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*Article*

# SLICE HOLOMORPHIC FUNCTIONS IN THE UNIT BALL HAVING BOUNDED *L*-INDEX IN DIRECTION

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**Abstract:** Let $\mathbf{b} \in \mathbb{C}^n \setminus \{\mathbf{0}\}$ be a fixed direction. We consider slice holomorphic functions of several complex variables in the unit ball, i.e. we study functions which are analytic in intersection of every slice $\{z^0 + t\mathbf{b} : t \in \mathbb{C}\}$ with the unit ball $\mathbb{B}^n = \{z \in \mathbb{C}^: |z| := \sqrt{|z|_1^2 + \ldots + |z_n|^2} < 1\}$ for any $z^0 \in \mathbb{B}^n$. For this class of functions there is introduced a concept of boundedness of $L$-index in the direction $\mathbf{b}$ where $\mathbf{L} : \mathbb{B}^n \to \mathbb{R}_+$ is a positive continuous function such that $L(z) > \frac{\beta|\mathbf{b}|}{1-|z|}$, where $\beta > 1$ is some constant. For functions from this class we describe local behavior of modulus of directional derivatives on every 'circle' $\{z + t\mathbf{b} : |t| = r/L(z)\}$ with $r \in (0; \beta]$, $t \in \mathbb{C}$, $z \in \mathbb{C}^n$. It is estimated by value of the function at center of the circle. Other propositions concern a connection between boundedness of $L$-index in the direction $\mathbf{b}$ of the slice holomorphic function $F$ and boundedness of $l_z$-index of the slice function $g_z(t) = F(z + t\mathbf{b})$ with $l_z(t) = L(z + t\mathbf{b})$. Also we show that every slice holomorphic and joint continuous function in the unit ball has bounded $L$-index in direction in any domain compactly embedded in the unit ball and for any continuous function $L : \mathbb{B}^n \to \mathbb{R}_+$.

**Keywords:** bounded index; bounded $L$-index in direction; slice function; analytic function; bounded $l$-index; unit ball; local behavior; maximum modulus

**MSC:** 32A10, 32A17, 32A37

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## 1. Introduction

Theory of entire functions of bounded index was initiated by paper of B. Lepson [18]. An entire function $f : \mathbb{C} \to \mathbb{C}$ is called a function of bounded index [18,19] if there exists $m_0 \in \mathbb{Z}_+$ such that for all $z \in \mathbb{C}$ and for all $p \in \mathbb{N}$ one has $\frac{|f^{(p)}(z)|}{p!} \le \max\limits_{0\le k\le m_0} \{\frac{|f^{(k)}(z)|}{k!}\}$. This theory has applications in analytic theory of differential equations [7,17] and its system [20] and value distribution theory [12,13].

Let $\mathbf{b} \in \mathbb{C}^n \setminus \{\mathbf{0}\}$ be a fixed direction. Recently, there was proposed a generalization of notion of bounded index [3,4,10] for so-called slice holomorphic functions in $\mathbb{C}^n$. There was considered two classes of these functions: 1) $\widetilde{\mathcal{H}}_{\mathbf{b}}(\mathbb{B}^n)$ is a class of functions which are holomorphic on every slices $\{z^0 + t\mathbf{b} : t \in S_{z^0}\}$ for each $z^0 \in \mathbb{B}^n$; 2) $\mathcal{H}_{\mathbf{b}}(\mathbb{B}^n)$ be a class of functions from $\widetilde{\mathcal{H}}_{\mathbf{b}}(\mathbb{B}^n)$ which are joint continuous. Note that joint continuity and slice holomophy (in one direction $\mathbf{b}$) do not imply holomorphy in whole $n$-dimensional complex space (see examples in [3]). For these classes there was constructed theory of bounded index in direction. Particularly, there was obtained growth estimates and was described local behavior of holomorphic solutions of some partial differential equations [10]. These slice holomophic functions in $\mathbb{C}^n$ are some generalization of entire functions of several complex

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variables. Together with class of entire functions the analytic functions in the unit ball or in the polydisc are very important objects of investigations in multidimensional complex analysis. W. Rudin [23] wrote that 'The ball is the prototype of two important classes of regions that have been studied in depth, namely the strictly pseudoconvex domains and the bounded symmetric ones'. Thus, it leads to **a general problem** to construct of to construct theory of bounded index for slice holomorphic functions in a bounded symmetric domain. In the paper, we consider this problem for the unit ball because it is an important model example of bounded symmetric domain. Thus, we will study functions which are slice holomorphic in such a bounded domain as unit ball. Its symmetry simplifies many proofs and helps to select main ideas with a minimum of fuss and bother.

Let us introduce some notations and definitions.

Let $\mathbf{0} = (0, \dots, 0)$, $\mathbb{R}_+ = (0, +\infty)$, $\mathbb{R}_+^* = [0, +\infty)$, $\mathbf{b} = (b_1, \dots, b_n) \in \mathbb{C}^n \setminus \{\mathbf{0}\}$ be a given direction, $\mathbb{B}^n = \{z \in \mathbb{C}^n : |z| < 1\}$, $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, $L:\ \mathbb{B}^n \to \mathbb{R}_+$ be a continuous function such that for all $z \in \mathbb{B}^n$

$$L(z) > \frac{\beta|\mathbf{b}|}{1-|z|},\ \beta = \mathrm{const} > 1. \tag{1}$$

For a given $z \in \mathbb{B}^n$ we denote $S_z = \{t \in \mathbb{C} : z + t\mathbf{b} \in \mathbb{B}^n\}$. Clearly, $\mathbb{D} = \mathbb{B}^1$.

The slice functions on $S_z$ for fixed $z^0 \in \mathbb{B}^n$ we will denote as $g_{z^0}(t) = F(z^0 + t\mathbf{b})$ and $l_{z^0}(t) = L(z^0 + t\mathbf{b})$ for $t \in S_z$.

**Definition 1** ([1]). *An analytic function $F : \mathbb{B}^n \to \mathbb{C}$ is called a function of bounded $L$-index in a direction $\mathbf{b}$, if there exists $m_0 \in \mathbb{Z}_+$ such that for every $m \in \mathbb{Z}_+$ and for all $z \in \mathbb{B}^n$ one has*

$$\frac{|\partial_{\mathbf{b}}^m F(z)|}{m! L^m(z)} \le \max_{0\le k\le m_0} \frac{|\partial_{\mathbf{b}}^k F(z)|}{k! L^k(z)}, \tag{2}$$

*where $\partial_{\mathbf{b}}^0 F(z) = F(z), \partial_{\mathbf{b}} F(z) = \sum\limits_{j=1}^n \frac{\partial F(z)}{\partial z_j} b_j, \partial_{\mathbf{b}}^k F(z) = \partial_{\mathbf{b}}\left(\partial_{\mathbf{b}}^{k-1} F(z)\right), k \ge 2.$*

The least such integer number $m_0$, obeying (2), is called the $L$-index in the direction $\mathbf{b}$ of the function $F$ and is denoted by $N_{\mathbf{b}}(F, L)$. If such $m_0$ does not exist, then we put $N_{\mathbf{b}}(F, L) = \infty$, and the function $F$ is said to be of unbounded $L$-index in the direction $\mathbf{b}$ in this case. Let $l : \mathbb{D} \to \mathbb{R}_+$ be a continuous function such that $l(z) > \frac{\beta}{1-|z|}$. For $n = 1$, $\mathbf{b} = 1$, $L(z) \equiv l(z)$ $(z \in \mathbb{D})$ inequality (2) defines an analytic function in the unit disc of bounded $l$-index with the $l$-index $N(F, l) \equiv N_1(F, l)$ (see [21]). Let $N_{\mathbf{b}}(F, L, z^0)$ stands for the $L$-index in the direction $\mathbf{b}$ of the function $F$ at the point $z^0$, i.e., it is the least integer $m_0$, for which inequality (2) is satisfied at this point $z = z^0$. By analogy, the notation $N(f, l, z^0)$ is defined if $n = 1$, i.e., in the case of analytic functions in the unit disc.

There are many papers on entire and slice holomorphic functions of bounded $L$-index in direction. Methods of investigation of properties of these functions often use the restriction of the function to the slices $\{z^0 + t\mathbf{b}:\ t \in \mathbb{C}\}$. For fixed $\mathbf{b} \in \mathbb{C}^n \setminus \{0\}$ and $z^0 \in \mathbb{C}^n$, using considerations from the one-dimensional case, mathematicians obtain the estimates which are uniform in $z^0 \in \mathbb{C}^n$. This is a short description of main idea.

Please note that the positivity and continuity of the function $L$ are weak restrictions to deduce constructive results. Thus, we assume additional restrictions by the function $L$.

