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Article

On solution existence of MHD Casson Nanofluid transportation across an extending cylinder through porous media and evaluation of priori bounds

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Abstract: It is a theoretical exportation for mass transpiration and thermal transpiration of Casson nanofluid over an extending cylindrical surface. The Stagnation point flow through porous matrix is influenced by magnetic field of form strength. Appropriate similarity functions are availed to yield the transmuted system of leading differential equations. Existence for the solution of momentum equation is proved for various values of Casson parameter β, magnetic parameter M, porosity parameter K_p and Raynolds number Re in two situations of mass transpiration (suction/injuction). Moreover, uniqueness results are discussed and for skin friction factor are established to attain accuracy for large injection values. Thermal and concentration profiles are delineated numerically by applying Runge-Kutta method and shooting technique.

Keywords: Casson fluid; Magneto-hydrodynamics; Nanofluid; Porous medium; Extending cylinder

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1. Introduction

Non Newtonian fluids do not satisfy the Newton's law of viscosity e.g. juice of apple, few oils, cream, honey, blood, toothpaste etc. Casson is a different type of fluid among all of them. claimed that for some fluids the rheological model is better as compared to the viscoelastic model. And he also favored this model for blood as well as for chocolate. Basically, the sample of casson fluids is made up due to the connections or interactions between the phases of liquids and solid. When yield stress becomes compulsory and it is lower than the shear stress, Cason fluids behaves like solids. e.g. Soup, tomato, honey, etc. Human blood is also an example of Casson fluid. Shah et al. [1] investigated the flow of casson nano fluid along with activation energy as well as the chemical reaction by using the stretched surface. Shah et al. [2] discussed the flow of casson fluid by stretched sheet along with the impact of radiation into contemplation. Hamid et al. [3] investigated the stretched sheet with Mhd casson fluid flow and it's effect on thermal radiation which is acting linearly. Priyadharshini and Ponalagusamy [4] discussed the impact of MHD on the paramerers of blood in stenosed artery along with the magnetic nanoparticals.

Some analytical uses of straight-line flows along with the stretching/shrinking sheet or by the regular string consist in different processing of collecting i.e. industry of polymer, a porous stretching/shrinking of plastic films, artificial filaments, fibers of counterfeit, melting of metals, expulsion of metals, persistent throwing, glass blowing etc. [5]. Firstly, the problem of the stretching sheet was discussed by Sakiadis [[6], [7]]. Awaludin et.al. [8] discussed the boundary layer flow of magnetohydrodynamic over stretching and shrinking sheet. Dzulkifli et. al. [9] analyzed the flow of stagnation point as well as relocation of heat over stretching and stretching sheet by using the nano fluid along with the impact of slip velocity. Bakar et.al.[10] disussed on analysis of relocation of heat

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along with the nanofluid by using stretching / shrinking surface with the impact of suction. Malvandi et al. [11] discussed about the flow of stagnation point by using the nonlinear stretching/shrinking sheet which is a porous surface.

In 1942, Haanas Alfren introduced terminology of MHD "Magnetohydrodynamic". Large number scholars has done researches to understand the properties of MHD and to check these properties impact with various terms of nanofluid Now a days using in various fields of life such as astrophysics, medical science, geography, and many other. Impact of activation energy of Arrhenius over a nonlinear stretching surface with convective third grade nanofluid in MHD flow investigated by Hayat et al. [12]. Nanomaterials treatment regardless of the imposition of MHD streamline considering the melting sheet reviewed by Dinh et al. [13]. Explored MHD nanofluid flow over a porous formation of shrinking walls of entropy conducted by Rashid et al. [14]. Research taken on magnetohydrodynamic current of nanofluid through a vertical permeable plate that flows semi-infinitely by Pavar et al. [15]. Chen et al. [16] studied Mixed convection nanofluid stream in vertical channel entropy production in magnetohydrodynamic.

Nano liquids are potential heat exchange fluids with improved thermos-physical properties and heat trade execution can be associated with various tools for better exhibitions Work nowadays in the area of nano-materials grown rapidly due to its comprehensive implementations in variety of fields. Scholars paid so interest in recent array in this field due to their various applications, heat and mass transfer In the engineering and industrial appliances sector, for example Nuclear reactor cooling, furnace, coolant, polymer Process, filament plastics. Improving fluid thermal conductivity of nanoparticles studied by Choi and Eastman [17]. Nanofluid jet cooling fluid flow and heat transfer analysis on a hot surface with varying roughness studded by Mahdavi et al. [18]. Review the laminar-turbulent transition zone of the heat and fluid streaming of an Al2O3-water nanofluid conducted by Baclot et al [19]. Saif et al. [20] condected research on Jeffrey nanofluid magnetohydrodynamic flow becuse of a curved stretching surface. Three dimensional casson-carreau nanofluid flow numerical scrutinization interrogated by Shahid et al. [21].

Transportation of heat and mass for Casson nano fluids across extending cylindrical surface is elaborated herein. The non-linearity involved in the formulation is tackled by numeric simulation on matlab platform by employing R-K method combined with shooting technique. The innovation of the work highlighted the existence of solution with uniqueness of results and bounds for skin friction.

2. Mathematical Formulation

In the segment, we are concerned with the following incompressible Casson Nanofluid model

$$\left.\begin{aligned} \partial_t\rho + u.\nabla\rho = 0 \\ \rho(\partial_t u + u.\nabla u) = -\nabla\rho - (1+\frac{1}{\beta})\mu\Delta u - f \\ div\ u = 0 \\ \partial_t T + div(u\ T) = \frac{k}{\rho C_p}\nabla^2 T + \tau(D_B\ \partial_r T\ \partial_r C + \frac{D_T}{T_\infty}(\partial_r T)^2) \\ \partial_t C + div(u\ C) = D_B\ \nabla^2 C + + \frac{D_T}{T_\infty}(\partial_r T)^2 \end{aligned}\right\} \tag{1}$$

Consider an incompressible and electrically conducting fluid which flows steady state across an axially extending cylinder (radius = R). There is a non varying magnetic field of intensity B_o acts normally to the axis of symmetry. The temperature T_w and concentration C_w are taken at the cylinder and T_∞ and

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C_∞ are the far field temperature and concentration. Casson fluid parameter is β and k' is the porosity of medium. The formulation in (r, θ, z) is constituted keeping in cantact with the assumptions.

