Matrix equalities equivalent to the reverse order law $(AB)^{\dagger} = B^{\dagger}A^{\dagger}$

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Abstract. This note shows that the well-known reverse order law $(AB)^{\dagger} = B^{\dagger}A^{\dagger}$ for the Moore–Penrose inverse of matrix product is equivalent to many other equalities for that are composed of multiple products $(AB)^{\dagger}$ and $B^{\dagger}A^{\dagger}$ by means of the definition of the Moore–Penrose inverse and orthogonal projector theory.

Keywords: matrix product, Moore–Penrose inverse, reverse order law, orthogonal projector

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1 Introduction

Let $\mathbb{C}^{m\times n}$ denote the collection of all $m \times n$ complex matrices, A^* denote the conjugate transpose; $r(A)$ denote the rank of A, i.e., the maximum order of the invertible submatrix of A; $\mathscr{R}(A) = \{Ax \,|\, x \in \mathbb{C}^{n\times 1}\}$ and $\mathscr{N}(A) = \{x \in \mathbb{C}^{n\times 1} \,|\, Ax = 0\}$ denote the range and the null space of a matrix $A \in \mathbb{C}^{m\times n}$, respectively; I_m denote the identity matrix of order m; and $[A,\, B]$ denote a columnwise partitioned matrix consisting of two submatrices A and B. A matrix $A \in \mathbb{C}^{m\times m}$ is said to be Hermitian if $A = A^*$; to be idempotent if $A^2 = A$; to be orthogonal projector if $A^2 = A = A^*$. The Moore–Penrose inverse of $A \in \mathbb{C}^{m\times n}$, denoted by $A^{\dagger}$, is the unique matrix $X \in \mathbb{C}^{n\times m}$ that satisfies the four Penrose equations

$$(1)\ AXA = A, \quad (2)\ XAX = X, \quad (3)\ (AX)^* = AX, \quad (4)\ (XA)^* = XA. \tag{1.1}$$

A matrix X is called a $\{i,\ldots,j\}$-generalized inverse of A, denoted by $A^{(i,\ldots,j)}$, if it satisfies the ith,..., jth equations in (1.1). The collection of all $\{i,\ldots,j\}$-generalized inverses of A is denoted by $\{A^{(i,\ldots,j)}\}$ (cf. [2,3]).

In this note, the author reconsiders the following two well-known reverse order laws:

$$ABB^{\dagger}A^{\dagger}AB = AB, \quad (AB)^{\dagger} = B^{\dagger}A^{\dagger} \tag{1.2}$$

for the Moore–Penrose inverses of the matrix products, which are direct extensions of the two fundamental reverse order laws $ABB^{-1}A^{-1}AB = AB$ and $(AB)^{-1} = B^{-1}A^{-1}$ for the product of two invertible matrices of the same size, and can be used to simplify various matrix expressions that involve products of matrices and their products in matrix theory and applications.

Because the non-commutativity of matrix algebra, and also because $AA^{\dagger} \neq I_m$, $A^{\dagger}A \neq I_n$, $BB^{\dagger} \neq I_n$, and $B^{\dagger}B \neq I_p$ for two singular matrices A and B, the two matrix equalities in (1.2) do not necessarily hold. Hence, there have been a heavy of approaches of characterizing relationships between A and B that satisfy (1.2) since 1960s. The two well-known classical results in the theory of generalized inverses concerning the two matrix equalities in (1.2) are given by

$$ABB^{\dagger}A^{\dagger}AB = AB \Leftrightarrow B^{\dagger}A^{\dagger}ABB^{\dagger}A^{\dagger} = B^{\dagger}A^{\dagger} \Leftrightarrow r[A^*,\, B] = r(A) + r(B) - r(AB), \tag{1.3}$$

$$(AB)^{\dagger} = B^{\dagger}A^{\dagger} \Leftrightarrow \mathscr{R}(A^*AB) \subseteq \mathscr{R}(B) \text{ and } \mathscr{R}(BB^*A^*) \subseteq \mathscr{R}(A^*); \tag{1.4}$$

see e.g., [1,4,9]. In addition, it has been noticed that there are many matrix equalities that are equivalent to the reverse order laws in (1.2). One of such known results is given by

$$(AB)^{\dagger} = B^{\dagger}A^{\dagger} \Leftrightarrow (AB)^{\dagger} = B^{\dagger}(A^{\dagger}ABB^{\dagger})^{\dagger}A^{\dagger} \quad \text{and} \quad (A^{\dagger}ABB^{\dagger})^{\dagger} = BB^{\dagger}A^{\dagger}A; \tag{1.5}$$

see e.g., [6,7,9] for their expositions and many other equivalent facts about (1.2).