Let us denote

$$\lambda_{\mathbf{b}}(\eta) = \sup_{z\in\mathbb{B}^n} \sup_{t_1,t_2\in S_z} \left\{ \frac{L(z+t_1\mathbf{b})}{L(z+t_2\mathbf{b})} : |t_1 - t_2| \le \frac{\eta}{\min\{L(z+t_1\mathbf{b}), L(z+t_2\mathbf{b})\}} \right\}.$$

By $Q_{\mathbf{b}}(\mathbb{B}^n)$ we denote a class of positive continuous functions $L : \mathbb{B}^n \to \mathbb{R}_+$, satisfying the condition

$$\forall \eta \in [0; \beta] :\ \lambda_{\mathbf{b}}(\eta) < +\infty. \tag{3}$$

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Moreover, it is sufficient to require validity of (3) for one value $\eta \in (0;\beta]$.

Besides, we denote by $\langle a, c\rangle = \sum\limits_{j=1}^{n} a_j\overline{c_j}$ the scalar product in $\mathbb{B}^n$, where $a, c \in \mathbb{B}^n$.

Let $\widetilde{\mathcal{H}}_{\mathbf{b}}(\mathbb{B}^n)$ be a class of functions which are holomorphic on every slices $\{z^0 + t\mathbf{b} : t \in S_{z^0}\}$ for each $z^0 \in \mathbb{B}^n$ and let $\mathcal{H}_{\mathbf{b}}(\mathbb{B}^n)$ be a class of functions from $\widetilde{\mathcal{H}}_{\mathbf{b}}(\mathbb{B}^n)$ which are joint continuous. The notation $\partial_{\mathbf{b}}F(z)$ stands for the derivative of the function $g_z(t)$ at the point 0, i.e., for every $p \in \mathbb{N}$ $\partial_{\mathbf{b}}^p F(z) = g_z^{(p)}(0)$, where $g_z(t) = F(z + t\mathbf{b})$ is analytic function of complex variable $t \in \mathbb{C}$ for given $z \in \mathbb{B}^n$. In this research, we will often call this derivative as directional derivative because if $F$ is analytic function in $\mathbb{B}^n$ then the derivatives of the function $g_z(t)$ matches with directional derivatives of the function $F$.

Please note that if $F \in \mathcal{H}_{\mathbf{b}}(\mathbb{B}^n)$ then for every $p \in \mathbb{N}$ $\partial_{\mathbf{b}}F \in \mathcal{H}_{\mathbf{b}}(\mathbb{B}^n)$. It can be proved by using of Cauchy's formula.

Together the hypothesis on joint continuity and the hypothesis on holomorphy in one direction **do not imply** holomorphy in whole $n$-dimensional complex unit ball. We give some examples to demonstrate it. For $n = 2$ let $f : \mathbb{D} \to \mathbb{C}$ be an analytic function, $g : \mathbb{D} \to \mathbb{C}$ be a continuous function. Then $f(z_1)g(z_2)$, $f(z_1) \pm g(z_2)$ are functions which are holomorphic in the direction $(1, 0)$ and are joint continuous in $\mathbb{B}^2$. Moreover, the function $f(z_1 \cdot g(z_2))$ has the same properties if $|g(z)| = 1$. If, in addition, we have performed an affine transformation

$$\begin{cases} z_1 = b_2 z_1' + b_1 z_2', \\ z_2 = b_2 z_1' - b_1 z_2' \end{cases}$$

then the new functions are also holomorphic in the direction $(b_1, b_2)$ and are joint continuous in $\mathbb{B}^2$, where $|b_1 b_2| = 1/2$.

**Definition 2.** *A function $F \in \widetilde{\mathcal{H}}_{\mathbf{b}}(\mathbb{B}^n)$ is said to be of bounded $L$-index in the direction $\mathbf{b}$, if there exists $m_0 \in \mathbb{Z}_+$ such that for all $m \in \mathbb{Z}_+$ and each $z \in \mathbb{B}^n$ inequality (2) is true.*

All notations, introduced above for analytic functions of bounded $L$-index in direction, keep for functions from $\widetilde{\mathcal{H}}_{\mathbf{b}}(\mathbb{B}^n)$.

## 2. Sufficient Sets

Now we prove several assertions that establish a connection between functions of bounded $L$-index in direction and functions of bounded $l$-index of one variable. The similar results for analytic functions in the unit ball were obtained in [2] and for slice holomorphic functions in $\mathbb{C}^n$ [3]. The next proofs use ideas from the mentioned papers.

**Proposition 1.** *If a function $F \in \widetilde{\mathcal{H}}_{\mathbf{b}}(\mathbb{B}^n)$ has bounded $L$-index in the direction $\mathbf{b}$ then for every $z \in \mathbb{B}^n$ the analytic function $g_z(t)$ is of bounded $l_z$-index and $N(g_z, l_z) \le N_{\mathbf{b}}(F, L)$.*

**Proof.** Let $z \in \mathbb{B}^n$, $g(t) \equiv g_z(t)$, $l(t) \equiv l_z(t)$. As for all $p \in \mathbb{N}$

$$g^{(p)}(t) = \partial_{\mathbf{b}}^p F(z + t\mathbf{b}), \tag{4}$$

then by the definition of boundedness of $L$-index in the direction $\mathbf{b}$ for all $t \in S_z$ and $p \in \mathbb{Z}_+$ we obtain

$$\frac{|g^{(p)}(t)|}{p!l^p(t)} = \frac{|\partial_{\mathbf{b}}^p F(z^0 + t\mathbf{b})|}{p!L^p(z^0 + t\mathbf{b})} \le \max\left\{\frac{|\partial_{\mathbf{b}}^k F(z^0 + t\mathbf{b})|}{k!L^k(z^0 + t\mathbf{b})} : 0 \le k \le N_{\mathbf{b}}(F, L)\right\} =$$
$$= \max\left\{\frac{|g^{(k)}(t)|}{k!l^k(t)} : 0 \le k \le N_{\mathbf{b}}(F, L)\right\}.$$

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Hence, we obtain that $g(t)$ is of bounded $l$-index and $N(g,l) \le N_{\mathbf{b}}(F,L)$. Proposition 1 is proved. □

Equality (4) implies that the proposition holds.

**Proposition 2.** *If a function $F \in \widetilde{\mathcal{H}}_{\mathbf{b}}(\mathbb{B}^n)$ has bounded L-index in the direction $\mathbf{b}$ then*

$$N_{\mathbf{b}}(F,L) = \max\{N(g_z,l_z) : z \in \mathbb{B}^n\}.$$

**Theorem 1.** *A function $F \in \widetilde{\mathcal{H}}_{\mathbf{b}}(\mathbb{B}^n)$ has bounded L-index in the direction $\mathbf{b}$ if and only if there exists a number $M > 0$ such that for all $z \in \mathbb{B}^n$ the function $g_z(t)$ is of bounded $l_z$-index with $N(g_z,l_z) \le M < +\infty$, as a function of variable $t \in \mathbb{C}$. Thus, $N_{\mathbf{b}}(F,L) = \max\{N(g_z,l_z) : z \in \mathbb{B}^n\}$.*

**Proof.** The necessity follows from Proposition 1.
*Sufficiency.* Since $N(g_z,l_z) \le M$, there exists $\max\{N(g_z,l_z) : z \in \mathbb{B}^n\}$. We denote $N_{\mathbf{b}}(F,L) = \max\{N(g_z,l_z) : z \in \mathbb{B}^n\} < +\infty$. Suppose that $N_{\mathbf{b}}(F,L)$ is not the $L$-index in the direction $\mathbf{b}$ of the function $F(z)$. It means that there exists $n^* > N_{\mathbf{b}}(F,L)$ and $z^* \in \mathbb{B}^n$ such that

$$\frac{|\partial_{\mathbf{b}}^{n^*}F(z^*)|}{n^*!L^{n^*}(z^*)} > \max\left\{\frac{|\partial_{\mathbf{b}}^{k}F(z^*)|}{k!L^{k}(z^*)} : 0 \le k \le N_{\mathbf{b}}(F,L)\right\}. \tag{5}$$

Since for $g_z(t) = F(z+t\mathbf{b})$ we have $g_z^{(p)}(t) = \partial_{\mathbf{b}}^p F(z+t\mathbf{b})$, inequality (5) can be rewritten as $\frac{\left|g_{z^*}^{(n^*)}(0)\right|}{n^*!l_{z^*}^{n^*}(0)} > \max\left\{\frac{|g_{z^*}^{(k)}(0)|}{k!l_{z^*}^{k}(0)} : 0 \le k \le N_{\mathbf{b}}(F,L)\right\}$, but it is impossible (it contradicts that all $l_z$-indices $N(g_{z_0},l_z)$ are not greater than $N_{\mathbf{b}}(F,L)$). Thus, $N_{\mathbf{b}}(F,L)$ is the $L$-index in the direction $\mathbf{b}$ of the function $F(z)$. Theorem 1 is proved. □

However, maximum can be calculated on a set $A$ with a property $\bigcup_{z^0 \in A}\{z^0 + t\mathbf{b} : t \in S_{z^0}\} = \mathbb{B}^n$. Thus, the following assertion is valid.