$$\left.\begin{aligned} \frac{\partial(rw)}{\partial z} + \frac{\partial(ru)}{\partial r} = 0, \\ w\frac{\partial w}{\partial z} + u\frac{\partial w}{\partial r} = \nu(1+\frac{1}{\beta})(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r}) - \frac{\sigma B_0^2 w}{\rho} - \frac{\nu}{k'}w, \\ w\frac{\partial T}{\partial z} + u\frac{\partial T}{\partial r} = \alpha(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r}) + \tau(D_B\frac{\partial T}{\partial r}\frac{\partial C}{\partial r} + (\frac{\partial T}{\partial y})^2\frac{D_T}{T_\infty}), \\ w\frac{\partial C}{\partial z} + u\frac{\partial C}{\partial y} = D_B(\frac{\partial^2 C}{\partial r^2} + \frac{1}{r}\frac{\partial C}{\partial r}) + \frac{D_T}{T_\infty}\frac{1}{r}(\frac{\partial}{\partial r}(r\frac{\partial T}{\partial r})). \end{aligned}\right\} \tag{2}$$

with boundary conditions:

$$\left.\begin{aligned} u = U_w,\ w = w_w,\ T = T_w,\ C = C_w,\ n - n_w = 0,\ at\ r = a, \\ w \to 0,\ T \to T_\infty,\ C \to C_\infty,\ as\ r \to \infty. \end{aligned}\right\} \tag{3}$$

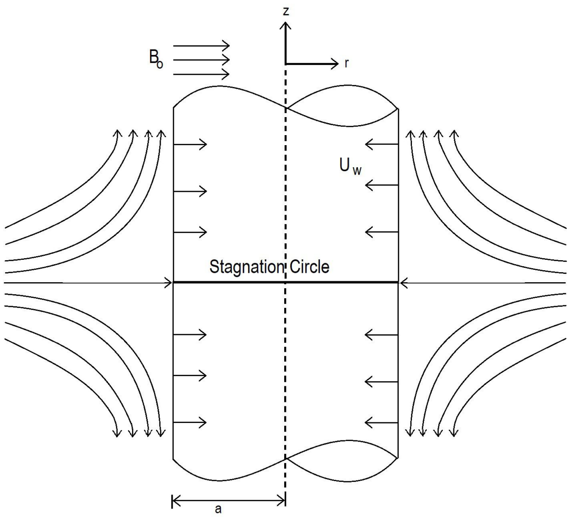

Figure 1. Physical configuration and coordinate system

In order to yield dimensionless form, similarity transformations are entitled as:

$$\xi = (\frac{r}{a})^2,\ u = -ca\frac{f(\xi)}{\sqrt{\xi}},\ w = 2cf'(\xi)z,\ \theta(\xi) - \frac{T - T_\infty}{T_w - T_\infty} = 0,\ \phi(\xi) - \frac{C - C_\infty}{C_w - C_\infty} = 0. \Big\} \tag{4}$$

The first expression in (2) becomes an identity and the remaining's take the form as follow:

$$(1+\frac{1}{\beta})\xi f''' + f'' - Re[f'^2 - ff''] - (M + K_p)f' = 0 \tag{5}$$

$$\xi\theta'' + (1 + PrRef)\theta' + \xi Pr[Nb\theta'\phi' + Nt\theta'^2] = 0 \tag{6}$$

$$\xi\phi'' + (1 + LeRef)\phi' + \frac{Nt}{Nb}[\xi\theta'' + \theta'] = 0 \tag{7}$$

Where the expression (3) are transformed:

$$\left.\begin{aligned} f(1) = \gamma,\ f'(1) = 1,\ \theta(1) = 1,\ \phi(1) = 1, \\ f'(\infty) \to 0,\ \theta(\infty) \to 0,\ \phi(\infty) \to 0, \end{aligned}\right\} \tag{8}$$

3. Existence

Consider the BVP (boundary value problem)

$$(1 + \frac{1}{\beta})\xi f''' + f'' - Re[f'^2 - ff''] - (M + K_p)f' = 0 \tag{9}$$

with

$$f(1) = \gamma,\ f'(1) = 1, f'(\infty) \to 0.$$

In order to get the corresponding IVP (initial value problem), the missing initial condition is assumed to be

$$f''(1) = \epsilon, \tag{10}$$

Here ϵ, as a free parameter is relevant to skin friction parameter $f(\xi;\epsilon)$ denotes the solution. It is because an IVP can be uniquely solved (locally). Thus, a topological shooting argument for some choice of ϵ will yield. For convince, the dependence of f on ϵ may be skipped for some time. The existence of $f'(\xi;\epsilon)$ for all $\xi > 1$ to satisfy Eq. (8). It may yield a solution to BVP. Two sets X and Y are taken as:

$$X = \epsilon < 0|\ a\ first\ point\ \xi_X > 1\ is\ such\ that\ f'(\xi) > 0\ and\ f''(\xi_X) = 0\ on\ [1, \xi_X]$$

and

$$Y = \epsilon < 0|\ a\ first\ point\ \xi_Y > 1\ such\ that\ f'(\xi) < 0\ and\ f''(\xi_Y) = 0\ on\ [1, \xi_Y]$$

Both of these sets are shown to be open and non-empty in the two Lemmas below:
Lemma 1. The set X is non-empty and open.
Proof. From Eqs (9) and (10), for $\xi = 1$,

$$(1 + \frac{1}{\beta})f'''(1) = Re - \epsilon[(1 + \frac{1}{\beta}) + R\gamma] + (M + k_p) \tag{11}$$

When $\epsilon = 0$, it implies that $f'''(1) = Re > 0$. Then initially $f' > 1$ and $f'' > 0$ on $(1, 1+\delta]$ for some $\delta > 0$. The continuity of the solutions of IVP and for $\epsilon < 0$ approaching zero, $f'(\xi;\epsilon)$ approaches $f'(\xi;0)$, i.e., $f'(\xi;\epsilon) > 0$ on $(1, 1+\delta]$ with $f(1+\delta;\epsilon) > 1$. But $f'(\xi;\epsilon) < 1$ and non-increasing for $\xi \in (1, 1+\delta_1)$ for some $0 < \delta_1 < \delta$. f' is to have a minimum if it is to go over 1. So the existence of first point ξ_X such that $f''(\xi_X;\epsilon) = 0$ and $f'(\xi_X;\epsilon) > 0$ on $[1, \xi_X]$. Therefore in case of $\epsilon < 0$ approaching to 0 this implies that ϵ belong to A. In order to show that X is open, let $\bar{\epsilon} \in X$ is open, let $\bar{\in} X$. It is to show that all ϵ approaching $\bar{\epsilon}$ are in X. Then $f''(\xi_X) = 0$ and $0 < f'(\xi_X) < 1$. At $\xi_X(\bar{\epsilon})$, the Eq. (5) yields

$$f'''(\xi_X) = \frac{1}{(1 + \frac{1}{\beta})\xi_X}[Ref'^2(\xi_X) + (M + K_p)f'(\xi_X)] > 0.$$

As the situation for IVP is continuous in its initial conditions, ϵ is approaching close to $\bar{\epsilon}$, $f''(\xi;\epsilon)$ has a root $\xi X(\epsilon)$, near $\xi X(\bar{\epsilon})$ with $f'(\xi;\epsilon) > 0$. Thus $\epsilon \in X$. We are there with the only possibility that $f'' = 0$ and $f' = 0$ simultaneously. When these values are substituted in Eq. (5), then $f''' = 0$ to imply $f'(\xi) = 0$ for all ξ. This is a contradiction to Eq.(8).