Noticing that the multiple matrix products

$$ABB^{\dagger}A^{\dagger}, \quad B^{\dagger}A^{\dagger}AB, \quad A^{\dagger}ABB^{\dagger}, \quad BB^{\dagger}A^{\dagger}A, \quad ABB^{\dagger}A^{\dagger}AB, \quad B^{\dagger}A^{\dagger}ABB^{\dagger}A^{\dagger}$$

occur in (1.2)–(1.5), it is necessary to give a systematic research to these matrix products and their variations in order to get deeper understanding to (1.2). The purpose of this note is to show that the two reverse order laws (1.2) are equivalent to several matrix equalities that are composed of the above multiple matrix products by means of the definition of the Moore–Penrose inverse and orthogonal projector theory.

We shall use the following preliminary results in order to derive our main results. The following two lemmas are well known in linear algebra.

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Lemma 1.1. *Let $A \in \mathbb{C}^{m\times n}$. Then the rank equality $r(A) = r(A^*)$ holds.*

Lemma 1.2. *The matrix rank inequality $r(PAQ) \le r(A)$ always holds. If the two matrix equalities $P_1AQ_1 = B$ and $A = P_2BQ_2$ hold, then the rank equality $r(A) = r(B)$ holds.*

The following lemma regarding Moore–Penrose inverses can be found in [2,3].

Lemma 1.3. *Let $A \in \mathbb{C}^{m\times n}$. Then the following equalities hold*

$$(A^\dagger)^* = (A^*)^\dagger, \quad (A^\dagger)^\dagger = A, \tag{1.6}$$
$$A^\dagger = A^*(AA^*)^\dagger = (A^*A)^\dagger A^* = A^*(A^*AA^*)^\dagger A^*, \tag{1.7}$$
$$(A^*)^\dagger A^* = (AA^\dagger)^* = AA^\dagger, \quad A^*(A^*)^\dagger = (A^\dagger A)^* = A^\dagger A, \tag{1.8}$$
$$(AA^*)^\dagger = (A^\dagger)^*A^\dagger, \quad (A^*A)^\dagger = A^\dagger(A^\dagger)^*, \quad (AA^*A)^\dagger = A^\dagger(A^\dagger)^*A^\dagger. \tag{1.9}$$

Lemma 1.4. *Let $A \in \mathbb{C}^{m\times m}$, and assume that $s > t \ge 1$.*

(a) *If $A^* = A$, then $AA^\dagger = A^\dagger A$ and $A^s(A^t)^\dagger = (A^t)^\dagger A^s = A^{s-t}$.*

(b) *If $A^* = A$ and $A^s = A^t$, then $A^{s-t+1} = A$.*

(c) *If A is positive semi-definite and $A^s = A^t$, then $A^2 = A$.*

(d) *If $A^* = A$, then $r(A^k) = r(A)$ for all $k \ge 1$.*

Proof. If $A^* = A$, then we obtain by the definition of the Moore–Penrose inverse and (1.6) that $AA^\dagger = (AA^\dagger)^* = (A^\dagger)^*A^* = (A^*)^\dagger A^* = A^\dagger A$, establishing the first equality in (a). The second equality in (a) follows from the definition of the Moore–Penrose inverse and (1.9). In this case, pre-multiplying the second equality with $(A^{t-1})^\dagger$ yields the last equality in (b). Both positive semi-definiteness and $A^s = A^t$ mean that the eigenvalues of A are 0 or 1, so that $A^2 = A$. □

Lemma 1.5. *Let $A,\ B \in \mathbb{C}^{m\times m}$ be two orthogonal projectors, and assume that $s > t \ge 1$. Then*

(a) *$(ABA)^s = (ABA)^t$ if and only if $AB = BA$.*

(b) *$(AB)^s = (AB)^t$ if and only if $AB = BA$.*

Proof. Since ABA is Hermitian and positive semi-definite, $(ABA)^s = (ABA)^t$ implies $(ABA)^2 = ABABA = ABA$ by Lemma 1.4(c). In this case,

$$(ABA - AB)(ABA - AB)^* = ABABA - ABABA - ABABA + ABA = 0,$$

that is, $ABA = AB = BA$, establishing the first equality in (a). Post-multiplying the first equality with A yields $(ABA)^s = (ABA)^t$, which is equivalent to $AB = BA$ by (a). □