**Lemma 1.** *If a function $F \in \widetilde{\mathcal{H}}_{\mathbf{b}}(\mathbb{B}^n)$ has bounded L-index in the direction $\mathbf{b}$ and $j_0$ is chosen with $b_{j_0} \ne 0$ then $N_{\mathbf{b}}(F,L) = \max\{N(g_{z^0},l_{z^0}) : z^0 \in \mathbb{C}^n, z^0_{j_0} = 0\}$ and if $\sum_{j=1}^n b_j \ne 0$ then $N_{\mathbf{b}}(F,L) = \max\{N(g_{z^0},l_{z^0}) : z^0 \in \mathbb{C}^n, \sum_{j=1}^n z_j^0 = 0\}$.*

**Proof.** We prove that for every $z \in \mathbb{B}^n$ there exist $z^0 \in \mathbb{C}^n$ and $t \in S_{z^0}$ with $z = z^0 + t\mathbf{b}$ and $z^0_{j_0} = 0$. Put $t = z_{j_0}/b_{j_0}$, $z_j^0 = z_j - t\mathbf{b}_j$, $j \in \{1,2,\ldots,n\}$. Clearly, $z^0_{j_0} = 0$ for this choice.

However, the point $z^0$ may not be contained in $\mathbb{B}^n$. But there exists $t \in \mathbb{C}$ that $z^0 + t\mathbf{b} \in \mathbb{B}^n$. Let $z^0 \notin \mathbb{B}^n$ and $|z| = R_1 < 1$. Therefore, $|z^0 + t\mathbf{b}| = |z - \frac{z_{j_0}}{b_{j_0}}\mathbf{b} + t\mathbf{b}| = |z + (t - \frac{z_{j_0}}{b_{j_0}})\mathbf{b}| \le |z| + |t - \frac{z_{j_0}}{b_{j_0}}| \cdot |\mathbf{b}| \le R_1 + |t - \frac{z_{j_0}}{b_{j_0}}| \cdot |\mathbf{b}| < 1$. Thus, $|t - \frac{z_{j_0}}{b_{j_0}}| < \frac{1-R_1}{|\mathbf{b}|}$.

In the second part we prove that for every $z \in \mathbb{B}^n$ there exist $z^0 \in \mathbb{C}^n$ and $t \in S_{z^0}$ such that $z = z^0 + t\mathbf{b}$ and $\sum_{j=1}^n z_j^0 = 0$. Put $t = \frac{\sum_{j=1}^n z_j}{\sum_{j=1}^n b_j}$ and $z_j^0 = z_j - t\mathbf{b}_j$, $1 \le j \le n$. Thus, the following equality is valid $\sum_{j=1}^n z_j^0 = \sum_{j=1}^n (z_j - tb_j) = \sum_{j=1}^n z_j - \sum_{j=1}^n b_j t = 0$.
Lemma 1 is proved. □

Note that for a given $z \in \mathbb{B}^n$ we can pick uniquely $z^0 \in \mathbb{C}^n$ and $t \in S_{z^0}$ such that $\sum_{j=1}^n z_j^0 = 0$ and $z = z^0 + t\mathbf{b}$.

**Remark 1.** *If for some $z^0 \in \mathbb{C}^n$ $\{z^0 + t\mathbf{b} :\ t \in \mathbb{C}\} \bigcap \mathbb{B}^n = \varnothing$ then we put $N(g_{z^0},l_{z^0}) = 0$.*

**Theorem 2.** *Let $A_0 \subset \mathbb{C}^n$ be such that $\bigcup_{z \in A_0}\{z + t\mathbf{b} : t \in S_z\} = \mathbb{B}^n$. A function $F \in \widetilde{\mathcal{H}}_{\mathbf{b}}(\mathbb{B}^n)$ is of bounded L-index in the direction $\mathbf{b}$ if and only if there exists $M > 0$ such that for all $z^0 \in A_0$ the*

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*function $g_{z^0}(t)$ is of bounded $l_{z^0}$-index with $N(g_{z^0}, l_{z^0}) \leq M < +\infty$, as a function of variable $t \in S_{z^0}$ and $N_{\mathbf{b}}(F, L) = \max\{N(g_{z^0}, l_{z^0}) : z^0 \in A_0\}$.*

**Proof.** By Theorem 1 the analytic function $F$ is of bounded $L$-index in the direction $\mathbf{b}$ if and only if there exists number $M > 0$ such that for every $z^0 \in \mathbb{B}^n$ the function $g_{z^0}(t)$ is of bounded $l_{z^0}$-index $N(g_{z^0}, l_{z^0}) \leq M < +\infty$, as a function of variable $t \in S_{z^0}$. But in view of property of the set $A_0$ for every $z^0 + t\mathbf{b}$ there exist $\tilde{z}^0 \in A_0$ and $\tilde{t} \in \mathbb{B}_{\tilde{z}^0}$ such that $z^0 + t\mathbf{b} = \tilde{z}^0 + \tilde{t}\mathbf{b}$. In other words, for all $p \in \mathbb{Z}_+$ $(g_{z_0}(t))^{(p)} = (g_{\tilde{z}_0}(\tilde{t}))^{(p)}$. But $\tilde{t}$ depends on $t$. Thus, the condition that $g_{z^0}(t)$ is of bounded $l_{z^0}$-index for all $z^0 \in \mathbb{B}^n$ is equivalent to the condition $g_{\tilde{z}^0}(t)$ is of bounded $l_{\tilde{z}^0}$-index for all $\tilde{z}^0 \in A_0$. □

**Remark 2.** *An intersection of arbitrary hyperplane $H = \{z \in \mathbb{C}^n : \langle z, c\rangle = 1\}$ and the set $\mathbb{B}^n_{\mathbf{b}} = \{z + \frac{1-\langle z,\mathbf{c}\rangle}{\langle \mathbf{b},c\rangle}\mathbf{b} \colon z \in \mathbb{B}^n\}$, where $\langle \mathbf{b}, c\rangle \neq 0$, satisfies conditions of Theorem 2.*

We prove that for every $w \in \mathbb{B}^n$ there exist $z \in H \bigcap \mathbb{B}^n_{\mathbf{b}}$ and $t \in \mathbb{C}$ such that $w = z + t\mathbf{b}$. Choosing $z = w + \frac{1-\langle w,c\rangle}{\langle \mathbf{b},c\rangle}\mathbf{b} \in H \bigcap \mathbb{B}^n_{\mathbf{b}}$, $t = \dfrac{\langle w, c\rangle - 1}{\langle \mathbf{b}, c\rangle}$, we obtain

$$z + t\mathbf{b} = w + \frac{1 - \langle w, c\rangle}{\langle \mathbf{b}, c\rangle}\mathbf{b} + \frac{\langle w, c\rangle - 1}{\langle \mathbf{b}, c\rangle}\mathbf{b} = w.$$

Theorem 3 requires replacement of the space $\widetilde{\mathcal{H}}_{\mathbf{b}}(\mathbb{B}^n)$ by the space $\mathcal{H}_{\mathbf{b}}(\mathbb{B}^n)$. In other words, we use joint continuity in its proof.

**Theorem 3.** *Let $\overline{A} = \mathbb{B}^n$, i.e., $A$ be an everywhere dense set in $\mathbb{B}^n$ and let a function $F \in \mathcal{H}_{\mathbf{b}}(\mathbb{B}^n)$. The function $F$ is of bounded $L$-index in the direction $\mathbf{b}$ if and only if there exists $M > 0$ such that for all $z^0 \in A$ a function $g_{z^0}(t)$ is of bounded $l_{z^0}$-index $N(g_{z^0}, l_{z^0}) \leq M < +\infty$ and $N_{\mathbf{b}}(F, L) = \max\{N(g_{z^0}, l_{z^0}) : z^0 \in A\}$.*

**Proof.** The necessity follows from Theorem 1.

*Sufficiency.* Since $\overline{A} = \mathbb{B}^n$, then for every $z^0 \in \mathbb{B}^n$ there exists a sequence $z^{(m)}$, that $z^{(m)} \to z^0$ as $m \to +\infty$ and $z^{(m)} \in A$ for all $m \in \mathbb{N}$. However, $F(z + t\mathbf{b})$ is of bounded $l_z$-index for all $z \in \overline{A}$ as a function of variable $t$. That is why in view the definition of bounded $l_z$-index there exists $M > 0$ that for all $z \in A$, $t \in \mathbb{C}$, $p \in \mathbb{Z}_+$ $\frac{|g_z^{(p)}(t)|}{p! l^p(t)} \leq \max\left\{\frac{|g_z^{(k)}(t)|}{k! l_z^k(t)} : 0 \leq k \leq M\right\}$.