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Lemma 2. The set Y is open and non-empty.
Proof. The Eq. (5) after integration yields as:

$$(1+\frac{1}{\beta})\xi f''(\xi) = (1+\frac{1}{\beta})\epsilon + Re\int_1^{\xi}(f'^2(z) - f(z)f''(z))dz + (M+K_p)(f(\xi)-\gamma) \tag{12}$$

and a subsequent integration by parts yields

$$(1+\frac{1}{\beta})\xi f''(\xi) = (1+\frac{1}{\beta})\epsilon + 2Re\int_1^{\xi} f'^2(z)dz + Re[\gamma - f(\xi)f'(\xi) + (M+K_p)(f(\xi)-\gamma)] \tag{13}$$

It is to show that there is $\epsilon < 0$, such that f' is equated to zero in the interval (1,2], say, before $f'' = 0$ in strict. Suppose this assertion is not true and consider.
Case (1A). Taking $f'' < 0, 0 < f' < 1$ for $\xi \in (1,2]$, when $\gamma \geq 0$: By integrating $0 < f' < 1$ yields $\gamma < f < \gamma + \xi - 1$ on (1,2]. Then Eq. (13), provides:

$$f'' < [\frac{\epsilon}{2} + 2Re + Re\gamma + (M+k_p)](1+\frac{1}{\beta})$$

By selecting $\epsilon < -2(M+K_p) - 2Re\gamma - 4Re - 2$ to have $f'' < -1(1+\frac{1}{\beta})$ on (1,2] and thus $f'(2) < 0$ which contradicts $f' > 0$ on (1,2].
Case (1B). $f'' < 0, 0 < f' < 1$ for $\xi \in (1,2], \gamma < 0$. Also, the integration of $0 < f' < 1$ on (1,2] yields $\gamma < f < \gamma + \xi - 1$ on (1,2]. By employing these conditions in Eq. (13) to get

$$f'' < [\frac{\epsilon}{2} + 2Re + Re\gamma + (M+k_p)](1+\frac{1}{\beta})$$

Choosing $\epsilon < -2(M+K_p) - 4Re - 2$ then $f'' < -1(1+\frac{1}{\beta})$ on (1,2] and $f'(2) < 0$, it is a contradiction to $f' > 0$ on (1,2].
Case (2). If there is first point $\xi_1 \in (1,2]$ when $f''(\xi_1) = 0$ with $f'' < 0$ on $(1,\xi_1)$. By taking conditions on f'' as in case (1), it results in

$$\left.\begin{aligned} f'' &< [2Re + \frac{\epsilon}{2}], when \gamma < 0, \\ f'' &< [Re\gamma + 2Re + \frac{\epsilon}{2}], when \gamma \geq 0. \end{aligned}\right\}$$

for $\xi \in (1,\xi_1]$. Choosing

$$\left.\begin{aligned} f'' &< [-4Re, when \gamma < 0, \\ f'' &< -(2Re\gamma + 4Re), when \gamma \geq 0. \end{aligned}\right\}$$

implies that $f''(\xi_1) < 0$ it contradicts $f''(\xi_1) = 0$.
Case (3). We are left with options that $f'' = 0$ and $f' = 0$, but the process of Lemma 1, yields that $f' \equiv 0$ to contradict Eq. (8).
Hence Y is non void. Now it is to see that Y is open, let $\bar{\epsilon} \in Y$ with existence of $\xi_Y(\bar{\epsilon})$ such that $f''\xi_Y(\bar{\epsilon}) < 0$ and $f'\xi_Y(\bar{\epsilon}) = 0$. The continuity of the solution of IVP, for ϵ close to $\bar{\epsilon}$, there exist $\xi_Y(\epsilon)$ with $f''\xi_Y(\epsilon) < 0$ and $f'\xi_Y(\epsilon) = 0$, and so, Y is open.
Thus X and Y are non empty, disjoint and open sets, but $(-\infty, 0)$ is connected and so $XUY \neq (-\infty, 0)$. Thus, there is ϵ^* such that $\epsilon^* \notin X$ and $\epsilon^* \notin Y$. As already noticed it in not possible to have $f' = 0$ and $f'' = 0$ simultaneously; thus, thus only choice is $f''(\xi;\epsilon^*) < 0$ and $f'(\xi;\epsilon^*) > 0$ for all $\xi > 1$.
Since f' is bounded below and decreasing, $f'(\infty;\epsilon^*) = Z$ exists where $0 \leq Z < 1$. It is to see that

$Z = 0$. We let $0 \leq Z < 1$. As $f'' < 0$ for $\xi > 1$, f' is bounded below by $Z > 0$, and so, f approaches to positive infinity. Finally the term ff'' is negative. The Eq. (5) provides as below:

$$\xi f'''(\xi) = [-f''(\xi) - (1+\frac{1}{\beta})^{-1}[Re(ff'' - f'^2) + (M+K_p)f'(\xi)]] > ReC^2 = K > 0$$

for ξ to be large enough, there exists a point $\xi_2 > 1$ and $\xi > \xi_2$ to imply that

$$\xi f'''(\xi) > \frac{K}{2}$$

By integrating the above expression

$$f''(\xi) > f''(\xi_2) + \frac{K}{2}[In\,\xi - In\,\xi_2]\ for\ \xi >\ \xi_2,$$

Let $\xi \to \infty$ then $f'' \to \infty$, it contradicts to the fact that $f'' < 0$. Hence we have $f'(\infty;\epsilon^*) = 0$ the following theorem is established.

Theorem 1. There exists a solution to the boundary value problem for any $Re > 0$ and $-\infty < \gamma < \infty$, to satisfy $f'(\xi) > 0$ and $f''(\xi) < 0$ for all $\xi > 1$.

4. Uniqueness

Now, we prove uniqueness of results:

Theorem 2, If $-\infty < \gamma < \infty$ and $Re > 0$, then we cannot have two solutions for BVP (see 8)

$$(1+\frac{1}{\beta})\xi f'''(\xi) + f''(\xi) - Re(f'^2 - ff'') - (M+K_p)f'(\xi) = 0$$

when $f'(\xi) > 0$.

Proof. From Eq. (5), $f'(\xi;\epsilon^*)$ cannot attain maximum. Thus for a solution with $f'(\xi;\epsilon^*) > 0$, $f''(\xi;\epsilon^*) < 0$. So for any positive solution $0 < f'(\xi;\epsilon^*) < 1$.