Lemma 1.6 ([2,3])**.** *Let $A \in \mathbb{C}^{m\times n}$ and $G \in \mathbb{C}^{n\times m}$. Then*

$$G \in \{A^{(1)}\} \Leftrightarrow AGA = A, \tag{1.10}$$
$$G \in \{A^{(1,2)}\} \Leftrightarrow AGA = A \text{ and } r(G) = r(A), \tag{1.11}$$
$$G \in \{A^{(1,3)}\} \Leftrightarrow AG = AA^\dagger \Leftrightarrow A^*AG = A^*, \tag{1.12}$$
$$G \in \{A^{(1,4)}\} \Leftrightarrow GA = A^\dagger A \Leftrightarrow GAA^* = A^*, \tag{1.13}$$
$$G \in \{A^{(1,2,3)}\} \Leftrightarrow A^*AG = A^* \text{ and } r(G) = r(A) \Leftrightarrow A^*AG = A^* \text{ and } GE_A = 0, \tag{1.14}$$
$$G \in \{A^{(1,2,4)}\} \Leftrightarrow GAA^* = A^* \text{ and } r(G) = r(A) \Leftrightarrow GAA^* = A^* \text{ and } F_AG = 0, \tag{1.15}$$
$$G \in \{A^{(1,3,4)}\} \Leftrightarrow A^*AG = A^* \text{ and } GAA^* = A^*, \tag{1.16}$$
$$\begin{aligned} G = A^\dagger &\Leftrightarrow G \in \{A^{(1,3)}\},\ G \in \{A^{(1,4)}\},\ \text{and } r(G) = r(A) \\ &\Leftrightarrow AG = AA^\dagger,\ GA = A^\dagger A,\ \text{and } r(G) = r(A) \\ &\Leftrightarrow A^*AG = A^*,\ GAA^* = A^*,\ \text{and } r(G) = r(A) \\ &\Leftrightarrow AG = AA^\dagger,\ GA = A^\dagger A,\ GE_A = 0,\ \text{and } F_AG = 0. \end{aligned} \tag{1.17}$$

Lemma 1.7 ([5,8])**.** *Let $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{m\times k}$, $C \in \mathbb{C}^{l\times n}$, and $D \in \mathbb{C}^{l\times k}$. Then*

$$\max_{A^{(1)}} r(D - CA^{(1)}B) = \min\left\{ r[\,C,\ D\,],\ \ r\begin{bmatrix} B \\ D \end{bmatrix},\ \ r\begin{bmatrix} A & B \\ C & D \end{bmatrix} - r(A) \right\}, \tag{1.18}$$
$$\max_{A^{(1,2)}} r(D - CA^{(1,2)}B) = \min\left\{ r(A) + r(D),\ r[\,C,\ D\,],\ r\begin{bmatrix} B \\ D \end{bmatrix},\ r\begin{bmatrix} A & B \\ C & D \end{bmatrix} - r(A) \right\}. \tag{1.19}$$

2 Main results

It is easy to see that if (1.2) holds, then the following matrix equality

$$(AB)^{\dagger} = B^{\dagger}A^{\dagger}ABB^{\dagger}A^{\dagger} \tag{2.1}$$

holds as well by the definition of the Moore–Penrose inverse. So that it would be of interest to see whether (2.1) also implies (1.2). The answer is positive although the implication is not obvious (cf. [9]). This fact motivates us to consider more implications of equalities of words composed of multiple mixed products of AB and $B^{\dagger}A^{\dagger}$, and obtain the following three groups of results.

Theorem 2.1. *Let $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}^{n\times p}$, $k \geq 1$, and denote $P = AB$ and $Q = B^{\dagger}A^{\dagger}$. Then the rank equalities hold*

$$r((PQ)^kP) = r((QP)^kQ) = r(PQ) = r(QP) = r(Q) = r(P), \tag{2.2}$$

the following range, and null space equalities hold

$$\mathscr{R}((PQ)^kP) = \mathscr{R}(PQ) = \mathscr{R}(P), \tag{2.3}$$
$$\mathscr{R}((QP)^kQ) = \mathscr{R}(QP) = \mathscr{R}(Q), \tag{2.4}$$
$$\mathscr{N}((PQ)^kP) = \mathscr{N}(QP) = \mathscr{N}(P), \tag{2.5}$$
$$\mathscr{N}((QP)^kQ) = \mathscr{N}(PQ) = \mathscr{N}(Q), \tag{2.6}$$

and the following matrix equalities hold

$$((PQ)^kP)((PQ)^kP)^{\dagger} = (PQ)(PQ)^{\dagger} = PP^{\dagger},\ ((PQ)^kP)^{\dagger}((PQ)^kP) = (QP)^{\dagger}(QP) = P^{\dagger}P, \tag{2.7}$$
$$((QP)^kQ)((QP)^kQ)^{\dagger} = (QP)(QP)^{\dagger} = QQ^{\dagger},\ ((QP)^kQ)^{\dagger}((QP)^kQ) = (PQ)^{\dagger}(PQ) = Q^{\dagger}Q. \tag{2.8}$$