Substituting instead of $z$ a sequence $z^{(m)} \in A$, $z^{(m)} \to z^0$, we obtain that for every $m \in \mathbb{N}$

$$\frac{|\partial^p_{\mathbf{b}} F(z^{(m)} + t\mathbf{b})|}{p! L^p(z^{(m)} + t\mathbf{b})} \leq \max\left\{\frac{|\partial^k_{\mathbf{b}} F(z^{(m)} + t\mathbf{b})|}{k! L^k(z^{(m)} + t\mathbf{b})} : 0 \leq k \leq M\right\}.$$

However, $F$ and $\partial^p_{\mathbf{b}} F$ are continuous in $\mathbb{B}^n$ for all $p \in \mathbb{N}$ and $L$ is a positive continuous function. Thus, in the obtained expression the limiting transition is possible as $m \to +\infty$ $(z^{(m)} \to z^0)$. Evaluating the limit as $m \to +\infty$ we obtain that for all $z^0 \in \mathbb{B}^n$, $t \in \mathbb{C}$, $m \in \mathbb{Z}_+$

$$\frac{|\partial^p_{\mathbf{b}} F(z^0 + t\mathbf{b})|}{p! L^p(z^0 + t\mathbf{b})} \leq \max\left\{\frac{|\partial^k_{\mathbf{b}} F(z^0 + t\mathbf{b})|}{k! L^k(z^0 + t\mathbf{b})} : 0 \leq k \leq M\right\}.$$

This inequality implies that $F(z + t\mathbf{b})$ is of bounded $L(z + t\mathbf{b})$-index as a function of variable $t$ for every given $z \in \mathbb{B}^n$. Applying Theorem 1 we obtain the desired conclusion. Theorem 3 is proved. □

Remark 2 and Theorem 3 yield the following corollary.

**Corollary 1.** *Let $A_0$ be such that its closure is $\overline{A_0} = \{z \in \mathbb{C}^n : \langle z, c\rangle = 1\} \bigcap \mathbb{B}^n_{\mathbf{b}}$, where $\langle c, \mathbf{b}\rangle \neq 0$, $\mathbb{B}^n_{\mathbf{b}} = \{z + \frac{1-\langle z,\mathbf{c}\rangle}{\langle \mathbf{b},c\rangle}\mathbf{b} \colon z \in \mathbb{B}^n\}$. A function $F \in \mathcal{H}_{\mathbf{b}}(\mathbb{B}^n)$ is of bounded $L$-index in the direction $\mathbf{b}$ if and*

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*only if there exists number $M > 0$ such that for all $z^0 \in A_0$ the function $g_{z^0}(t)$ is of bounded $l_{z^0}$-index with $N(g_{z^0}, l_{z^0}) \le M < +\infty$, as a function of variable $t \in S_{z^0}$. And $N_{\mathbf{b}}(F, L) = \max\{N(g_{z^0}, l_{z^0}) : z^0 \in A_0\}$.*

**Proof.** In view of Remark 2 in Theorem 2 we can take $B_0 = \{z \in \mathbb{C}^n : \langle z, c\rangle = 1\} \cap \{z + \frac{1-\langle z,\mathbf{c}\rangle}{\langle \mathbf{b},c\rangle}\mathbf{b} : z \in \mathbb{B}^n\}$, where $\langle c, \mathbf{b}\rangle \neq 0$. Let $A_0$ be a dense set in $B_0$, $\overline{A_0} = B_0$. Repeating considerations of Theorem 3, we obtain the desired conclusion.

Indeed, the necessity follows from Theorem 1 (in this theorem same condition is satisfied for all $z^0 \in \mathbb{C}^n$, and we need this condition for all $z^0 \in A_0$.

To prove the sufficiency, we use the density of the set $A_0$. Obviously, for every $z^0 \in B_0$ there exists a sequence $z^{(m)} \to z^0$ and $z^{(m)} \in A_0$. But $g_z(t)$ is of bounded $l_z$-index for all $z \in A_0$. Taking the conditions of Corollary 1 into account, for some $M > 0$ and for all $z \in A_0$, $t \in \mathbb{C}$, $p \in \mathbb{Z}_+$ the following inequality holds $\frac{g_z^{(p)}(t)}{p! l_z^p(t)} \le \max\left\{\frac{|g_z^{(k)}(t)|}{k! l_z^k(t)} : 0 \le k \le M\right\}$.

Substituting an arbitrary sequence $z^{(m)} \in A$, $z^{(m)} \to z^0$ instead of $z \in A^0$, we have $\frac{|g^{(p)}_{z^{(m)}}(t)|}{p! l^p_{z^{(m)}}(t)} \le \max\left\{\frac{|g^{(k)}_{z^{(m)}}(t)|}{k! l^k_{z^{(m)}}(t)} : 0 \le k \le M\right\}$, that is

$$\frac{\left|\partial_{\mathbf{b}}^p F(z^{(m)} + t\mathbf{b})\right|}{L^p(z^{(m)} + t\mathbf{b})} \le \max_{0 \le k \le M} \frac{\left|\partial_{\mathbf{b}}^k F(z^{(m)} + t\mathbf{b})\right|}{k! L^k(z^{(m)} + t\mathbf{b})}.$$

However, $F$ is an analytic function in $\mathbb{B}^n$, $L$ is a positive continuous. So we calculate a limit as $m \to +\infty$ $(z^m \to z)$. For all $z^0 \in B_0$, $t \in S_{z^0}$, $m \in \mathbb{Z}_+$ we have

$$\frac{\left|\partial_{\mathbf{b}}^p F(z^0 + t\mathbf{b})\right|}{L^p(z^0 + t\mathbf{b})} \le \max_{0 \le k \le M} \frac{\left|\partial_{\mathbf{b}}^k F(z^0 + t\mathbf{b})\right|}{k! L^k(z^0 + t\mathbf{b})}.$$

Therefore, $F(z^0 + t\mathbf{b})$ is of bounded $L(z^0 + t\mathbf{b})$-index as a function of $t$ at each $z^0 \in \mathbb{B}^n$. By Theorem 3 and Remark 2 $F$ is of bounded $L$-index in the direction $\mathbf{b}$. □

**Proposition 3.** *Let $(r_p)$ be a positive sequence such that $r_p \to 1$ as $p \to \infty$, $D_p = \{z \in \mathbb{C}^n : |z| = r_p\}$, $A_p$ be a dense set in $D_p$ (i.e. $\overline{A}_p = D_p$) and $A = \bigcup_{p=1}^{\infty} A_p$. A function $F \in \mathcal{H}_{\mathbf{b}}(\mathbb{B}^n)$ is of bounded $L$-index in the direction $\mathbf{b}$ if and only if there exists number $M > 0$ such that for all $z^0 \in A$ the function $g_{z^0}(t)$ is of bounded $l_{z^0}$-index $N(g_{z^0}, l_{z^0}) \le M < +\infty$, as a function of variable $t \in S_{z^0}$. And $N_{\mathbf{b}}(F, L) = \max\{N(g_{z^0}, l_{z^0}) : z^0 \in A\}$.*

**Proof.** Theorem 1 implies the necessity of this theorem.

*Sufficiency.* It is easy to prove $\{z + t\mathbf{b} : t \in S_z,\ z \in A\} = \mathbb{B}^n$. Further, we repeat arguments with the proof of sufficiency in Theorem 3 and obtain the desired conclusion. □

## 3. Local Behavior of Directional Derivative

The following proposition is important in theory of functions of bounded index. It initializes series of propositions which are necessary to prove logarithmic criterion of index boundedness. It was first obtained by G. H. Fricke [15] for entire functions of bounded index. Later the proposition was generalized for entire functions of bounded $l$-index [22], analytic functions of bounded $l$-index [16], entire functions of bounded $L$-index in direction [11], functions analytic in a polydisc [5] or in a ball [6] with bounded **L**-index in joint variables, for slice holomorphic functions in $\mathbb{C}^n$ [3].

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**Theorem 4.** *Let $L \in Q_{\mathbf{b}}(\mathbb{B}^n)$. A function $F \in \widetilde{\mathcal{H}}_{\mathbf{b}}(\mathbb{B}^n)$ is of bounded L-index in the direction* **b** *if and only if for each $\eta \in (0;\beta]$ there exist $n_0 = n_0(\eta) \in \mathbb{Z}_+$ and $P_1 = P_1(\eta) \geq 1$ such that for every $z \in \mathbb{B}^n$ there exists $k_0 = k_0(z) \in \mathbb{Z}_+, 0 \leq k_0 \leq n_0$, and*

$$\max\left\{\left|\partial_{\mathbf{b}}^{k_0} F(z+t\mathbf{b})\right| : |t| \leq \frac{\eta}{L(z)}\right\} \leq P_1 \left|\partial_{\mathbf{b}}^{k_0} F(z)\right|. \tag{6}$$

**Proof.** Our proof is based on the proof of appropriate theorem for analytic functions in the unit ball having bounded $L$-index in direction [1] and for slice holomorphic functions in $\mathbb{C}^n$ [3].