Let $v = \frac{\partial f}{\partial \alpha}$. The differentiation of Eq. (5) with respect to ξ yields:

$$(1+\frac{1}{\beta})\xi f^{iv} + f''' - Re[2f'f'' - ff''' - f'f''] - (M+K_p)f'' = 0 \tag{14}$$

$$(1+\frac{1}{\beta})\xi v''' + v'' - Re[2f'v' - vf'' - fv''] - (M+K_p)v' = 0 \tag{15}$$

associated with

$$v(1) = v'(1) = 0, v''(1) = 1. \tag{16}$$

Thus for $\xi > 1$, we have v' positive and increasing and $v' > 0$ and increasing for $\xi > 1$.

It is to show a positive maximum does not exists for $v'(\xi,\epsilon^*)$. Let a maximum exists at first point for which $v > 0$, $v' > 0$, $v'' = 0$ and $v''' \leq 0$. Substituting $v'' = 0$ into Eq. (15) yields

$$(1+\frac{1}{\beta})\xi v''' = Re[2f'v' - vf''] + (M+K_p)v' > 0 \tag{17}$$

It becomes contrary and hence v' cannot have a maxima. So $v' = \frac{\partial f'}{\partial \alpha} > 0$.

IF we let two solutions $f'(\xi;\epsilon^*)$ and $f'(\xi;\epsilon^{**})$ with $\epsilon^{**} > \epsilon^*$, and using Mean Value Theorem

$$f'(\xi;\epsilon^{**}) - f'(\xi;\epsilon^*) = (\frac{\partial f'}{\partial \epsilon})_{\epsilon=\hat{\epsilon}}(\epsilon^{**} - \epsilon^*) = v'(\xi;\hat{\epsilon})(\epsilon^{**} - \epsilon^*) \tag{18}$$

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where $\epsilon^* < \hat{\epsilon} < \epsilon^{**}$. Now v' is bounded below by $L > 0$ for ξ large as it cannot have a maximum. Suppose $M = L(\epsilon^{**} - \epsilon^*)$ and $\xi \to \infty$, From Eq. (18),
$0 = 1 - 1 = f'(\xi; \epsilon^{**}) - f'(\xi; \epsilon^*) = v'(\xi; \hat{\epsilon})(\epsilon^{**} - \epsilon^*) > M > 0$
It becomes contrary.

5. Bounds for skin friction factor.

Bounds are derived for coefficient of skin friction $f''(1) = \epsilon^*$. As $f'(\xi; \epsilon^*)$ be a solution of the BVP to satisfy $f''(1; \epsilon^*) = \epsilon^* < 0$ and cannot have a maximum. It is claimed that for a solution to company the boundary condition (8), yields

$$f'''(1) = \frac{1}{(1 + \frac{1}{\beta})}[Re - \epsilon(1 + Re\gamma) + (M + K_p)] > 0 \tag{19}$$

Consider,
Case-1: Solutions with $f'(\xi; \epsilon^*) > 0$ for $\xi > 1$: let $f'''(1) < 0$ as f' is down concave initially. To satisfy Eq. (8), f' must change concavity at some point. For some ξ_3 such that $f'(\xi_3) > 0$, $f''(\xi_3) < 0$, and $f'''(\xi_3) = 0$ with $f^{iv}(\xi_3) \geq 0$. Differentiating Eq. (5), yields:

$$(1 + \frac{1}{\beta})\xi f^{iv} + (2 + \frac{1}{\beta} + Ref)f''' - Ref'f'' - (M + K_p)f'' = 0,\ 1 < \xi < \infty, \tag{20}$$

From Eq. (20), at $\xi = \xi_3$

$$(1 + \frac{1}{\beta})\xi_3 f^{iv}(\xi_3) = Ref'(\xi_3)f''(\xi_3) + (M + K_p)f''(\xi_3) < 0 \tag{21}$$

Also, seen in Lemma 1, $f'''(\xi_3) = f''(\xi_3) = 0$, so it becomes contrary. Next let $f'''(1) = 0$, in Eq. (20) to get:

$$f^{iv}(1) = \frac{1}{(1 + \frac{1}{\beta})}[Re + (M + K_p)]\epsilon < 0 \tag{22}$$

Then initially, $f''' < 0$ for $\xi > 1$, and f''' cannot change sign.
Case-2: Solution for which $f'(\xi; \epsilon^*) < 0$: let $f'''(1) < 0$ and f' is down concave initially. Because there exist a first point ξ_4 such that $f'(\xi_4) = 0$ and $f''(\xi_4) < 0$, f' is not positive for all ξ. Also, f' should be concave up to satisfy Eq.(8) for some $\xi > \xi_4$ and it attained a minimum. As f' does not attain maximum, f' necessarily increase from its minimum monotonically, and then tends to 0 from below to become concave down.
It becomes clear that, f''' must change sign from minus to plus and back to minus. Thus a point ξ_5 is such that f''' has a positive max, i.e., $f'''(\xi_5) > 0$, $f^{iv}(\xi_5) = 0$, and $f^{(v)}(\xi_5) \leq 0$. The Eq. (20) is differentiated and evaluated at ξ_5 to produce,

$$\xi_5 f^{(v)}(\xi_5) = \frac{1}{(1 + \frac{1}{\beta})}Re(f''(\xi_5))^2 \geq 0 \tag{23}$$

If $f''(\xi_5) \neq 0$, contradiction is arrived: Taking the case $f''(\xi_5) = f^v(\xi_5) = 0$. The Eq. (20) is differentiated two times to have $f^{vi}(\xi_5) = 0$. Then Eq. (20) is differentiated thrice to get a result for $\xi = \xi_5$

$$\xi_5 f^{(vii)}(\xi_5) = \frac{1}{(1 + \frac{1}{\beta})}2Re(f'''(\xi_5))^2 > 0.$$

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So finally, $f^{iv}(\xi_5) = f^v(\xi_5) = f^{vi}(\xi_5) = 0$ with $f^{vii}(\xi_5) > 0$. For ξ nearly greater than ξ_5, f^{iv} is positive and f''' is increasing to contradict if f''' is to possess maximum at ξ_5. We have

$$f'''(1) = \frac{1}{(1+\frac{1}{\beta})}[Re - \epsilon^*(1 + Re\gamma) + (M + K_p)] > 0 \tag{24}$$

This bounds provides useful information, if $\gamma \geq -\frac{1}{Re}$. However, we have

$$\frac{Re + (M + K_p)}{1 + Re\gamma} < \epsilon^*,\ if\ \gamma < -\frac{1}{Re} \tag{25}$$

then an upper bound on ϵ^* can be attained if $\gamma \leq -\frac{2}{R}$. At this stage , it is assumed that

$$f^{iv}(1) = \frac{1}{(1+\frac{1}{\beta})}(Re + M + K_p)\epsilon - \frac{1}{(1+\frac{1}{\beta})}(2 + \frac{1}{\beta} + Re\gamma)[Re - (1 + Re\gamma)\epsilon + (M + K_p)] < 0 \tag{26}$$