Proof. Let $X = A^{\dagger}A$ and $Y = BB^{\dagger}$. Then by the definition of the Moore–Penrose inverse and Lemma 1.2,

$$r((PQ)^kP) = r(A^{\dagger}(PQ)^kPB^{\dagger}) = r((XY)^{k+1}), \tag{2.9}$$
$$r((QP)^kQ) = r(B((QP)^kQ)A) = r((YX)^{k+1}), \tag{2.10}$$
$$r(PQ) = r(A^{\dagger}PQA) = r(XYX), \tag{2.11}$$
$$r(QP) = r(BQPB^{\dagger}) = r(YXY), \tag{2.12}$$
$$r(Q) = r(BQA) = r(YX), \tag{2.13}$$
$$r(P) = r(A^{\dagger}PB^{\dagger}) = r(XY). \tag{2.14}$$

Since X and Y are two orthogonal projectors, the two triple matrix products XYX and YXY are positive semi-definite, So that we obtain by from Lemmas 1.1, 1.2, and 1.4(d) the following rank inequalities

$$r((XY)^{k+1}) = r((YX)^{k+1}) \leq r(XY) = r(YX) = r(AB), \tag{2.15}$$
$$r((XY)^{k+1}) \geq r((XYX)^{k+1}) = r(XYX) = r((XY)(XY)^*) = r(XY) = r(AB), \tag{2.16}$$
$$r((YX)^{k+1}) \geq r((YXY)^{k+1}) = r(YXY) = r((YX)(YX)^*) = r(YX) = r(AB). \tag{2.17}$$

Combining (2.9)–(2.17) leads to the rank equalities in (2.2). Eqs. (2.3)–(2.8) are direct consequences of (2.2). □

Theorem 2.2. *Let $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}^{n\times p}$, and denote $P = AB$ and $Q = B^{\dagger}A^{\dagger}$. Then the following four matrix set inclusions always hold*

$$\{Q(QPQ)^{(1)}Q\} \subseteq \{P^{(1)}\},\quad \{P(PQP)^{(1)}P\} \subseteq \{Q^{(1)}\}, \tag{2.18}$$
$$\{Q(QPQ)^{(1,2)}Q\} \subseteq \{P^{(1,2)}\},\quad \{P(PQP)^{(1,2)}P\} \subseteq \{Q^{(1,2)}\}. \tag{2.19}$$

Proof. By the definition of $\{1\}$-generalized inverse,

$$\{Q(QPQ)^{(1)}Q\} \subseteq \{P^{(1)}\} \Leftrightarrow PQ(QPQ)^{(1)}QP = P \text{ for all } (QPQ)^{(1)}, \tag{2.20}$$
$$\{P(PQP)^{(1)}P\} \subseteq \{Q^{(1)}\} \Leftrightarrow QP(PQP)^{(1)}PQ = Q \text{ for all } (PQP)^{(1)}. \tag{2.21}$$

Applying (1.18) to $P - PQ(QPQ)^{(1)}QP$ and $Q - QP(PQP)^{(1)}PQ$, and simplifying by (2.2), we obtain

$$\max_{(QPQ)^{(1)}} r(P - PQ(QPQ)^{(1)}QP) = \min\left\{ r[P,\ PQ],\ \ r\begin{bmatrix} P \\ QP \end{bmatrix},\ \ r\begin{bmatrix} QPQ & QP \\ PQ & P \end{bmatrix} - r(PQP) \right\}$$
$$= \min\left\{ r(P),\ \ r\begin{bmatrix} 0 & 0 \\ 0 & P \end{bmatrix} - r(P) \right\} = 0, \tag{2.22}$$
$$\max_{(PQP)^{(1)}} r(Q - QP(PQP)^{(1)}PQ) = \min\left\{ r[Q,\ QP],\ \ r\begin{bmatrix} Q \\ PQ \end{bmatrix},\ \ r\begin{bmatrix} PQP & PQ \\ QP & Q \end{bmatrix} - r(PQP) \right\}$$
$$= \min\left\{ r(Q),\ \ r\begin{bmatrix} 0 & 0 \\ 0 & Q \end{bmatrix} - r(Q) \right\} = 0. \tag{2.23}$$