*Necessity.* Let $N_{\mathbf{b}}(F;L) \equiv N < +\infty$. Let $[a]$, $a \in \mathbb{R}$, stands for the integer part of the number $a$ in this proof. We denote

$$q(\eta) = [2\eta(N+1)(\lambda_{\mathbf{b}}(\eta))^{2N+1}] + 1.$$

For $z \in \mathbb{B}^n$ and $p \in \{0, 1, \ldots, q(\eta)\}$ we put

$$R_p^{\mathbf{b}}(z,\eta) = \max\left\{\frac{|\partial_{\mathbf{b}}^k F(z+t\mathbf{b})|}{k! L^k(z+t\mathbf{b})} : |t| \leq \frac{p\eta}{q(\eta)L(z)}, 0 \leq k \leq N\right\},$$

$$\widetilde{R}_p^{\mathbf{b}}(z,\eta) = \max\left\{\frac{|\partial_{\mathbf{b}}^k F(z+t\mathbf{b})|}{k! L^k(z)} : |t| \leq \frac{p\eta}{q(\eta)L(z)}, 0 \leq k \leq N\right\}.$$

However, $|t| \leq \frac{p\eta}{q(\eta)L(z)} \leq \frac{\eta}{L(z)}$, then $\lambda_{\mathbf{b}}\left(\frac{p\eta}{q(\eta)}\right) \leq \lambda_{\mathbf{b}}(\eta)$. It is clear that $R_p^{\mathbf{b}}(z,\eta)$, $\widetilde{R}_p^{\mathbf{b}}(z,\eta)$ are well-defined. Moreover,

$$\begin{aligned} & R_p^{\mathbf{b}}(z,\eta) = \\ &= \max\left\{\frac{|\partial_{\mathbf{b}}^k F(z+t\mathbf{b})|}{k! L^k(z)}\left(\frac{L(z)}{L(z+t\mathbf{b})}\right)^k : 0 \leq k \leq N, |t| \leq \frac{p\eta}{q(\eta)L(z)}\right\} \leq \\ &\leq \max\left\{\frac{|\partial_{\mathbf{b}}^k F(z+t\mathbf{b})|}{k! L^k(z)}\left(\lambda_{\mathbf{b}}\left(\frac{p\eta}{q(\eta)}\right)\right)^k : |t| \leq \frac{p\eta}{q(\eta)L(z)}, 0 \leq k \leq N\right\} \leq \\ &\leq \max\left\{\frac{|\partial_{\mathbf{b}}^k F(z+t\mathbf{b})|}{k! L^k(z)}(\lambda_{\mathbf{b}}(\eta))^k : |t| \leq \frac{p\eta}{q(\eta)L(z)}, 0 \leq k \leq N\right\} \leq \\ &\leq (\lambda_{\mathbf{b}}(\eta))^N \max\left\{\frac{|\partial_{\mathbf{b}}^k F(z+t\mathbf{b})|}{k! L^k(z)} : |t| \leq \frac{p\eta}{q(\eta)L(z)}, 0 \leq k \leq N\right\} = \\ &= \widetilde{R}_p^{\mathbf{b}}(z,\eta)(\lambda_{\mathbf{b}}(\eta))^N, \end{aligned} \tag{7}$$

$$\begin{aligned} & \widetilde{R}_p^{\mathbf{b}}(z,\eta) = \\ &= \max\left\{\frac{|\partial_{\mathbf{b}}^k F(z+t\mathbf{b})|}{k! L^k(z+t\mathbf{b})}\left(\frac{L(z+t\mathbf{b})}{L(z)}\right)^k : |t| \leq \frac{p\eta}{q(\eta)L(z)}, 0 \leq k \leq N\right\} \leq \\ &\leq \max\left\{\frac{|\partial_{\mathbf{b}}^k F(z+t\mathbf{b})|}{k! L^k(z+t\mathbf{b})}\left(\lambda_{\mathbf{b}}\left(\frac{p\eta}{q(\eta)}\right)\right)^k : |t| \leq \frac{p\eta}{q(\eta)L(z)}, 0 \leq k \leq N\right\} \leq \\ &\leq \max\left\{(\lambda_{\mathbf{b}}(\eta))^k \frac{|\partial_{\mathbf{b}}^k F(z+t\mathbf{b})|}{k! L^k(z+t\mathbf{b})} : |t| \leq \frac{p\eta}{q(\eta)L(z)}, 0 \leq k \leq N\right\} \leq \\ &\leq (\lambda_{\mathbf{b}}(\eta))^N \max\left\{\frac{|\partial_{\mathbf{b}}^k F(z+t\mathbf{b})|}{k! L^k(z+t\mathbf{b})} : |t| \leq \frac{p\eta}{q(\eta)L(z)}, 0 \leq k \leq N\right\} = \\ &= R_p^{\mathbf{b}}(z,\eta)(\lambda_{\mathbf{b}}(\eta))^N. \end{aligned} \tag{8}$$

Let $k_p^z \in \mathbb{Z}$, $0 \leq k_p^z \leq N$, and $t_p^z \in S_z$, $|t_p^z| \leq \frac{p\eta}{q(\eta)L(z)}$, be such that

$$\widetilde{R}_p^{\mathbf{b}}(z,\eta) = \frac{|\partial_{\mathbf{b}}^{k_p^z} F(z+t_p^z\mathbf{b})|}{k_p^z! L^{k_p^z}(z)}. \tag{9}$$

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However, for every given $z \in \mathbb{B}^n$ the function $g_z(t) = F(z + t\mathbf{b})$ and its derivatives are analytic as functions of variable $t$. Then by the maximum modulus principle, equality (9) holds for $t_p^z$ such that $|t_p^z| = \frac{p\eta}{q(\eta)L(z)}$. We set $\widetilde{t}_p^z = \frac{p-1}{p} t_p^z$. Then

$$|\widetilde{t}_p^z| = \frac{(p-1)\eta}{q(\eta)L(z)}, \tag{10}$$

$$|\widetilde{t}_p^z - t_p^z| = \frac{|t_p^z|}{p} = \frac{\eta}{q(\eta)L(z)}. \tag{11}$$

It follows from (10) and the definition of $\widetilde{R}_{p-1}^{\mathbf{b}}(z,\eta)$ that

$$\widetilde{R}_{p-1}^{\mathbf{b}}(z,\eta) \geq \frac{|\partial_{\mathbf{b}}^{k_p^z} F(z + \widetilde{t}_p^z \mathbf{b})|}{k_p^z! L^{k_p^z}(z)}.$$

Therefore,

$$\begin{aligned} 0 \leq \widetilde{R}_p^{\mathbf{b}}(z,\eta) - \widetilde{R}_{p-1}^{\mathbf{b}}(z,\eta) \leq \frac{\left|\partial_{\mathbf{b}}^{k_p^z} F(z+t_p^z\mathbf{b})\right| - \left|\partial_{\mathbf{b}}^{k_p^z} F(z+\widetilde{t}_p^z\mathbf{b})\right|}{k_p^z! L^{k_p^z}(z)} = \\ = \frac{1}{k_p^z! L^{k_p^z}(z)} \int_0^1 \frac{d}{ds} \left|\partial_{\mathbf{b}}^{k_p^z} F(z + (\widetilde{t}_p^z + s(t_p^z - \widetilde{t}_p^z))\mathbf{b})\right| ds. \end{aligned} \tag{12}$$

For every analytic complex-valued function of real variable $\varphi(s)$, $s \in \mathbb{R}$, the inequality $\frac{d}{ds}|\varphi(s)| \leq \left|\frac{d}{ds}\varphi(s)\right|$ holds, where $\varphi(s) \neq 0$. Applying this inequality to (12) and using the mean value theorem we obtain

$$\begin{aligned} \widetilde{R}_p^{\mathbf{b}}(z,t_0,\eta) - \widetilde{R}_{p-1}^{\mathbf{b}}(z,t_0,\eta) \leq \\ \leq \frac{|t_p^z - \widetilde{t}_p^z|}{k_p^z! L^{k_p^z}(z)} \int_0^1 \left|\partial_{\mathbf{b}}^{k_p^z+1} F(z + (\widetilde{t}_p^z + s(t_p^z - \widetilde{t}_p^z))\mathbf{b})\right| ds = \\ = \frac{|t_p^z - \widetilde{t}_p^z|}{k_p^z! L^{k_p^z}(z)} \left|\partial_{\mathbf{b}}^{k_p^z+1} F(z + (\widetilde{t}_p^z + s^*(t_p^z - \widetilde{t}_p^z))\mathbf{b})\right| = \\ = L(z)(k_p^z+1)|t_p^z - \widetilde{t}_p^z| \frac{|\partial_{\mathbf{b}}^{k_p^z+1} F(z + (\widetilde{t}_p^z + s^*(t_p^z - \widetilde{t}_p^z))\mathbf{b})|}{(k_p^z+1)! L^{k_p^z+1}(z)}, \end{aligned}$$

where $s^* \in [0,1]$. The point $\widetilde{t}_p^z + s^*(t_p^z - \widetilde{t}_p^z)$ belongs to the set

$$\left\{ t \in \mathbb{C} : |t| \leq \frac{p\eta}{q(\eta)L(z)} \leq \frac{\eta}{L(z)} \right\}.$$