First if $f^{iv}(1) > 0$, then there exists a first point ξ_6 such that $f^{vi}(\xi_6) = 0$ with $f^v(\xi_6) \leq 0$; otherwise,

$$f^{iv}(\xi) > 0,\ for\ \xi > 1 \tag{27}$$

It will leads to a contradiction. Integration of Eq. (4.19) yields:

$$f'''(\xi) > K,\ for\ \xi > 1 \tag{28}$$

where $K = \frac{1}{(1+\frac{1}{\beta})}[Re - \epsilon(1 + Re\gamma) + (M + K_p)] > 0$. Integrating second time

$$f''(\xi) > \epsilon + K(\xi - 1),\ for\ \xi > 1 \tag{29}$$

When $\xi \to \infty$, let $f'' \to \infty$ then $f' \to 0$ as needed for Eq. (8).

Table 1. Values of $f''(1)$ for $Re = 1, M = K_p = 0.1\ and\ \beta = 10$.

γ	Lower bounds on $f''(1)$	$f''(1)$ num. approx	Upper bounds on $f''(1)$
-0.5	NA	-1.0007	NA
-1.0	NA	-0.8389	NA
-3.0	-0.600	-0.4432	-0.402
-4.0	-0.400	-0.3414	-0.340
-6.0	-0.240	-0.2260	-0.220
-8.0	-0.170	-0.1663	-0.160
-10.0	-0.133	-0.1320	-0.131

Thus f^{iv} goes to decrease from 0 at some point ξ_6. Differentiate Eq. (20) and evaluate at ξ_6 to get

$$\xi_6 f^{(v)}(\xi_6) = Re(f''(\xi_6))^2 \geq 0$$

If $f''(\xi_6) \neq 0$, it makes a contradiction. If $f''(\xi_6) = 0$, then a similar procedure as above provides $f^{vi}(\xi_6) = 0$ and $f^{vii}(\xi_6) > 0$. Thus $f^{iv} > 0$ for right interval of ξ_6, it is not negative as needed, so $f^{iv}(1) \not> 0$.
If $f^{iv}(1) = 0$ then Eq. (24) becomes $f^v(1) = R\epsilon^2 > 0$ then contradiction is attained through above arguments. Solving for ϵ in Eq. (26) and using Eq. (25) yields.

$$\frac{Re + (M + K_p)}{1 + Re\gamma} < \epsilon^* < \frac{(Re + M + K_p)(2 + \frac{1}{\beta} + Re\gamma)}{(1+\frac{1}{\beta})(R - M - K_p) + (2 + \frac{1}{\beta} + R\gamma)(1 + R\gamma)},\ if\ \gamma \leq -\frac{2}{R} \tag{30}$$

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It can be noticed that both bounds converge to zero, and so, $f''(1)$ converges to zero as γ $(\gamma < 0)$ tends to infinity. Computations of skin friction coefficient $f''(1) = \epsilon^*$ are provided in Table 1. Here sharpening of the bounds on $f''(1)$ is elucidated for a fixed a $Re = 1$, as the parameter γ enhances. The bound are acceptable for the solutions of the BVP if $\gamma \le -\frac{2}{R}$. Now we discuss the bounds for $\gamma > -\frac{1}{2R}$. firstly for $f'(\xi;\epsilon^*) > 0$ when $\xi > 1$, and secondly fot $f'(\xi;\epsilon^* < 0)$. A lemma is presented for proof of Theorem 3.

Lemma 3. Suppose $f'(\xi;\epsilon^*) > 0$ is solution of Eq.(5) with associated conditions (8). If $\gamma > -\frac{1}{2Re}$, then

$$\lim_{\xi\to\infty}[-\xi(f''(\xi))^2 + \frac{2Re}{3}(f'(\xi))^3 + (M+K_p)(f'(\xi))^2] = 0.$$

Proof. From theorem 1 we take $f'(\xi;\epsilon^*) > 0$ for $\xi > 1$ and $f''(\xi;\epsilon^*) < 0$ for $\xi > 1$. Then f is increasing and f' is decreasing function. As $\gamma > -\frac{1}{2Re}$, then $1 - \frac{1}{\beta} + 2Re\gamma > 0$ for $\xi > 1$. Multiplication of Eq.(5) with $f''(\xi)$ and integrating to get

$$\left.\begin{aligned} &\int_1^{\xi}(1-\frac{1}{\beta}+2Ref(z))(f''(z))^2dz - (1+\frac{1}{\beta})\epsilon^2 + \frac{2Re}{3} + (M+K_p) = \\ &\qquad -(1+\frac{1}{\beta})\xi(f''(\xi))^2 + \frac{2Re}{3}(f'(\xi))^3 + (M+K_p)(f'(\xi))^2 \end{aligned}\right\} \tag{31}$$

Here, the LHS of equation Eq.(31) is an increasing function and similarly the RHS. As $f'(\xi;\epsilon^*)$ is a solution to the B.V.P, we have $f' \to 0$ as $\xi \to \infty$. As $-(1+\frac{1}{\beta})\xi(f''(\xi))^2$ increases and bounded above by 0, its limit as $\xi \to \infty$ exists.
Also let limit is $l \neq 0$. Since $\lim_{\xi\to\infty} f'(\xi) = 0$ and $-(1+\frac{1}{\beta})\xi(f''(\xi))^2 < 0$ for $\xi > 1$, we must have $l < 0$. Suppose $l = -m$. Keeping in view That RHS of Eq.(31) is increasing, we have

$$-(1+\frac{1}{\beta})\xi(f''(\xi))^2 + \frac{2R}{3}(f'(\xi))^3 + (M+K_p)(f'(\xi))^2 < -m \; for \; \xi \ge 1$$

and by skipping second term on LHS to get:

$$(1+\frac{1}{\beta})\xi(f''(\xi))^2 > m \; for \; \xi \ge 1.$$

It implies as:

$$(f''(\xi) - \sqrt{\frac{m}{(1+\frac{1}{\beta})\xi}})(f''(\xi) + \sqrt{\frac{m}{(1+\frac{1}{\beta})\xi}}) > 0 \; for \; \xi \ge 1,$$

As the second term on the left is negative,

$$f''(\xi) < \sqrt{\frac{m}{(1+\frac{1}{\beta})\xi}} \; for \; \xi \ge 1.$$

Integration of this inequality provides as:

$$f'(\xi) < 1 - 2\sqrt{\frac{m}{(1+\frac{1}{\beta})}}(\sqrt{\xi} - 1) \; for \; \xi \ge 1,$$

and let $\xi \to \infty$ then $f' \to -\infty$ which is contradiction to Eq. (8).