Combining (2.22) and (2.23) with (2.20) and (2.21) yields the two matrix set inclusions in (2.18). By (1.11),

$$\{Q(QPQ)^{(1,2)}Q\} \subseteq \{P^{(1,2)}\}$$
$$\Leftrightarrow PQ(QPQ)^{(1,2)}QP = P \text{ and } r(Q(QPQ)^{(1,2)}Q) = r(P) \text{ for all } (QPQ)^{(1,2)}, \tag{2.24}$$
$$\{P(PQP)^{(1,2)}P\} \subseteq \{Q^{(1,2)}\}$$
$$\Leftrightarrow QP(PQP)^{(1,2)}PQ = Q \text{ and } r(P(PQP)^{(1,2)}P) = r(Q) \text{ for all } (PQP)^{(1,2)}. \tag{2.25}$$

where by (1.19) and (2.2),

$$\max_{(QPQ)^{(1,2)}} r(P - PQ(QPQ)^{(1,2)}QP)$$
$$= \min\left\{ r(P) + r(QPQ),\ \ r[P,\ PQ],\ \ r\begin{bmatrix} P \\ QP \end{bmatrix},\ \ r\begin{bmatrix} QPQ & QP \\ PQ & P \end{bmatrix} - r(PQP) \right\}$$
$$= \min\left\{ r(P),\ \ r\begin{bmatrix} 0 & 0 \\ 0 & P \end{bmatrix} - r(P) \right\} = 0, \tag{2.26}$$
$$\max_{(PQP)^{(1,2)}} r(Q - QP(PQP)^{(1,2)}PQ)$$
$$= \min\left\{ r(Q) + r(PQP),\ \ r[Q,\ QP],\ \ r\begin{bmatrix} Q \\ PQ \end{bmatrix},\ \ r\begin{bmatrix} PQP & PQ \\ QP & Q \end{bmatrix} - r(PQP) \right\}$$
$$= \min\left\{ r(Q),\ \ r\begin{bmatrix} 0 & 0 \\ 0 & Q \end{bmatrix} - r(Q) \right\} = 0, \tag{2.27}$$

and

$$r(Q(QPQ)^{(1,2)}Q) = r(QPQ(QPQ)^{(1,2)}QPQ) = r(QPQ) = r(P) \ \text{ for all } (QPQ)^{(1,2)}, \tag{2.28}$$
$$r(P(PQP)^{(1,2)}P) = r(PQP(PQP)^{(1,2)}PQP) = r(PQP) = r(Q) \ \text{ for all } (PQP)^{(1,2)}. \tag{2.29}$$

Combining (2.26)–(2.29) with (2.24) and (2.25) yields the two matrix set inclusions in (2.19). □

Theorem 2.3. *Let $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}^{n\times p}$, and denote $P = AB$ and $Q = B^{\dagger}A^{\dagger}$. Then the following statements are equivalent:*

(a) $PQP = P$, *i.e.*, $Q \in \{P^{(1)}\}$.

(b) $QPQ = Q$, *i.e.*, $P \in \{Q^{(1)}\}$.

(c) $(PQ)^2 = PQ$ *and/or* $(QP)^2 = QP$.

(d) $(PQ)^3 = PQ$ *and/or* $(QP)^3 = QP$.

(e) $QPQ \in \{P^{(1)}\}$.

(f) $PQP \in \{Q^{(1)}\}$.

(g) $\{(PQP)^{(1)}\} \subseteq \{P^{(1)}\}$.

(h) $\{(QPQ)^{(1)}\} \subseteq \{Q^{(1)}\}$.

(i) $\{QP(PQP)^{(1)}PQ\} \subseteq \{P^{(1)}\}$.

(j) $\{PQ(QPQ)^{(1)}QP\} \subseteq \{Q^{(1)}\}$.

(k) $Q \in \{P^{(2)}\}$.

(l) $P \in \{Q^{(2)}\}$.

(m) $QPQ \in \{P^{(2)}\}$.

(n) $PQP \in \{Q^{(2)}\}$.

(o) $Q \in \{P^{(1,2)}\}$.

(p) $P \in \{Q^{(1,2)}\}$.

(q) $QPQ \in \{P^{(1,2)}\}$.

(r) $PQP \in \{Q^{(1,2)}\}$.

(s) $\{(QPQ)^{(1,2)}\} \subseteq \{P^{(1,2)}\}$.

(t) $\{(PQP)^{(1,2)}\} \subseteq \{Q^{(1,2)}\}$.

(u) $\{QP(PQP)^{(1,2)}PQ\} \subseteq \{P^{(1,2)}\}$.

