_{n+1}-\left(\overline{\mathbb{L}}_n\right)^2=5(-1)^{n+1}(1+26\mathbf{i}-28\boldsymbol{\epsilon}+14\mathbf{h}),$$

$$\overline{\mathbb{L}}_{n+1}\overline{\mathbb{L}}_{n-1}-\left(\overline{\mathbb{L}}_n\right)^2=5(-1)^{n+1}(1+34\mathbf{i}-12\boldsymbol{\epsilon}+6\mathbf{h}).$$

4. Conclusion

In this paper, we have introduced hybrid numbers with Fibonacci and Lucas hybrid number coefficients. We have given the Binet formulas, generating functions, exponential generating functions, some summation formulas for these numbers. Then we have defined an associate matrix for these numbers. Using this matrix, we have given two different versions of Cassini identitiy of these numbers. For the interest of the readers of our paper, the results given here have the potential to motivate further researchers of the subject of the higher order hybrid numbers with Fibonacci and Lucas hybrid number coefficients.

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