Theorem 3. Let $f'(\xi;\epsilon^*) > 0$ is a solution of Eq.(5) associated with boundary conditions (8). If $\gamma > -\frac{1}{2R}$, then $\epsilon^* < -\sqrt{\frac{1}{(1+\frac{1}{\beta})}[\frac{2Re}{3} + (M+K_p)]}$

Proof. Using Lemma 3 results and letting $\xi \to \infty$ in Eq. (31)

$$\int_1^{\xi}(1-\frac{1}{\beta}+2Ref(z))(f''(z))^2dz = (1+\frac{1}{\beta})\epsilon^2 - \frac{2Re}{3} + (M+K_p) > 0,$$

since $1-\frac{1}{\beta}+2Rf > 0$ for $\xi > 1$. Thus
$\epsilon^* < -\sqrt{\frac{1}{(1+\frac{1}{\beta})}[\frac{2Re}{3}+(M+K_p)]}$.
Although the existence of solutions where $f'(\xi;\epsilon^*) < 0$ is yet an open problem. Suppose such solution exist, then a bound on the skin friction coefficient is established in next Theorem 4. Two lemmas are required for the proof of this bounds

Lemma 4. suppose there exist a solution of Eq.(5) associated with boundary conditions (8) where $f'(\xi;\epsilon^*) < 0$. Then $\lim_{\xi\to\infty}(1+\frac{1}{\beta})\xi f''(\xi) = 0$.

Proof. In preview of the case $\gamma \le -\frac{2}{Re}$, f' must attain a negative minimum and then turn concave down as $f' \to 0$ from below. Thus there exist a point ξ_7 such that $f' < 0$, $f'' > 0$, and $f''' < 0$ for $\xi > \xi_7$. By using these inequalities and rearranging Eq. (5) into the form

$$(1+\frac{1}{\beta})\xi f''' + (1+Ref)f'' - Re(f')^2 - (M+K_p)f' = 0,\ 1 < \xi < \infty, \tag{32}$$

It is concluded that
$f(\xi) > -\frac{1}{Re}$ for $\xi > \xi_7$.
Hence f is decreasing and bounded below for $\xi > \xi_7$, and so, $f(\infty) = l \ge -\frac{1}{Re}$ where l is finite. This results in

$$\lim_{\xi\to\infty} Ref(\xi)f'(\xi) = 0. \tag{33}$$

Hence for all $\epsilon_1 > 0$, there is $\bar{\xi} > \xi_7$ to yield:

$$-\frac{\epsilon_1}{4} < Ref(\xi)f'(\xi) < \frac{\epsilon_1}{4}\ for\ \xi > \bar{\xi}_7. \tag{34}$$

Keeping in view of contradiction, suppose that $\lim_{\xi\to\infty}(1+\frac{1}{\beta})\xi f''(\xi) \ne 0$, there exist an $\epsilon_1 > 0$ and a sequence $\xi_i \to \infty$ such that
$|(1+\frac{1}{\beta})\xi_i f''(\xi_i)| \ge \epsilon_1$ for $i = 1, 2, ...$.
and since $f'' > 0$ for $\xi > \bar{\xi}_7$, we have

$$(1+\frac{1}{\beta})\xi_i f''(\xi_i) \ge \epsilon_1\ for\ \xi_i > \xi_7. \tag{35}$$

For any positive integer N, the inequalities (34)-(35) hold where $\xi_N > \bar{\xi} > \xi_7$. We get

$$(1+\frac{1}{\beta})\xi_i f''(\xi_i) + Ref(\xi_i)f'(\xi_i) > \epsilon_1 - \frac{\epsilon_1}{4} = \frac{3\epsilon_1}{4}\ for\ \xi_i \ge \xi_N. \tag{36}$$

Arrangements of Eq. (13) yields

$$2Re\int_1^{\xi}(f'(z))^2dz + Re\gamma + (1+\frac{1}{\beta})\epsilon = (1+\frac{1}{\beta})\xi f''(\xi) + Ref(\xi)f'(\xi) + Re[(M+K_p)(f(\xi)-\gamma)], \tag{37}$$

here LHS is increasing. It is concluded that the inequality (36) stands for all $\xi \ge \xi_N$ and (36) becomes

$$(1+\frac{1}{\beta})\xi f''(\xi) \ge \frac{3\epsilon_1}{4} - Ref(\xi)f'(\xi)\ for\ \xi \ge \xi_N \tag{38}$$

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and using (34) in (38) yields
$(1+\frac{1}{\beta})\xi f''(\xi) \geq \frac{\epsilon_1}{2}$ for $\xi \geq \xi_N$.
Dividing both sides by ξ and integrating results in
$f'(\xi) \geq f'(\xi_N) + \frac{\epsilon_1}{2}[In\xi - In\xi_N]$ for $\xi \geq \xi_N$.
Finally, suppose $\xi \to \infty$ and $f' \to \infty$ which contradict Eq. (8) and thus proof of lemma is complete.

Lemma 5. Let there exists a solution of Eq.(5) with boundary conditions (8) when $f'(\xi;\epsilon^*) < 0$ provided that $\gamma > -\frac{1}{2R}$, $\int_1^\infty (1-\frac{1}{\beta}+2Rf(z))(f''(z))^2dz > 0$

Proof. It is sufficient to show that $1-\frac{1}{\beta}+2Rf > 0$ for $\xi \geq 1$. From Lemma 4,it is seen that $f' < 0$, $f'' > 0$ and $f''' < 0$ for $\xi > \xi_7$. Hence $f'' > 0$ and decreasing and $f'(\infty)$ exists, then $f''(\infty) = 0$. Suppose $\xi \to \infty$ in Eq.(31) and using Lemma 4 to get

$$\lim_{\xi\to\infty}[-\xi(f''(\xi))^2 + \frac{2Re}{3}(f'(\xi))^3 + (M+K_p)(f'(\xi))^2] = 0,$$

and thus

$$\int_1^\infty (1-\frac{1}{\beta}+2Ref(z))(f''(z))^2dz = (1+\frac{1}{\beta})\epsilon^2 - \frac{2Re}{3} + (M+K_p). \tag{39}$$

Also, we have

$$\left.\begin{aligned}\int_1^\xi (1-\frac{1}{\beta}+2Rf(z))(f''(z))^2dz - (1+\frac{1}{\beta})\epsilon^2 + \frac{2Re}{3} + (M+K_p) = \\ -(1+\frac{1}{\beta})\xi(f''(\xi))^2 + (M+K_p)(f'(\xi))^2 + \frac{2R}{3}(f'(\xi))^3 < 0\ for\ \xi > \xi_7\end{aligned}\right\} \tag{40}$$

It is to note that both terms on the right are negative, and so,

$$\int_1^\xi (1-\frac{1}{\beta}+2Ref(z))(f''(z))^2dz = (1+\frac{1}{\beta})\epsilon^2 - \frac{2Re}{3} + +(M+K_p)\ for\ \xi > \xi_7. \tag{41}$$

Thus $\int_1^\xi (1-\frac{1}{\beta}+2Ref(z))(f''(z))^2dt$ tends to infinity from below, and $1-\frac{1}{\beta}+2Ref$ is to be positive for large values of ξ. Since $\gamma > -\frac{1}{2Re}$, and $1-\frac{1}{\beta}+2Ref$ starts out positive because f' has only one sign change − from positive to negative $-f$ attains one maximum and so does $1-\frac{1}{\beta}+2Ref$. Thus $1-\frac{1}{\beta}+2Ref > 0$ for $\xi \geq 1$ and hence, the proof of lemma.