(v) $\{PQ(QPQ)^{(1,2)}QP\} \subseteq \{Q^{(1,2)}\}$.

(w) $A^{\dagger}ABB^{\dagger} = BB^{\dagger}A^{\dagger}A$.

(x) $r[A^{*},\, B] = r(A) + r(B) - r(AB)$.

Proof. Let $X = A^{\dagger}A$ and $Y = BB^{\dagger}$, both of which are orthogonal projectors. Pre- and post-multiplying the first equality with $A^{\dagger}$ and $B^{\dagger}$ in (a), and pre- and post-multiplying the first equality with B and A in (b) yield

$$\begin{aligned} A^{\dagger}PQPB^{\dagger} &= A^{\dagger}ABB^{\dagger}A^{\dagger}ABB^{\dagger} = (XY)^2 = XY = A^{\dagger}ABB^{\dagger} = A^{\dagger}PB^{\dagger}, \\ BQPQA &= BB^{\dagger}A^{\dagger}ABB^{\dagger}A^{\dagger}A = (YX)^2 = YX = BB^{\dagger}A^{\dagger}A = BQA, \end{aligned}$$

which, by Lemma 1.5(b), is equivalent to $A^{\dagger}ABB^{\dagger} = XY = YX = BB^{\dagger}A^{\dagger}A$, i.e., (w) holds. Conversely, if (w) holds, then

$$\begin{aligned} PQP &= ABB^{\dagger}A^{\dagger}AB = AA^{\dagger}ABB^{\dagger}B = AB = P, \\ QPQ &= B^{\dagger}A^{\dagger}ABB^{\dagger}A^{\dagger} = B^{\dagger}BB^{\dagger}A^{\dagger}AA^{\dagger} = B^{\dagger}A^{\dagger} = Q, \end{aligned}$$

establishing the equivalences of (a), (b), and (w).

The equivalences of (c)–(f), (k)–(n), and (w) can be established by a similar approach.

By (2.2) and (1.18),

$$\begin{aligned} \max_{(PQP)^{(1)}} r(P - P(PQP)^{(1)}P) &= \min\left\{ r[P,\, P],\ r\begin{bmatrix} P \\ P \end{bmatrix},\ r\begin{bmatrix} PQP & P \\ P & P \end{bmatrix} - r(PQP) \right\} \\ &= \min\left\{ r(P),\ r\begin{bmatrix} PQP - P & 0 \\ 0 & P \end{bmatrix} - r(P) \right\} = r(PQP - P), \\ \max_{(QPQ)^{(1)}} r(Q - Q(QPQ)^{(1)}Q) &= \min\left\{ r[Q,\, Q],\ r\begin{bmatrix} Q \\ Q \end{bmatrix},\ r\begin{bmatrix} QPQ & Q \\ Q & Q \end{bmatrix} - r(QPQ) \right\} \\ &= \min\left\{ r(Q),\ r\begin{bmatrix} QPQ - Q & 0 \\ 0 & Q \end{bmatrix} - r(Q) \right\} = r(QPQ - Q). \end{aligned}$$

Setting both sides of the two rank equalities equal to zero leads to the equivalences (a), (b), (g), and (h).

By the definition of $\{1\}$-generalized inverse,

$$\{QP(PQP)^{(1)}PQ\} \subseteq \{P^{(1)}\} \Leftrightarrow PQP(PQP)^{(1)}PQP = PQP = P, \tag{2.30}$$

$$\{PQ(QPQ)^{(1)}QP\} \subseteq \{Q^{(1)}\} \Leftrightarrow QPQ(QPQ)^{(1)}QPQ = QPQ = Q, \tag{2.31}$$

establishing the equivalences of (a) and (i), and (b) and (j), respectively.

The equivalences of (a), (b), (e), (f), and (o)–(r) follow from (1.11) and (2.2), respectively.

By (1.19) and (2.2),

$$\begin{aligned}
&\max_{(PQP)^{(1,2)}} r(P - P(PQP)^{(1,2)}P) \\
&= \min\left\{ r(P) + r(PQP),\ r[P,\ P],\ r\begin{bmatrix} P \\ P \end{bmatrix},\ r\begin{bmatrix} PQP & P \\ P & P \end{bmatrix} - r(PQP) \right\} \\
&= \min\left\{ r(P),\ \ r\begin{bmatrix} PQP - P & 0 \\ 0 & P \end{bmatrix} - r(P) \right\} = r(PQP - P), \\
&\max_{(QPQ)^{(1,2)}} r(Q - Q(QPQ)^{(1,2)}Q) \\
&= \min\left\{ r(Q) + r(QPQ),\ r[Q,\ Q],\ r\begin{bmatrix} Q \\ Q \end{bmatrix},\ r\begin{bmatrix} QPQ & Q \\ Q & Q \end{bmatrix} - r(QPQ) \right\} \\
&= \min\left\{ r(Q),\ \ r\begin{bmatrix} QPQ - Q & 0 \\ 0 & Q \end{bmatrix} - r(Q) \right\} = r(QPQ - Q).
\end{aligned}$$