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Using the definition of boundedness of $L$-index in direction, the definition of $q(\eta)$, inequalities (7) and (11), for $k_p^z \le N$ we have

$$\widetilde{R}_p^{\mathbf{b}}(z,\eta) - \widetilde{R}_{p-1}^{\mathbf{b}}(z,\eta) \le \frac{|\partial_{\mathbf{b}}^{k_p^z+1} F(z + (\widetilde{t}_p^z + s^*(t_p^z - \widetilde{t}_p^z))\mathbf{b})|}{(k_p^z+1)! L^{k_p^z+1}(z + (\widetilde{t}_p^z + s^*(t_p^z - \widetilde{t}_p^z))\mathbf{b})} \times$$
$$\times \left( \frac{L(z+(\widetilde{t}_p^z+s^*(t_p^z-\widetilde{t}_p^z))\mathbf{b})}{L(z)} \right)^{k_p^z+1} L(z)(k_p^z+1)|t_p^z - \widetilde{t}_p^z| \le \eta \frac{N+1}{q(\eta)} (\lambda_{\mathbf{b}}(\eta))^{N+1} \times$$
$$\times \max\left\{ \frac{|\partial_{\mathbf{b}}^k F(z + (\widetilde{t}_p^z + s^*(t_p^z - \widetilde{t}_p^z))\mathbf{b})|}{k! L^k(z + (\widetilde{t}_p^z + s^*(t_p^z - \widetilde{t}_p^z))\mathbf{b})} : 0 \le k \le N \right\} \le \eta \frac{N+1}{q(\eta)} (\lambda_{\mathbf{b}}(\eta))^{N+1} R_p^{\mathbf{b}}(z,\eta) \le$$
$$\le \frac{\eta(N+1)(\lambda_{\mathbf{b}}(\eta))^{2N+1}}{[2\eta(N+1)(\lambda_{\mathbf{b}}(\eta))^{2N+1}]+1} \widetilde{R}_p^{\mathbf{b}}(z,\eta) \le \frac{1}{2} \widetilde{R}_p^{\mathbf{b}}(z,\eta)$$

It follows that $\widetilde{R}_p^{\mathbf{b}}(z,\eta) \le 2\widetilde{R}_{p-1}^{\mathbf{b}}(z,\eta)$. Using inequalities (7) and (8), we obtain for $R_p^{\mathbf{b}}(z,\eta)$

$$R_p^{\mathbf{b}}(z,\eta) \le 2(\lambda_{\mathbf{b}}(\eta))^N \widetilde{R}_{p-1}^{\mathbf{b}}(z,\eta) \le 2(\lambda_{\mathbf{b}}(\eta))^{2N} R_{p-1}^{\mathbf{b}}(z,\eta).$$

Hence,

$$\begin{gathered} \max\left\{ \frac{|\partial_{\mathbf{b}}^k F(z+t\mathbf{b})|}{k! L^k(z+t\mathbf{b})} : |t| \le \frac{\eta}{L(z)}, 0 \le k \le N \right\} = R_{q(\eta)}^{\mathbf{b}}(z,\eta) \le \\ \le 2(\lambda_{\mathbf{b}}(\eta))^{2N} R_{q(\eta)-1}^{\mathbf{b}}(z,\eta) \le (2(\lambda_{\mathbf{b}}(\eta))^{2N})^2 R_{q(\eta)-2}^{\mathbf{b}}(z,\eta) \le \\ \le \cdots \le (2(\lambda_{\mathbf{b}}(\eta))^{2N})^{q(\eta)} R_0^{\mathbf{b}}(z,\eta) = \\ = (2(\lambda_{\mathbf{b}}(\eta))^{2N})^{q(\eta)} \max\left\{ \frac{|\partial_{\mathbf{b}}^k F(z)|}{k! L^k(z)} : 0 \le k \le N \right\}. \end{gathered} \tag{13}$$

Let $k_z \in \mathbb{Z}, 0 \le k_z \le N$, and $\widetilde{t}_z \in \mathbb{C}$, $|\widetilde{t}_z| = \frac{\eta}{L(z)}$ be such that

$$\frac{|\partial_{\mathbf{b}}^{k_z} F(z)|}{k_z! L^{k_z}(z)} = \max_{0 \le k \le N} \frac{|\partial_{\mathbf{b}}^k F(z)|}{k! L^k(z)},$$

and

$$|\partial_{\mathbf{b}}^{k_z} F(z + \widetilde{t}_z \mathbf{b})| = \max\{|\partial_{\mathbf{b}}^{k_z} F(z + t\mathbf{b})| : |t| \le \eta / L(z)\}.$$

Inequality (13) implies

$$\frac{|\partial_{\mathbf{b}}^{k_z} F(z + \widetilde{t}_z \mathbf{b})|}{k_z! L^{k_z}(z + \widetilde{t}_z \mathbf{b})} \le \max\left\{ \frac{|\partial_{\mathbf{b}}^{k_z} F(z + t\mathbf{b})|}{k_z! L^{k_z}(z + t\mathbf{b})} : |t| = \frac{\eta}{L(z)} \right\} \le$$
$$\le \max\left\{ \frac{|\partial_{\mathbf{b}}^k F(z + t\mathbf{b})|}{k! L^k(z + t\mathbf{b})} : |t| = \frac{\eta}{L(z)}, 0 \le k \le N \right\} \le$$
$$\le (2(\lambda_{\mathbf{b}}(\eta))^{2N})^{q(\eta)} \frac{|\partial_{\mathbf{b}}^{k_z} F(z)|}{k_z! L^{k_z}(z)}.$$

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Hence,

$$\max\left\{|\partial_{\mathbf{b}}^{k_z}F(z+t\mathbf{b})|:|t|\le\eta/L(z)\right\}\le$$
$$\le(2(\lambda_{\mathbf{b}}(\eta))^{2N})^{q(\eta)}\frac{L^{k_z}(z+\widetilde{t_z}\mathbf{b})}{L^{k_z}(z)}|\partial_{\mathbf{b}}^{k_z}F(z)|\le$$
$$\le(2(\lambda_{\mathbf{b}}(\eta))^{2N})^{q(\eta)}(\lambda_{\mathbf{b}}(\eta))^{N}|\partial_{\mathbf{b}}^{k_z}F(z)|\le$$
$$\le(2(\lambda_{\mathbf{b}}(\eta))^{2N})^{q(\eta)}(\lambda_{\mathbf{b}}(\eta))^{N}|\partial_{\mathbf{b}}^{k_z}F(z)|.$$

Thus, we obtain (6) with $n_0=N_{\mathbf{b}}(F,L)$ and

$$P_1(\eta)=(2(\lambda_{\mathbf{b}}(\eta))^{2N})^{q(\eta)}(\lambda_{\mathbf{b}}(\eta))^{N}>1.$$

*Sufficiency.* Suppose that for each $\eta\in(0;\beta]$ there exist $n_0=n_0(\eta)\in\mathbb{Z}_+$ and $P_1=P_1(\eta)\ge1$ such that for every $z\in\mathbb{B}^n$ there exists $k_0=k_0(z)\in\mathbb{Z}_+$, $0\le k_0\le n_0$, for which inequality (6) holds. We choose $\eta>1$ and $j_0\in\mathbb{N}$ such that $P_1\le\eta^{j_0}$. For given $z\in\mathbb{B}^n$, $k_0=k_0(z)$ and $j\ge j_0$ by Cauchy's formula for $F(z+t\mathbf{b})$ as a function of one variable $t$

$$\partial_{\mathbf{b}}^{k_0+j}F(z)=\frac{j!}{2\pi i}\int_{|t|=\eta/L(z)}\frac{\partial_{\mathbf{b}}^{k_0}F(z+t\mathbf{b})}{t^{j+1}}dt.$$

Therefore, in view of (6) we have

$$\frac{|\partial_{\mathbf{b}}^{k_0+j}F(z)|}{j!}\le\frac{L^j(z)}{\eta^j}\max\left\{|\partial_{\mathbf{b}}^{k_0}F(z+t\mathbf{b})|:|t|=\frac{\eta}{L(z)}\right\}\le P_1\frac{L^j(z)}{\eta^j}|\partial_{\mathbf{b}}^{k_0}F(z)|,$$

Hence, for all $j\ge j_0$, $z\in\mathbb{B}^n$

$$\frac{|\partial_{\mathbf{b}}^{k_0+j}F(z)|}{(k_0+j)!L^{k_0+j}(z)}\le\frac{j!k_0!}{(j+k_0)!}\frac{P_1}{\eta^j}\frac{|\partial_{\mathbf{b}}^{k_0}F(z)|}{k_0!L^{k_0}(z)}\le\eta^{j_0-j}\frac{|\partial_{\mathbf{b}}^{k_0}F(z)|}{k_0!L^{k_0}(z)}\le\frac{|\partial_{\mathbf{b}}^{k_0}F(z)|}{k_0!L^{k_0}(z)}.$$

Since $k_0\le n_0$, the numbers $n_0=n_0(\eta)$ and $j_0=j_0(\eta)$ are independent of $z$ and $t_0$, this inequality means that a function $F$ has bounded $L$-index in the direction $\mathbf{b}$ and $N_{\mathbf{b}}(F,L)\le n_0+j_0$. The proof of Theorem 4 is complete. □

Theorem 4 implies the next proposition that describes the boundedness of $L$-index in direction for an equivalent function to $L$.