Theorem 4. Let there is a solution for Eq.(5) associated with the boundary conditions (8) where $f'(\xi;\epsilon^*) < 0$. If $\gamma > -\frac{1}{2Re}$, then $\epsilon^* < min[-\sqrt{\frac{1}{(1+\frac{1}{\beta})}[\frac{2Re}{3}+(M+K_p)]}, -\frac{Re\gamma}{(1+\frac{1}{\beta})}]$

Proof. Suppose $\xi \to \infty$, using Lemma 4,Eq. (33) in Eq.(37) to achieve as below

$$\int_1^\infty (f''(z))^2dz = (\frac{(1+\frac{1}{\beta})\epsilon + Re\gamma}{2Re}) > 0,$$

$$\epsilon^* < -\frac{Re\gamma}{(1+\frac{1}{\beta})} \tag{42}$$

Using Lemma 5 in Eq. (39) to get

$$\epsilon^* < -\sqrt{\frac{1}{(1+\frac{1}{\beta})}[\frac{2Re}{3}+(M+K_p)]},\ if\ \gamma > -\frac{1}{2Re}, \tag{43}$$

From combining the inequalities (42)-(43) we get
$\epsilon^* < min[-\sqrt{\frac{1}{(1+\frac{1}{\beta})}[\frac{2Re}{3}+(M+K_p)]}, -\frac{Re\gamma}{(1+\frac{1}{\beta})}]$, if $\gamma > -\frac{1}{2Re}$.

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6. Results and Discussion

The current results are checked for validation in **Table 2** and **Table 3**. Their acceptable accord with those by Mastroberardino and Siddique [22] has eslablished the accuracy of the present numeric scheme. The outcomes for velocity $f'(\xi)$, temperature $\theta'(\xi)$ and concentration $\phi'(\xi)$ are sketched in **Figure 2** to **Figure 5** for two cases of $\gamma(\gamma = -0.5 and \gamma = 0.5)$ with the variation of other influential parameters. The velocity $f'(\xi)$ vividly decelerated against the increments in magnetic parameter M as well as that of porosity parameter K_p as seen in **Figure 2**. The strength of M means growth of electromagnetic resistive force (Lorentz force) which inherits the flow. Similarly, parameter of porous matrix (K_p) offers enhanced resistance to the velocity. The incremented values of Re and Casson parameter β also slowed the flow velocity $f'(\xi)$ as delineated in **Figure 3**. Here the viscous effects are enhanced (to oppose to) momentum. Furthermore, it is noticed that velocity of flow is faster in case of injection $(\gamma > 0)$ than for suction $(\gamma < 0)$. **Figure 4** exposed that the nanofluid diffusion parameters namely Nb (Brownian diffusion) and Nt (Thermophorsis diffusion) are responsible to raise the temperature function $\theta(\xi)$ but the progressive values of Pr reduced $\theta(\xi)$. The fluid temperature for suction is higher than for injection. The greater values of Le and Re diminish the nanoparticle concentration $\phi(\xi)$ in the boundary layer region as depicted in **Figure 5**.
The absolute values of skin friction are augmented in direct proportion with K_p, M, Re and β for three cases of γ $(\gamma < 0, \gamma = 0, \gamma > 0)$ as enumerated in **Table 4**. Physically, K_p signifies the resistance of porous matrix, M for electromagnetic resistive force, Re (Reynolds number) and β the non-newtonian viscous effects (for Casson fluid). Hence the drag force enhances. **Table 5** indicates that Nusselt number $-\theta'(0)$ increases with Pr and Nt but it diminishes against Nb. Also, the Sherwood number $-\phi'(0)$ exceeds directly with Le and Re.

Table 2. The skin friction coefficient by varying M and Re.

M	Re	Mastroberardino and Siddique[22]		Present Results	
		$\gamma = 0.5$	$\gamma = -0.5$	$\gamma = 0.5$	$\gamma = -0.5$
0	10	-6.62227	-1.67757	-6.6223	-1.6778
2		-6.88470	-1.92938	-6.8847	-1.9294
5	10	-7.24505	-2.27933	-7.2451	-2.2793
2	1	-2.21659	-1.72075	-2.2180	-1.7214
	5	-4.33228	-1.86364	-4.3330	-1.8638
	10	-6.88470	-1.92938	-6.8847	-1.9294

Table 3. Nusselt number table for varying γ, Re, M, and Pr.

γ	Re	M	Pr	Mastroberardino and Siddique[22]	Present Results
0.5	10	2	7	36.60283	36.6027
0.0				6.08375	6.0857
-0.5				0.00002	0.00002
0.5	1			4.57611	4.5741
	5			18.99556	18.9952
	10			36.60283	36.6027
	10	0		36.60105	36.6115
		2		36.60283	36.6027
		5		36.60551	36.5906
		2	0.7	4.18133	4.1801
			2	11.13801	11.1360
			7	36.60283	36.6027

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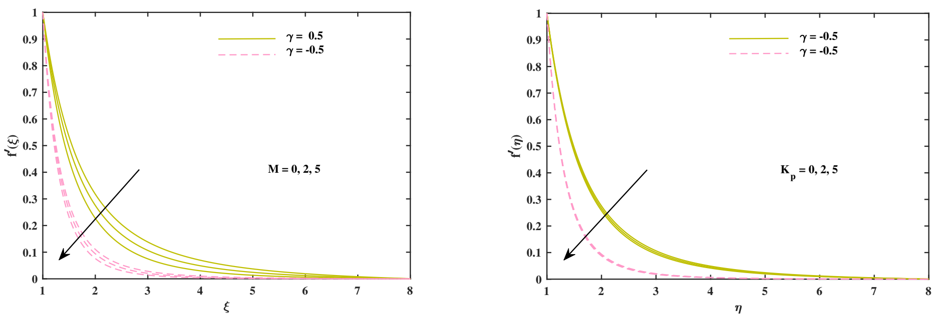
Table 4. Skin friction $-f''(0)$ for varying K_p, M, Re, and β.