Setting both sides of the two rank equalities equal to zero and noticing that $r(PQP)^{(1,2)} = r(QPQ)^{(1,2)} = r(P) = r(Q)$ by (2.2), we obtain the equivalences (a) and (s), (b) and (t), respectively.

By (1.11),

$$\begin{aligned}
&\{QP(PQP)^{(1,2)}PQ\} \subseteq \{P^{(1,2)}\} \\
&\Leftrightarrow PQP(PQP)^{(1,2)}PQP = PQP = P \text{ and } r(QP(PQP)^{(1,2)}PQ) = r(PQP) = r(P), \\
&\{PQ(QPQ)^{(1,2)}QP\} \subseteq \{Q^{(1,2)}\} \\
&\Leftrightarrow QPQ(QPQ)^{(1,2)}QPQ = QPQ = Q \text{ and } r(PQ(QPQ)^{(1,2)}QP) = r(QPQ) = r(Q),
\end{aligned}$$

establishing the equivalences of (a) and (u), and (b) and (v), respectively.

The equivalence (w) and (x) follows from (1.3). □

Theorem 2.4. *Let $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}^{n\times p}$, and denote $P = AB$ and $Q = B^\dagger A^\dagger$. Then the following statements are equivalent:*

(a) $P^\dagger = Q$.

(b) $P^\dagger = (QP)^k Q$ *for some/all integers* $k \geq 1$.

(c) $Q^\dagger = (PQ)^k P$ *for some/all integers* $k \geq 1$.

(d) $((PQ)^k)^\dagger = (PQ)^k$ *and* $((QP)^k)^\dagger = (QP)^k$ *for some/all integers* $k \geq 1$.

(e) $((PQ)^k P)^\dagger = (QP)^k Q$ *for some/all integers* $k \geq 1$.

(f) $PP^\dagger = (PQ)^{k+1}$ *and* $P^\dagger P = (QP)^{k+1}$ *for some/all integers* $k \geq 1$.

(g) $QQ^\dagger = (QP)^{k+1}$ *and* $Q^\dagger Q = (PQ)^{k+1}$ *for some/all integers* $k \geq 1$.

Proof. If (a) holds, then by the definition of the Moore–Penrose inverse,

$$P^\dagger = Q = (QQ^\dagger)^k Q = (QP)^k Q \quad\text{and}\quad Q^\dagger = P = (PP^\dagger)^k P = (PQ)^k P$$

hold. Hence (a) implies (b) and (c). Conversely, pre- and post-multiplying (b) with B and A, (c) with $A^\dagger$ and $B^\dagger$ yield

$$BP^\dagger A = B(QP)^k QA = (YX)^{k+1} \quad\text{and}\quad A^\dagger Q^\dagger B^\dagger = A^\dagger (PQ)^k PB^\dagger = (XY)^{k+1},$$

where $BP^\dagger A$ and $A^\dagger Q^\dagger B^\dagger$ are idempotent by the definition of the Moore–Penrose inverse. Hence

$$(YX)^{2k+2} = (YX)^{k+1} \quad\text{and}\quad (XY)^{2k+2} = (XY)^{k+1},$$

which, by Lemma 1.5(b), are equivalent to $A^\dagger ABB^\dagger = XY = YX = BB^\dagger A^\dagger A$. Hence

$$\begin{aligned}
P^\dagger &= (QP)^k Q = B^\dagger A^\dagger ABB^\dagger A^\dagger = B^\dagger BB^\dagger A^\dagger AA^\dagger = B^\dagger A^\dagger = Q, \\
Q^\dagger &= (PQ)^k P = ABB^\dagger A^\dagger AB = AA^\dagger ABB^\dagger B = AB = P,
\end{aligned}$$

establishing the equivalences of (a), (b), and (c).