**Theorem 5.** *Let $L\in Q_{\mathbf{b}}(\mathbb{B}^n)$, $\frac{1}{\beta}<\theta_1\le\theta_2<+\infty$, $\theta_1L(z)\le L^*(z)\le\theta_2L(z)$. A function $F\in\widetilde{\mathcal{H}}_{\mathbf{b}}(\mathbb{B}^n)$ is of bounded $L^*$-index in the direction $\mathbf{b}$ if and only if $F$ is of bounded $L$-index in the direction $\mathbf{b}$.*

**Proof.** Obviously, if $L\in Q_{\mathbf{b}}(\mathbb{B}^n)$ and $\theta_1L(z)\le L^*(z)\le\theta_2L(z)$, then $L^*\in Q_{\mathbf{b}}(\mathbb{B}^n)$ with $\beta^*\in[\theta_1\beta;\theta_2\beta]$ and $\beta^*>1$ instead $\beta>1$. Let $N_{\mathbf{b}}(F,L^*)<+\infty$. Therefore, by Theorem 4 for each $\eta^*$, $0<\eta^*<\beta\theta_2$, there exist $n_0(\eta^*)\in\mathbb{Z}_+$ and $P_1(\eta^*)\ge1$ such that for every $z\in\mathbb{B}^n$, $t_0\in S_z$ and some $k_0$, $0\le k_0\le n_0$, inequality (6) is valid with $L^*$ and $\eta^*$ instead of $L$ and $\eta$. Taking $\eta*=\theta_2\eta$ we obtain

$$P_1|\partial_{\mathbf{b}}^{k_0}F(z)|\ge\max\left\{|\partial_{\mathbf{b}}^{k_0}F(z+t\mathbf{b})|:|t|\le\eta^*/L^*(z)\right\}\ge$$
$$\ge\max\left\{|\partial_{\mathbf{b}}^{k_0}F(z+t\mathbf{b})|:\ |t|\le\eta/L(z)\right\}.$$

Therefore, by Theorem 4 the function $F(z)$ is of bounded $L$-index in the direction $\mathbf{b}$. The converse assertion is obtained by replacing $L$ on $L^*$. □

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**Theorem 6.** *Let $L \in Q_{\mathbf{b}}(\mathbb{B}^n)$, $m \in \mathbb{C} \setminus \{0\}$. A function $F \in \widetilde{\mathcal{H}}_{\mathbf{b}}(\mathbb{B}^n)$ is of bounded L-index in the direction $\mathbf{b} \in \mathbb{C}^n$ if and only if $F(z)$ is of bounded L-index in the direction $m\mathbf{b}$.*

**Proof.** Let a function $F \in \widetilde{\mathcal{H}}_{\mathbf{b}}(\mathbb{B}^n)$ be of bounded $L$-index in the direction $\mathbf{b}$. By Theorem 4 $(\forall \eta > 0)$ $(\exists n_0(\eta) \in \mathbb{Z}_+)$ $(\exists P_1(\eta) \geq 1)$ $(\forall z \in \mathbb{B}^n)$ $(\exists k_0 = k_0(z) \in \mathbb{Z}_+, 0 \leq k_0 \leq n_0)$, and the following inequality is valid

$$\max\left\{|\partial_{\mathbf{b}}^{k_0} F(z+t\mathbf{b})| : |t| \leq \eta/L(z)\right\} \leq P_1|\partial_{\mathbf{b}}^{k_0} F(z)|. \tag{14}$$

Since $\frac{\partial^k F}{\partial(m\mathbf{b})^k} = (m)^k \frac{\partial^k F}{\partial \mathbf{b}^k}$, inequality (14) is equivalent to the inequality

$$\max\left\{|m|^{k_0}|\partial_{\mathbf{b}}^{k_0} F(z+t\mathbf{b})| : |t| \leq \eta/L(z)\right\} \leq P_1|m|^{k_0}|\partial_{\mathbf{b}}^{k_0} F(z)|$$

as well as to the inequality

$$\max\left\{|\partial_{m\mathbf{b}}^{k_0} F(z+\frac{t}{m}m\mathbf{b})| : |t/m| \leq \eta/(|m|L(z))\right\} \leq P_1|\partial_{m\mathbf{b}}^{k_0} F(z)|.$$

Denoting $t^* = \frac{t}{m}$, $\eta^* = \frac{\eta}{|m|}$, we obtain

$$\max\left\{|\partial_{m\mathbf{b}}^{k_0} F(z+t^* m\mathbf{b})| : |t^*| \leq \eta^*/L(z)\right\} \leq P_1|\partial_{\mathbf{b}}^{k_0} F(z)|.$$

By Theorem 4 the function $F(z)$ is of bounded $L$-index in the direction $\mathbf{b}$. The converse assertion can be proved similarly. □

Please note that Proposition 5 can be slightly refined. The following proposition is easy deduced from (2).

**Proposition 4.** *Let $L_1(z)$, $L_2(z)$ be positive continuous functions, $F \in \widetilde{\mathcal{H}}_{\mathbf{b}}(\mathbb{B}^n)$ be a function of bounded $L_1$-index in the direction $\mathbf{b}$, for all $z \in \mathbb{B}^n$ the inequality $L_1(z) \leq L_2(z)$ holds. Then $N_{\mathbf{b}}(L_2, F) \leq N_{\mathbf{b}}(L_1, F)$.*

Using Fricke's idea [14], we deduce a modification of Theorem 4. Our proof is similar to proof in [9].

**Theorem 7.** *Let $L \in Q_{\mathbf{b}}(\mathbb{B}^n)$, $F \in \widetilde{\mathcal{H}}_{\mathbf{b}}(\mathbb{B}^n)$. If there exist $\eta \in (0, \beta]$, $n_0 = n_0(\eta) \in \mathbb{Z}_+$ and $P_1 = P_1(\eta) \geq 1$ such that for any $z \in \mathbb{B}^n$ there exists $k_0 = k_0(z) \in \mathbb{Z}_+, 0 \leq k_0 \leq n_0$, and*

$$\max\{|\partial_{\mathbf{b}}^{k_0} F(z+t\mathbf{b})| : |t| \leq \eta/L(z)\} \leq P_1|\partial_{\mathbf{b}}^{k_0} F(z)|,$$

*then the function $F$ has bounded L-index in the direction $\mathbf{b} \in \mathbb{C}^n \setminus \{\mathbf{0}\}$.*

**Proof.** Besides the mentioned paper of Fricke [14], our proof is similar to proofs in [3] (slice holomorphic functions in $\mathbb{C}^n$).

Assume that there exist $\eta \in (0, \beta]$, $n_0 = n_0(\eta) \in \mathbb{Z}_+$ and $P_1 = P_1(\eta) \geq 1$ such that for any $z \in \mathbb{B}^n$ there exists $k_0 = k_0(z) \in \mathbb{Z}_+, 0 \leq k_0 \leq n_0$, and

$$\max\{|\partial_{\mathbf{b}}^{k_0} F(z+t\mathbf{b})| : |t| \leq \frac{\eta}{L(z)}\} \leq P_1|\partial_{\mathbf{b}}^{k_0} F(z)|. \tag{15}$$

If $\eta \in (1, \beta]$, then we choose $j_0 \in \mathbb{N}$ such that $P_1 \leq \eta^{j_0}$. And for $\eta \in (0; 1]$ we choose $j_0 \in \mathbb{N}$ such that $\frac{j_0!k_0!}{(j_0+k_0)!}P_1 < 1$. The $j_0$ is well-defined because

$$\frac{j_0!k_0!}{(j_0+k_0)!}P_1 = \frac{k_0!}{(j_0+1)(j_0+2)\cdot \ldots \cdot (j_0+k_0)}P_1 \to 0,\ j_0 \to \infty.$$

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Applying integral Cauchy's formula to the function $g_z(t) = F(z + t\mathbf{b})$ as analytic function of one complex variable $t$ for $j \geq j_0$ we obtain that for every $z \in \mathbb{B}^n$ there exists $k_0 = k_0(z)$, $0 \leq k_0 \leq n_0$, and

$$\partial_{\mathbf{b}}^{k_0+j} F(z) = \frac{j!}{2\pi i} \int\limits_{|t|=\frac{\eta}{L(z)}} \frac{\partial_{\mathbf{b}}^{k_0} F(z + t\mathbf{b})}{t^{j+1}} dt.$$

Taking into account (15), we deduce

$$\frac{|\partial_{\mathbf{b}}^{k_0+j} F(z)|}{j!} \leq \frac{L^j(z)}{\eta^j} \max \left\{ |\partial_{\mathbf{b}}^{k_0} F(z + t\mathbf{b})| : \ |t| = \frac{\eta}{L(z)} \right\} \leq P_1 \frac{L^j(z)}{\eta^j} |\partial_{\mathbf{b}}^{k_0} F(z)|. \tag{16}$$

In view of choice $j_0$ with $\eta \in (1, \beta]$, for all $j \geq j_0$ one has

$$\frac{|\partial_{\mathbf{b}}^{k_0+j} F(z)|}{(k_0 + j)! L^{k_0+j}(z)} \leq \frac{j! k_0!}{(j + k_0)!} \frac{P_1}{\eta^j} \frac{|\partial_{\mathbf{b}}^{k_0} F(z)|}{k_0! L^{k_0}(z + t_0\mathbf{b})} \leq \eta^{j_0 - j} \frac{|\partial_{\mathbf{b}}^{k_0} F(z)|}{k_0! L^{k_0}(z)} \leq \frac{|\partial_{\mathbf{b}}^{k_0} F(z)|}{k_0! L^{k_0}(z)}.$$

Since $k_0 \leq n_0$, the numbers $n_0 = n_0(\eta)$ and $j_0 = j_0(\eta)$ do not depend of $z$, and $z \in \mathbb{B}^n$ is arbitrary, the last inequality is equivalent to the assertion that $F$ has boudned $L$-index in the direction $\mathbf{b}$ and $N_{\mathbf{b}}(F, L) \leq n_0 + j_0$.