K_p	M	Re	β	$\gamma = -0.5$	$\gamma = 0$	$\gamma = 0.5$
0.5	2	10	0.5	1.5608	2.2392	3.1970
1.0				1.6001	2.2818	3.2380
1.5				1.6386	2.3234	3.2781
0.5	0			1.3953	2.0576	3.0218
	2			1.5608	2.2392	3.1970
	5			1.7862	2.4814	3.4307
	2	1		1.2200	1.3002	1.3851
		5		1.4208	1.7861	2.2385
		10		1.5608	2.2392	3.1970
		10	0.5	1.5608	2.2392	3.1970
			1.0	1.7271	2.6988	4.1909
			1.5	1.8000	2.9378	4.7615

Table 5. Nusselt number $-\theta'(0)$ and Sharwood number $-\phi'(0)$ for varying Pr, Nb, Nt, Le and Re.

Pr	Nb	Nt	Le	Re	$-\theta'(0)$	$-\phi'(0)$
0.72	0.1	0.1	1	10	4.2980	
0.1					5.6546	
1.3					7.0521	
0.72	0.1				4.2980	
	0.2				4.1206	
	0.3				3.9488	
	0.1	0.1			4.2980	
		0.2			4.2028	
		0.3			4.1100	
		0.1	1			2.4484
			2			7.8881
			3			13.1081
			1	1		0.6743
				5		1.4877
				10		2.4484



(a) (b)

Figure 2. Plot for velocity profile $f'(\xi)$ with varying values of M and K_p.

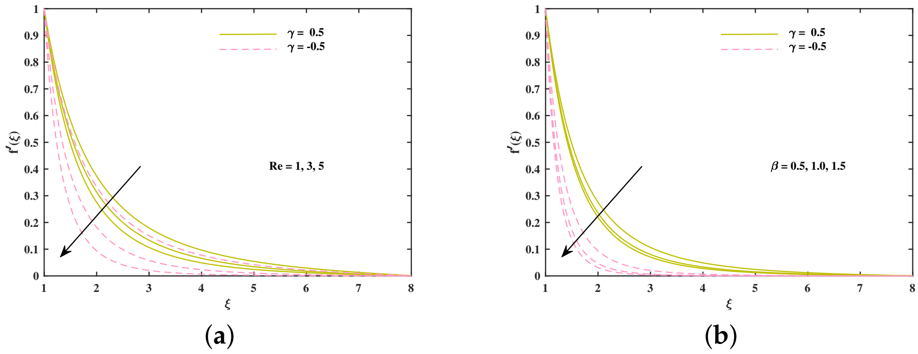


Figure 3. Plot for velocity profile $f'(\xi)$ with varying values of Re and β.

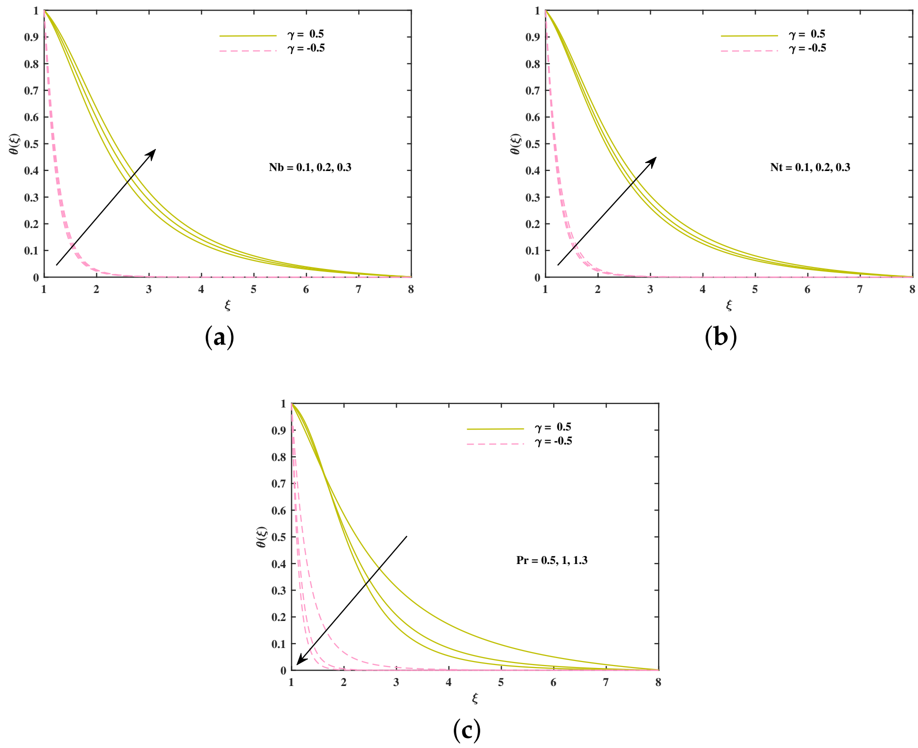


Figure 4. Plot for temperature profile $\theta(\xi)$ with varying values of Nb, Nt and Pr.

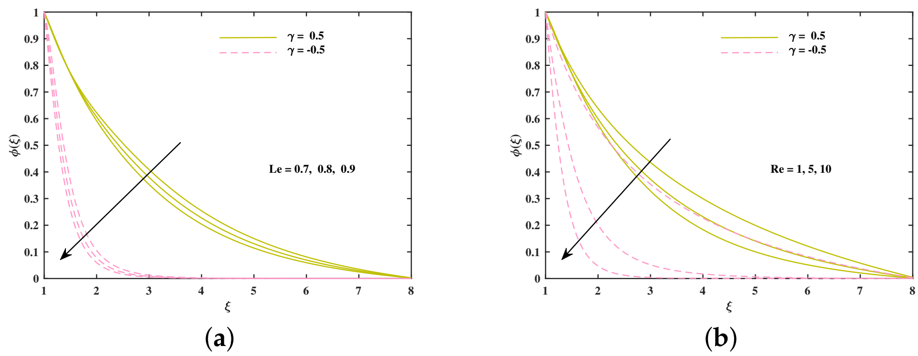


Figure 5. Plot for conentration profile $\phi(\xi)$ with varying values of Le and Re.

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7. Conclusions

Heat transportation under mass transpiration and magnetic field is studied in the flow of Casson nanofluid towards an extending cylinder. The varying parameters of velocity and temperature are elaborated when influential parameters are varied in ranges. We discussed the existence of solution and showed uniqueness of results The priori bounds on skin friction are also determined and explained.

Author Contributions: S.A modeled the problem and wrote the manuscript and thoroughly checked the mathematical modeling and English corrections. S.H. helped in MATLAB coding. I.S writing—review and editing. S.A contributed to the results and discussions. All authors finalized the manuscript after its internal evaluation.

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