If (a) holds, then by the definition of the Moore–Penrose inverse,

$$\begin{aligned}
((PQ)^k)^\dagger &= ((ABB^\dagger A^\dagger)^k)^\dagger = ((AB(AB)^\dagger)^k)^\dagger = (AB(AB)^\dagger)^k = (PQ)^k, \\
((QP)^k)^\dagger &= ((B^\dagger A^\dagger AB)^k)^\dagger = (((AB)^\dagger AB)^k)^\dagger = ((AB)^\dagger AB)^k = (QP)^k,
\end{aligned}$$

establishing (d). Conversely, by the definition of the Moore–Penrose inverse, (2.7), and (d),

$$P^{\dagger} = P^{\dagger}PP^{\dagger} = (QP)^{\dagger}QPP^{\dagger} = (QP)^{\dagger}QPQ(PQ)^{\dagger} = QPQPQPQ = (QP)^{3}Q,$$

establishing (b).

If (a) holds, then by the definition of the Moore–Penrose inverse,

$$((PQ)^{k}P)^{\dagger} = ((PP^{\dagger})^{k}P)^{\dagger} = P^{\dagger} = (P^{\dagger}P)^{k}P^{\dagger} = (QP)^{k}Q,$$

establishing (e). Conversely, by the definition of the Moore–Penrose inverse and (e),

$$(PQ)^{k}P(QP)^{k}Q(PQ)^{k}P = (PQ)^{3k+1}P = (PQ)^{k}P.$$

Pre- and post-multiplying the equalities with $A^{\dagger}$ and $B^{\dagger}$ yield

$$A^{\dagger}(PQ)^{3k+1}PB^{\dagger} = (XY)^{3k+2} = (XY)^{k+1} = A^{\dagger}(PQ)^{k}PB^{\dagger},$$

which, by Lemma 1.5(b), is equivalent to $A^{\dagger}ABB^{\dagger} = XY = YX = BB^{\dagger}A^{\dagger}A$. Hence,

$$\begin{aligned} &((PQ)^{k}P)^{\dagger} = (ABB^{\dagger}A^{\dagger}AB)^{\dagger} = (AA^{\dagger}ABB^{\dagger}B)^{\dagger} = P^{\dagger} \\ &\Rightarrow P^{\dagger} = (QP)^{k}Q \ \text{(by (e))} \\ &\Rightarrow P^{\dagger} = Q \ \text{(by (a) and (b))}, \end{aligned}$$

establishing the equivalences of (a) and (e).

Pre- and post-multiplying P and Q on the left- and right-sides of the equalities in (b) and (c) yield the equalities in (f) and (g). Conversely, (f) and (g) imply (b) and (c) by the definition of the Moore–Penrose inverse. □

In summary, we remark that the preceding work is some special cases of the following general two-direction matrix equality implication problems

$$\begin{aligned} &f(A,\, B,\, A^{*},\, B^{*},\, A^{\dagger},\, B^{\dagger},\, (AB)^{\dagger}) = 0 \Leftrightarrow ABB^{\dagger}A^{\dagger}AB = AB, \\ &f(A,\, B,\, A^{*},\, B^{*},\, A^{\dagger},\, B^{\dagger},\, (AB)^{\dagger}) = 0 \Leftrightarrow (AB)^{\dagger} = B^{\dagger}A^{\dagger}, \end{aligned}$$

where $f(\cdot)$ is certain algebraic operation of A, B, A^{*}, B^{*}, $A^{\dagger}$, $B^{\dagger}$, and $(AB)^{\dagger}$, namely, the ROLs on the right-hand sides are unique solutions of the matrix equations on the left-hand sides (cf. [9]). It is expected that many more concrete matrix equalities that are equivalent to (1.2) can be established from theoretical and applied points of view. Here we give the following two examples

$$\begin{aligned} &(AB)^{\dagger} = B^{\dagger}A^{\dagger} \Leftrightarrow (AB)^{\dagger} = B^{\dagger}(A^{\dagger}(B^{\dagger}A^{\dagger})^{\dagger}B^{\dagger})^{\dagger}A^{\dagger} \ \text{and} \ (B^{\dagger}A^{\dagger})^{\dagger} = A(BB^{\dagger}A^{\dagger}A)^{\dagger}B, \\ &(AB)^{\dagger} = B^{\dagger}A^{\dagger} \Leftrightarrow (AB)^{\dagger} = B^{\dagger}(A^{\dagger}(B^{\dagger}(A^{\dagger}(B^{\dagger}A^{\dagger})^{\dagger}B^{\dagger})^{\dagger}A^{\dagger})^{\dagger}B^{\dagger})^{\dagger}A^{\dagger} \ \text{and} \ (B^{\dagger}A^{\dagger})^{\dagger} = A(BB^{\dagger}A^{\dagger}A)^{\dagger}B \end{aligned}$$

for exercises.

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