If $\eta \in (0, 1)$, then from (16) it follows that for all $j \geq j_0$

$$\frac{|\partial_{\mathbf{b}}^{k_0+j} F(z)|}{(k_0 + j)! L^{k_0+j}(z)} \leq \frac{j! k_0! P_1}{(j + k_0)!} \frac{|\partial_{\mathbf{b}}^{k_0} F(z)|}{\eta^j k_0! L^{k_0}(z)} \leq \frac{|\partial_{\mathbf{b}}^{k_0} F(z)|}{\eta^j k_0! L^{k_0}(z)}$$

or in view of choice $j_0$

$$\frac{|\partial_{\mathbf{b}}^{k_0+j} F(z)|}{(k_0 + j)!} \frac{\eta^{k_0+j}}{L^{k_0+j}(z)} \leq \frac{|\partial_{\mathbf{b}}^{k_0} F(z)|}{k_0!} \frac{\eta^{k_0}}{L^{k_0}(z)}.$$

Thus, the function $F$ is of bounded $\tilde{L}$-index in the direction $\mathbf{b}$, where $\tilde{L}(z) = \frac{L(z)}{\eta}$. Then by Lemma 5 the function $F$ has bounded $L$-index in the direction $\mathbf{b}$, if $\eta\beta > 1$. When $\eta \leq \frac{1}{\beta}$, we choose arbitrary $\gamma > \frac{1}{\eta\beta}$. By Lemma 5 the function $F$ is of bounded $L_1$-index in the direction $\mathbf{b}$, where $L_1(z) = \eta\gamma\tilde{L}(z)$. Then be Lemma 6 the function $F$ has bounded $L_1$-index in the direction $\gamma\mathbf{b}$. Since $\partial_{\gamma\mathbf{b}}^k F = \gamma^k \partial_{\mathbf{b}}^k F$ and $L_1^k(z) = \gamma^k L^k(z)$, in inequality (2) with the definition of $L$-index boundedness in direction the corresponding multiplier $\gamma$ is reduced. Hence, the function $F$ is of bounded $L$-index in the direction $\mathbf{b}$. Theorem is proved. □

## 4. $L$-index in direction in a domain compactly embedded in the unit ball

Let $D$ be an arbitrary bounded domain in $\mathbb{B}^n$ such that $\operatorname{dist}(D, \mathbb{B}^n) > 0$. If inequality (2) holds for all $z \in D$ instead $\mathbb{B}^n$, then the function $F \in \widetilde{\mathcal{H}}_{\mathbf{b}}(\mathbb{B}^n)$ is called *a function of bounded $L$-index in the direction* $\mathbf{b}$ *in the domain $D$*. The least such integer $m_0$ is called the *$L$-index in the direction* $\mathbf{b} \in \mathbb{C}^n \setminus \{\mathbf{0}\}$ *in domain* $D$ and is denoted by $N_{\mathbf{b}}(F, L, D) = m_0$. The notation $\overline{D}$ stands for a closure of the domain $D$.

**Lemma 2.** *Let $D$ be a bounded domain in $\mathbb{B}^n$ such that $d = \operatorname{dist}(D, \mathbb{B}^n) = \inf_{z \in D}(1 - |z|) > 0$, $\beta > 1$, $\mathbf{b} \in \mathbb{C}^n \setminus \{\mathbf{0}\}$ be an arbitrary direction. If $L \colon \mathbb{B}^n \to \mathbb{R}_+$ is continuous function such that $L(z) \geq \frac{\beta|\mathbf{b}|}{d}$, and a function $F \in \mathcal{H}_{\mathbf{b}}(\mathbb{B}^n)$ be such that $(\forall z^0 \in \overline{D})\colon \ \ F(z^0 + t\mathbf{b}) \not\equiv 0$, then $N_{\mathbf{b}}(F, L, D) < \infty$.*

**Proof.** This proof is similar to proof in [3] for slice entire functions in $\mathbb{C}^n$.

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For every fixed $z^0 \in \overline{D}$ we expand the analytic function $F(z^0 + t\mathbf{b})$ in a power series by powers of $t$ in the disc $\{t \in \mathbb{C} : |t| \leq \frac{1}{L(z^0)}\}$

$$F(z^0 + t\mathbf{b}) = \sum_{m=0}^{\infty} \frac{\partial_{\mathbf{b}}^m F(z^0)}{m!} t^m. \tag{17}$$

The quantity $\frac{|\partial_{\mathbf{b}}^m F(z^0)|}{m!}$ is the modulus of a coefficient of the power series (17) at the point $t \in \mathbb{C}$ such that $|t| = \frac{1}{L(z^0)}$. Since $F(z)$ is analytic function, for every $z_0 \in \overline{D}$

$$\frac{|\partial_{\mathbf{b}}^m F(z^0)|}{m! L^m(z^0)} \to 0 \quad (m \to \infty),$$

i.e., there exists $m_0 = m(z^0, \mathbf{b})$ such that inequality (2) holds at the point $z = z^0$ for all $m \in \mathbb{Z}_+$.

We prove that $\sup\{m_0 \colon z^0 \in \overline{D}\} < +\infty$. On the contrary, we assume that the set of all values $m_0$ is unbounded in $z^0$, i.e., $\sup\{m_0 \colon z^0 \in \overline{D}\} = +\infty$. Hence, for every $m \in \mathbb{Z}_+$ there exists $z^{(m)} \in \overline{D}$ and $p_m > m$

$$\frac{|\partial_{\mathbf{b}}^{p_m} F(z^{(m)})|}{p_m! L^{p_m}(z^{(m)})} > \max\left\{\frac{|\partial_{\mathbf{b}}^k F(z^{(m)})|}{k! L^k(z^{(m)})} \colon \ 0 \leq k \leq m\right\}. \tag{18}$$

Since $\{z^{(m)}\} \subset \overline{D}$, there exists subsequence $z'^{(m)} \to z' \in \overline{G}$ as $m \to +\infty$. By Cauchy's integral formula

$$\frac{\partial_{\mathbf{b}}^p F(z)}{p!} = \frac{1}{2\pi i} \int_{|t|=r} \frac{F(z + t\mathbf{b})}{t^{p+1}} dt$$

for any $p \in \mathbb{N}$, $z \in D$. Rewrite (18) as following

$$\max\left\{\frac{|\partial_{\mathbf{b}}^k F(z^{(m)})|}{k! L^k(z^{(m)})} \colon \ 0 \leq k \leq m\right\} <$$
$$< \frac{1}{L^{p_m}(z^{(m)})} \int_{|t|=r/L(z^{(m)})} \frac{|F(z^{(m)} + t\mathbf{b})|}{|t|^{p_m+1}} |dt| \leq \frac{1}{r^{p_m}} \max\{|F(z)| : z \in D_r\}, \tag{19}$$

where $D_r = \bigcup_{z^* \in \overline{D}} \{z \in \mathbb{C}^n : |z - z^*| \leq \frac{|\mathbf{b}| r}{L(z^*)}\}$. We can choose $r \in (1, \beta)$, because $g_{z^{(m)}}(t) = F(z^{(m)} + t\mathbf{b})$ is an analytic function in $S_{z^{(m)}}$. Evaluating the limit for every directional derivative of fixed order in (19) as $m \to \infty$ we obtain

$$(\forall k \in \mathbb{Z}_+) \colon \quad \frac{|\partial_{\mathbf{b}}^k F(z')|}{k! L^k(z')} \leq \varlimsup_{m \to \infty} \frac{1}{r^{p_m}} \max\{|F(z)| : z \in D_r\} \leq 0.$$

Thus, all derivatives in the direction $\mathbf{b}$ of the function $F$ at the point $z'$ equals 0 and $F(z') = 0$. In view of (17) $F(z' + t\mathbf{b}) \equiv 0$. It is a contradiction.
□

## 5. Discussion

The proposed approach can be used in analytic theory of partial differential equations. For cases of entire functions and analytic functions it is known that similar results allow to deduce sufficient conditions by coefficients of partial differential equations and its systems providing index boundedness of every analytic solution. And it gives growth estimates, local behavior and some value distribution for these functions. By analogy, we hope that the obtained results allow to obtain similar applications for slice holomorphic functions in the unit ball in the future investigations.

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